

**Remark 1.** These solutions are there to help. Almost every problem is corrected, with many details for most of them. Some solutions are really short because the question is not hard nor tricky.

Some computations are abbreviated with  $\dots$ , of course if you solve it yourself (to practice, or during an exam), you have to do every step!

Sometimes, it might not be specify in which set  $x$  or  $y$  are taken, it is usually  $\mathbb{R}$ .

For this sheet, some problems are not (yet) corrected because they really are harder than we thought they will be. Any suggestion is welcome!

## 1 Solutions for “Exercises to do before tutorials”.

1. Let  $f$  and  $g$  be defined by  $f : x \mapsto \ln(x + \sqrt{1 + x^2})$  and  $g : x \mapsto \ln(\sqrt{1 + x^2} - x)$ .

For  $x \in \mathbb{R}$ ,  $\sqrt{1 + x^2} > |x|$  so  $x + \sqrt{1 + x^2} > 0$  and  $\sqrt{1 + x^2} - x > 0$ .  $f$  and  $g$  are defined, continuous and differentiable on  $\mathbb{R}$ .

For  $x \in \mathbb{R}$ ,  $f'(x) = \left(1 + \frac{x}{\sqrt{1 + x^2}}\right) \frac{1}{x + \sqrt{1 + x^2}} = \dots = \frac{1}{\sqrt{1 + x^2}} > 0$ . Then  $f' > 0$  everywhere implies that  $f$  is increasing on  $\mathbb{R}$ .

For  $x \in \mathbb{R}$ , we use the usual trick of conjugate quantities<sup>1</sup>  $g(x) = \ln(\sqrt{1 + x^2} - x) = \ln\left(\frac{(1 + x^2) - x^2}{\sqrt{1 + x^2} + x}\right) = \ln\left(\frac{1}{\sqrt{1 + x^2} + x}\right) = -\ln(\sqrt{1 + x^2} + x) = -f(x)$  as wanted.

- For any  $x$ ,  $\sinh(f(x)) = \frac{e^{f(x)} - e^{-f(x)}}{2}$ , but  $-f = g$  so  $\sinh(f(x)) = \frac{e^{f(x)} - e^{g(x)}}{2}$  and therefore  $\sinh(f(x)) = \frac{(x + \sqrt{1 + x^2}) - (-x + \sqrt{1 + x^2})}{2} = x$  so  $\sinh(f(x)) = x$ .
- Conversely, (with patience) one can prove that  $f(\sinh(x)) = x$  for any  $x \in \mathbb{R}$ .

Therefore, we recognize  $f$ : it is the **inverse hyperbolic sine**, ie. the inverse function of the function  $\sinh : x \mapsto \frac{e^x - e^{-x}}{2}$  called hyperbolic sine.

**Remark 2.** For a more complete study of  $f$ , also known as inverse hyperbolic sine, please read this *Wikipédia* page: [https://en.wikipedia.org/wiki/Inverse\\_hyperbolic\\_sine](https://en.wikipedia.org/wiki/Inverse_hyperbolic_sine).

2. Find the domain of definition, domain of continuity and domain of derivability of the following functions, and then compute their derivatives:

(1) For  $f_1 : x \mapsto \ln\left(\frac{1 - x}{1 + x}\right)$ .

- $f_1(x)$  is defined iff  $1 + x \neq 0$  and  $\frac{1 - x}{1 + x} > 0$ , so iff  $x \in (-1, 1)$ . So  $D_{f_1} = (-1, 1)$ , on which  $f_1$  is clearly continuous and differentiable.

<sup>1</sup>For  $a \neq b$ ,  $a + b = \frac{a^2 - b^2}{a - b}$  and if  $a \neq -b$ ,  $a - b = \frac{a^2 - b^2}{a + b}$ .

- On this domain,  $f_1'(x) = \left( \frac{d}{dx} \left( \frac{1-x}{1+x} \right) \right) \times \left( \frac{1}{\frac{1-x}{1+x}} \right) = \dots = \frac{-2x}{1-x^2}$ .
- (2) For  $f_2 : x \mapsto \ln \left( \frac{x^2 - 4x + 4}{-x^2 - x + 6} \right)$ .
- $f_2(x)$  is defined iff  $-x^2 - x + 6 \neq 0$ , and  $\left( \frac{x^2 - 4x + 4}{-x^2 - x + 6} \right) > 0$ . By solving these two quadratic equations in  $x$ , we find that  $x^2 - 4x + 4 = (x-2)^2$  and  $-x^2 - x + 6 = -(x+3)(x-2)$ , so  $f_2(x)$  is defined iff  $x \in (-3, 2)$ . So  $D_{f_2} = (-3, 2)$ , on which  $f_1$  is well continuous and differentiable.
  - On this domain,  $f_2'(x) = \dots = \frac{2}{x-2} + \frac{2x+1}{-x^2-x+6}$ .
- (3) For  $f_3 : x \mapsto \ln(\cos(x))$ .
- $f_3(x)$  is defined iff  $\cos(x) > 0$  so iff there is a  $k \in \mathbb{Z}$  such that  $x - 2k\pi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . So  $D_{f_3} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) + 2\pi\mathbb{Z}$ , on which  $f_3$  is indeed continuous and differentiable.
  - On this domain,  $f_3'(x) = \frac{d}{dx}(\cos(x)) \frac{1}{\cos(x)} = -\tan(x)$ .
  - Remark: this result proved that a *primitive* of  $\tan$  is the function  $(-\pi/2, \pi/2) \rightarrow \mathbb{R}, x \mapsto -\ln(|\cos(x)|) + C$  (for any  $C \in \mathbb{R}$ ).
- (4) For  $f_4 : x \mapsto e^{\frac{1}{1-x}}$ .
- $f_4(x)$  is defined iff  $1-x \neq 0$ . So  $D_{f_4} = \mathbb{R} \setminus \{1\}$ , on which  $f_4$  is clearly continuous and differentiable.
  - On this domain,  $f_4'(x) = \left( \frac{d}{dx} \left( \frac{1}{1-x} \right) \right) \times e^{\frac{1}{1-x}} = \frac{e^{\frac{1}{1-x}}}{(1-x)^2}$ .
- (5) For  $f_5 : x \mapsto \sin(x)^{\cos(x)}$ .
- $f_5(x)$  is defined iff  $\sin(x) > 0$ . And then,  $f_5(x) = \exp(\cos(x) \ln(\sin(x)))$ . But  $\sin(x) > 0$  iff  $x \bmod 2\pi \in (0, \pi)$ . So  $D_{f_5} = (0, \pi) + 2\pi\mathbb{Z}$ , on which  $f_5$  is indeed continuous and differentiable.
  - On this domain,  $f_5'(x) = \left( -\sin(x) \ln(\sin(x)) + \cos(x) \frac{\cos(x)}{\sin(x)} \right) \exp(\cos(x) \ln(\sin(x)))$ .
  - You can try to simplify it more if you want!
- (6) For  $f_6 : x \mapsto x^{(x^x)}$ .
- $f_6(x)$  is defined iff  $x > 0$ . So  $D_{f_6} = (0, +\infty)$ , on which  $f_6$  is continuous and differentiable. Additionally,  $f_6(x) = \exp(x^x \ln(x)) = \exp(\ln(x) \exp(x \ln(x)))$ .
  - On this domain,  $f_6'(x) = x^x \left( \frac{1}{x} + \ln(x) (\ln(x) + 1) \right) x^{(x^x)}$ .

3. Find<sup>2</sup> the following limit:  $\lim_{x \rightarrow +\infty} \frac{(x^x)^x}{x^{(x^x)}}$ .

Remember that for any  $a, b > 0$ ,  $a^b = \exp(b \ln(a))$ . Therefore, for  $x > 0$ ,  $x^x = \exp(x \ln(x))$ .

And  $x^{(x^x)} = \exp(x^x \ln(x)) = \exp(\exp(x \ln(x)) \ln(x))$ , and  $(x^x)^x = \exp(x \ln(x^x)) = \exp(x^2 \ln(x))$ .

So, for  $x > 0$ , the fraction is  $\frac{(x^x)^x}{x^{(x^x)}} = \frac{\exp(x^2 \ln(x))}{\exp(x^x \ln(x))} = \exp(\ln(x) (x^2 - x^x)) = \underline{\underline{\exp(\ln(x) x^2 (1 - x^{x-2}))}}$ .

But now, observe these three limits:  $\ln(x) \xrightarrow{x \rightarrow +\infty} +\infty$ ,  $x^2 \xrightarrow{x \rightarrow +\infty} +\infty$ , and  $(1 - x^{x-2}) \xrightarrow{x \rightarrow +\infty} -\infty$ .

And we know that  $\exp(y) \xrightarrow{y \rightarrow -\infty} 0$ , so by composition and product we conclude that

$$\lim_{x \rightarrow +\infty} \frac{(x^x)^x}{x^{(x^x)}} = 0.$$

**Remark 3.** Numerically, if we want to compare  $x^{(x^x)}$  and  $(x^x)^x$ , we can compute their values for some values of  $x$ :

$x$	$(x^x)^x$	$x^{(x^x)}$
0	1	0
1	1	1
2	16	16
3	19683	$7.6256e^{+12}$
3.4	$1.39282e^{+06}$	$1.20548e^{+34}$
3.6	$1.62062e^{+07}$	$9.45719e^{+55}$
3.8	$2.35546e^{+08}$	$3.66703e^{+92}$
4	$4.29497e^{+09}$	$1.34078e^{+154}$
4.2	$9.86546e^{+10}$	$2.56862e^{+258}$

After 4.2, the calculator used for computing this (the Python<sup>3</sup> v2.7.6 toplevel), stopped and yelled at me with a “**OverflowError**”:  $x^{(x^x)}$  becomes **too big**.

With these values, it is clear that the denominator  $x^{(x^x)}$  is **growing way faster** than the numerator  $(x^x)^x$ .

4. For  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ , compute<sup>4</sup> the next two sums:  $\sum_{k=0}^n \binom{n}{k} \cosh(kx)$ ,  $\sum_{k=0}^n \binom{n}{k} \sinh(kx)$ .

By definition<sup>5</sup>,  $\cosh(x) = \frac{e^x + e^{-x}}{2}$  and  $\sinh(x) = \frac{e^x - e^{-x}}{2}$ .

Therefore, if  $C_n(x) = \sum_{k=0}^n \binom{n}{k} \cosh(kx)$ , we can directly write that  $\cosh(kx) = \frac{e^{kx} + e^{-kx}}{2}$ , and so:

$$C_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{e^{kx} + e^{-kx}}{2} = \sum_{k=0}^n \binom{n}{k} \left( \frac{e^{kx}}{2} + \frac{e^{-kx}}{2} \right)$$

<sup>3</sup>This is the programming language we will use for CS 101 on the second semester! Go take a look to <https://www.python.org/about/> if you are curious about it!

<sup>5</sup>You can remember these definitions with this little trick: for any function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$  is the decomposition of  $f$  into an **even** part  $x \mapsto \frac{f(x) + f(-x)}{2}$  and an **odd** part  $x \mapsto \frac{f(x) - f(-x)}{2}$ . Well then  $\cosh$  is the even part of  $\exp$  (easy to remember, because  $\cos$  itself is even), and  $\sinh$  is the odd part of  $\exp$  ( $\sin$  is odd).

By using the basic property  $a^{(bc)} = (a^b)^c$ , we can develop:

$$C_n(x) = \frac{1}{2} \sum_{k=0}^n \binom{n}{k} \left( (e^x)^k + (e^{-x})^k \right)$$

So by using associativity, we can write this as:

$$C_n(x) = \left( \frac{1}{2} \sum_{k=0}^n \binom{n}{k} (e^x)^k \right) + \left( \frac{1}{2} \sum_{k=0}^n \binom{n}{k} (e^{-x})^k \right)$$

And then by two applications of the **binomial theorem**, in the form  $(1+a)^n$  with  $a = e^x$  and  $a = e^{-x}$ :

$$C_n(x) = \left( \frac{1}{2} (1 + e^x)^n \right) + \left( \frac{1}{2} (1 + e^{-x})^n \right)$$

Similarly, one can compute that

$$S_n(x) = \left( \frac{1}{2} (1 + e^x)^n \right) - \left( \frac{1}{2} (1 + e^{-x})^n \right).$$

**Remark 4.** One other method, simpler but longer, is to compute  $C_n(x) + S_n(x) = \dots = \frac{1}{2} (1 + e^x)^n$  and then  $C_n(x) - S_n(x) = \dots = \frac{1}{2} (1 - e^x)^n$ , and finally solve this simple system of two equations and two unknowns.

**Remark 5.** We can also write  $C_n(x)$  as  $\frac{1}{2} \cosh^n \left( \frac{x}{2} \right) \cosh \left( \frac{nx}{2} \right)$ .

And  $S_n(x) = \frac{1}{2} \cosh^n \left( \frac{x}{2} \right) \sinh \left( \frac{nx}{2} \right)$

5. Show that, if  $\alpha > 0$ , then  $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^\alpha} = -\infty$ .

**Bonus:** also prove that  $\lim_{x \rightarrow 0^+} \ln(x).x^\alpha = 0$ .

Let  $\alpha > 0$ . For  $x \rightarrow 0^+$ ,  $\ln(x) \rightarrow -\infty$  on one hand; and  $x^\alpha \rightarrow 0^+$ , so  $\frac{1}{x^\alpha} \rightarrow +\infty$  on the other hand. So by the product rule,  $\frac{\ln(x)}{x^\alpha} \rightarrow (-\infty)(+\infty) = -\infty$  as wanted (there is no trick here, just simply computing the limit as a product).

Conversely, for the bonus limit, we can write  $\ln(x).x^\alpha = -\alpha \frac{\ln(y)}{y}$  for  $y = x^{-\alpha}$ . But when  $x \rightarrow 0^+$ ,  $y \rightarrow +\infty$ , and by one use<sup>6</sup> of L'Hôpital rule, we know that  $\frac{\ln(y)}{y} \rightarrow 0$  when  $y \rightarrow +\infty$ .

6. By the method you like, determine the following limits:

<sup>6</sup>Indeed, both  $\ln(y)$  and  $y$  go to  $+\infty$ , but  $\frac{d}{dy} (\ln(y)) = \frac{1}{y}$  and  $\frac{d}{dy} (y) = 1$ , and  $\frac{\frac{1}{y}}{1} = \frac{1}{y} \rightarrow 0$  when  $y \rightarrow +\infty \dots$

- (a)  $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$ , by L'Hôpital rule,
- (b)  $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln(a)$  again by L'Hôpital rule (here  $a > 0$ ),
- (c)  $\lim_{x \rightarrow 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$  also with L'Hôpital rule (here,  $\alpha \in \mathbb{R}$ ).
- (d)  $\lim_{x \rightarrow +\infty} (\ln(x))^{\frac{1}{x}} = 1$  because  $(\ln(x))^{\frac{1}{x}} = \exp\left(\frac{\ln(\ln(x))}{x}\right)$  and  $\ln(\ln(x))/x \rightarrow 0$  for  $x \rightarrow +\infty$ ,
- (e)  $\lim_{x \rightarrow 0} (\cos(x))^{\frac{1}{\sin^2(x)}} = \frac{1}{\sqrt{e}}$ .  
 First  $(\cos(x))^{\frac{1}{\sin^2(x)}} = \exp\left(\frac{\ln(\cos(x))}{\sin^2(x)}\right)$ . But we also know that  $\cos(x) + \frac{x^2}{2} \rightarrow 1$  for  $x \rightarrow 0$ , and  $\ln(1+y)/y \rightarrow 1$  for  $y \rightarrow 0$ .  
 So  $\ln(\cos(x)) \simeq -\frac{x^2}{2}$  for  $x \rightarrow 0$  and  $\sin^2(x) \simeq x^2$  at 0. So  $\frac{\ln(\cos(x))}{\sin^2(x)} \rightarrow -\frac{1}{2}$ , and so  $(\cos(x))^{\frac{1}{\sin^2(x)}} \simeq \exp(-1/2) = \frac{1}{\sqrt{e}}$  which is the limit.
- (f)  $\lim_{x \rightarrow 0^+} (\sin(x))^{\frac{1}{\ln(x)}} = e$ .  
 Similarly, we write  $(\sin(x))^{\frac{1}{\ln(x)}} = \exp\left(\frac{\ln(\sin(x))}{\ln(x)}\right)$ . But  $\sin(x) \simeq x$  for  $x \rightarrow 0^+$ , so  $\ln(\sin(x))$  will behave like  $\ln(x)$ , therefore,  $\frac{\ln(\sin(x))}{\ln(x)} \rightarrow 1$  at  $0^+$  (ie, for  $x \rightarrow 0^+$ ). And  $\exp(1) = e$ .
- (g)  $\lim_{x \rightarrow 0^+} x^{\sin(x)} = 1$ .  
 We write  $x^{\sin(x)}$  as  $\exp(\ln(x) \sin(x))$ . At  $0^+$ , we know that  $x \ln(x) \rightarrow 0$  and  $\frac{\sin(x)}{x} \rightarrow 1$ , so  $\ln(x) \sin(x) \rightarrow 0$ , and so  $\exp(\ln(x) \sin(x)) \rightarrow \exp(0) = 1$ .
- (h)  $\lim_{x \rightarrow +\infty} (e^x - 1)^{\frac{1}{x}} = e$ .  
 As previously, we write  $(e^x - 1)^{\frac{1}{x}} = \exp\left(\frac{\ln(e^x - 1)}{x}\right)$ . But we know that  $\ln(e^x - 1) = x + \ln(1 - e^{-x})$ , so  $\frac{\ln(e^x - 1)}{x} = 1 + \frac{\ln(1 - e^{-x})}{x}$ .  
 For  $x \rightarrow +\infty$ ,  $1 - e^{-x} \rightarrow 1$ , so  $\ln(1 - e^{-x}) \rightarrow 0$ , and so  $\frac{\ln(1 - e^{-x})}{x} \rightarrow 0$ .  
 Therefore,  $\exp\left(\frac{\ln(e^x - 1)}{x}\right) \rightarrow \exp(1) = e$ .

7. For  $x, y \in \mathbb{R}_+^*$ , by using a simple convexity inequality with the correct function<sup>7</sup>, show

$$x \ln(x) + y \ln(y) \geq (x + y) \ln\left(\frac{x + y}{2}\right).$$

One thing we can say is that  $g : x \mapsto x \ln(x)$  is convex. Indeed,  $g$  is twice differentiable on  $\mathbb{R}_+^*$ , and  $g'(x) = \ln(x) + 1$  so  $g''(x) = \frac{1}{x} > 0$ .

And then, we just write the convexity inequality, with  $\lambda = \frac{1}{2}$ :  $g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y)$ . But  $g(x) = x \ln(x)$  and  $(1 - \lambda) = \frac{1}{2}$ , so we indeed have  $\frac{(x + y)}{2} \ln\left(\frac{x + y}{2}\right) \leq \frac{x}{2} \ln(x) + \frac{y}{2} \ln(y)$ , so  $(x + y) \ln\left(\frac{x + y}{2}\right) \leq x \ln(x) + y \ln(y)$ , which is exactly what we wanted.

## 2 Solutions for “Exercises to do during or after tutorials”.

1. Solve, on the appropriate set, the following equation:  $\ln|x + 2| + \ln 3 \leq \ln|3x - 4|$ .

We know that  $\ln(|x + 2|) + \ln(3) = \ln(3|x + 2|)$ , so:  $\ln|x + 2| + \ln 3 \leq \ln|3x - 4|$  iff  $\ln(3|x + 2|) \leq \ln(|3x - 4|)$ , but  $\ln$  is increasing, so  $\ln|x + 2| + \ln 3 \leq \ln|3x - 4|$  iff  $3|x + 2| \leq |3x - 4|$  iff ... iff  $x \leq \frac{-1}{3}$  (by plotting the two graphs, or just solving it in details).

Finally,  $\ln|x + 2| + \ln 3 \leq \ln|3x - 4|$  iff  $x \leq \frac{-1}{3}$ .

2. Precise the values (for  $x$ ) when the following expressions have a sense, and try to simplify them as much as possible:

- (1) Let  $x > 1$ . For  $x^{\frac{\ln(\ln(x))}{\ln(x)}}$ , we remember that  $a^b = \exp(b \ln(a))$  for  $a, b > 0$ . Here,  $x^{\frac{\ln(\ln(x))}{\ln(x)}} = \exp\left(\frac{\ln(\ln(x))}{\ln(x)} \ln(x)\right) = \exp(\ln(\ln(x))) = \ln(x)$ .
- (2) Again, let  $x > 1$ . For the second one,  $\ln_x(\ln_x(x^{(x^x)}))$ , we remember that  $\ln_x$  is the  $x$ -based **logarithm**. So  $\ln_x(\ln_x(x^{(x^x)})) = \ln_x(x^x) = x$ , directly!

3. Decide which of the two numbers is greater:

For this kind of problem, there is no magical recipe, just try to use what you know.  
*Any simpler solution which use less additional facts is welcome.*

- (a) For  $e^\pi$  or  $\pi^e$ :  
 We know that  $\pi^e = e^{e \ln(\pi)}$ . But  $e \simeq 2.7182$  and  $\pi < 3.1416$ , so  $\ln(\pi) < \ln(3.1416) \simeq 1.15$ , so  $e \ln(\pi) < 3.11 < \pi$ .  
 So  $\pi^e < e^\pi$ , which can be verified numerically with a calculator.
- (b) For  $2^{\sqrt{2}}$  or  $e$ : We know that  $2^{\sqrt{2}} = e^{\sqrt{2} \ln(2)}$ . But  $\ln(2) < 0.7$  and  $\sqrt{2} < 1.42$ , so  $\sqrt{2} \ln(2) < 0.993 < 1$ . Therefore,  $e^{\sqrt{2} \ln(2)} < e^{0.993} < e^1 = e$ .  
 Finally,  $2^{\sqrt{2}} < e$ , which can be verified numerically with a calculator.
- (c) For  $\ln(8)$  or  $2$ .  
 Well,  $\ln(8) = \ln(2^3) = 3 \ln(2) > 3 * 0.68 = 2.04 > 2$ , so  $\ln(8) > 2$ .

**Remark 6.** *To be more rigorous, one should (be able to) **prove** the facts that have been used :  $\ln(2) \simeq 0.693147$ ,  $\sqrt{2} \simeq 1.414213$ ,  $e \simeq 2.718281$  etc. More details on these ideas can be found in the footnotes and references of a page like this one:*  
[https://en.wikipedia.org/wiki/E\\_\(mathematical\\_constant\)#Known\\_digits](https://en.wikipedia.org/wiki/E_(mathematical_constant)#Known_digits).

4. Compute the derivatives of the following functions, of course you will not forget to precise their domains:

(1)  $f_1 : x \mapsto \arccos(x - 1)$ , is defined on  $[0, 2]$ , because  $\arccos$  is defined and continuous on  $[-1, 1]$ .

And  $f_1'(x) = \arccos'(x - 1) = \frac{-1}{\sqrt{1 - (x - 1)^2}} = \frac{-1}{\sqrt{2x - x^2}}$  on  $(0, 2)$ , because  $\arccos$  is differentiable on  $(-1, 1)$  (and we check indeed that this expression of  $f_1'$  is valid only for  $x \in (-1, 1)$ ).

(2) For  $f_2 : x \mapsto \arctan|x|$ , we know that  $\arctan$  is defined and continuous on  $\mathbb{R}$ , so  $f_2$  is also defined and continuous on  $\mathbb{R}$ . And  $f_2$  is differentiable on  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ . On this domain<sup>8</sup>,  $f_2'(x) = \frac{|x|}{x} \frac{1}{1 + (|x|)^2}$ , so  $f_2'(x) = \frac{1}{1 + x^2}$  if  $x > 0$  and  $f_2'(x) = \frac{-1}{1 + x^2}$  if  $x < 0$ .

(3) For  $f_3 : x \mapsto \operatorname{arcsinh}(\sqrt{x^2 - 1})$ , we know that  $\operatorname{arcsinh}$  is defined and continuous on  $\mathbb{R}$  (because  $\sinh$  is a bijection from  $\mathbb{R}$  to  $\mathbb{R}$ ), and so  $f_3(x)$  is defined *iff*  $x^2 - 1 \geq 0$  *iff*  $|x| \geq 1$  *iff*  $x \in (-\infty, -1] \cup [1, \infty)$ . And  $f_3$  is differentiable on this domain, except when the square root is 0, ie  $|x| = 1$ , so  $f_3$  is differentiable on  $(-\infty, -1) \cup (1, \infty)$ . There,  $f_3'(x) = \left( \frac{d}{dx}(\sqrt{x^2 - 1}) \right) \operatorname{arcsinh}'(\sqrt{x^2 - 1}) = \frac{x}{\sqrt{x^2 - 1}} \times \frac{1}{\sqrt{1 + (\sqrt{x^2 - 1})^2}}$ , so  $f_3'(x) = \frac{x}{\sqrt{x^2 - 1}} \times \frac{1}{|x|} = \frac{x}{|x|\sqrt{x^2 - 1}}$ .

Remark: for  $x > 1$ ,  $h(x) = \operatorname{arccosh}(x)$  in fact, thanks to the classic identity  $\cosh^2(x) - \sinh^2(x) = 1$  true for all  $x \in \mathbb{R}$ .

5. (**Hard**) For  $a, b \in \mathbb{R}$ , with  $a \neq b$ , by using the concavity of  $\tanh : [0, +\infty) \rightarrow \mathbb{R}, x \mapsto \tanh x = \frac{e^x - 1}{e^x + 1}$ , show that

$$\frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2}.$$

Assume  $a < b$  (if  $b < a$ , we can just swap  $a$  and  $b$ ). Then lets solve this inequality, with patience:

$$\begin{aligned} \frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2} &\iff e^b - e^a < \frac{b - a}{2} (e^b + e^a) \\ &\iff e^{b-a} - 1 < \frac{b - a}{2} (e^{b-a} + 1) \end{aligned}$$

<sup>8</sup>Note that  $\frac{|x|}{x}$  is usually written  $\operatorname{sign}(x)$ , the **sign** function ( $-1$  if  $x < 0$ ,  $+1$  if  $x > 0$ ).

By dividing by  $e^a$  on both sides, because  $e^a \neq 0$ . So we have:

$$\begin{aligned} \frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2} &\iff \frac{e^{b-a} - 1}{e^{b-a} + 1} < \frac{b - a}{2} \\ &\iff \frac{e^x - 1}{e^x + 1} < \frac{x}{2} \text{ if we define } x = b - a > 0 \\ &\iff \left( \frac{e^{x/2} - e^{-x/2}}{2} \right) \left( \frac{2}{e^{x/2} + e^{-x/2}} \right) < \frac{x}{2} \\ &\iff \frac{\sinh(x/2)}{\cosh(x/2)} < \frac{x}{2} \\ \text{So } \frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2} &\iff \tanh(x/2) < \frac{x}{2} \text{ with } x = b - a > 0. \end{aligned}$$

But  $\tanh(t) < t$  for all  $t > 0$ , as we will prove in details here:

Indeed,  $\tanh$  is twice differentiable, and for  $t > 0$ , we have  $\tanh(t) < 1$ ,  $\tanh'(t) = 1 - \tanh^2(t) > 0$  and  $\tanh''(t) = -2 \tanh'(t) \tanh(t) < 0$ . And one property we know tells us that  $\tanh'' < 0$  implies that  $\tanh$  is (strictly) concave on  $[0, +\infty)$ .

So graphically, the graph of  $\tanh$  on  $[0, +\infty)$  will be smaller than any of its tangent (this is also one property of the concavity).

But the tangent to this graph, at  $t = 0$ , is exactly the line  $y = x$ , so if the graph is smaller than its tangent, we conclude that  $\tanh(t) < t$ .

Remark: this proof is the same as the one we did to show that  $\sin(x) < x$  for any  $x > 0$ .

Finally, we indeed have  $\tanh(x/2) < x/2$  for any  $x > 0$ , so the initial inequality is always true, if  $a < b$ .

And this inequality is **symmetric** in  $a$  and  $b$ , so proving it for  $a < b$  is enough.

**Conclusion:** so we have, as wanted,  $\frac{e^b - e^a}{b - a} < \frac{e^b + e^a}{2}$ , for any  $a, b \in \mathbb{R}$  such that  $a \neq b$ .

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**Remark 7.** Another method, using only the strict convexity of  $\exp$ , is given here, thanks to *Prov. Vijaysekhar*.

Without loss of generality, let  $a < b$  and let  $x \in (a, b)$ . Now, define  $\lambda \equiv \frac{x - a}{b - a}$  and note that  $\lambda \in (0, 1)$ . Since the exponential is a strict convex function it follows that  $e^{(\lambda b + (1-\lambda)a)} < \lambda e^b + (1 - \lambda)e^a$ , or, equivalently (substituting for  $\lambda$ ),  $e^x < f(x)$ , where  $f(x) = \left(\frac{x - a}{b - a}\right) e^b + \left(\frac{b - x}{b - a}\right) e^a$ . Since,  $x \in (a, b)$  it follows that  $f(x) - e^x > 0$  for all  $x \in (a, b)$ . Hence,

$$\int_a^b (f(x) - e^x) dx > 0. \quad (2.1)$$

Now, note that  $\int_a^b f(x) dx = (b - a)(e^a + e^b)/2$  and  $\int_a^b e^x dx = e^b - e^a$ , and hence, (2.1) implies that  $(b - a)(e^a + e^b)/2 > e^b - e^a$ , yielding the desired result.



6. We say that  $f : [0, +\infty) \rightarrow \mathbb{R}$  is *subadditive*<sup>9</sup> if, for any  $x_1, x_2 \in (0, +\infty)$ ,  $f$  satisfies  $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ . Prove that:

(a) Assume that if  $x \mapsto \frac{f(x)}{x}$  is decreasing on  $(0, +\infty)$ , then lets prove that  $f$  is subadditive on  $[0, +\infty)$ . Let  $x_1 \leq x_2 \in [0, +\infty)$ .

First, assume that  $x_1 \neq 0$ . We first have  $\frac{f(x_1 + x_2)}{x_1 + x_2} \leq \frac{f(x_2)}{x_2}$  because  $x_1 + x_2 \geq x_2$  and the assumption on the variation of  $x \mapsto \frac{f(x)}{x}$ . So  $f(x_1 + x_2) \leq \left(1 + \frac{x_1}{x_2}\right) f(x_2)$ .

But  $f(x_2) \leq \frac{x_2}{x_1} f(x_1)$  again because  $x \mapsto \frac{f(x)}{x}$  is decreasing, and  $x_1 \leq x_2$  by assumption.

So  $\left(1 + \frac{x_1}{x_2}\right) f(x_2) \leq f(x_2) + f(x_1)$ . We can then conclude that  $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ .

And now if  $x_1 = 0$ ,  $f(0)$  is finite, so necessarily  $\frac{f(x)}{x} \xrightarrow{x \rightarrow 0} \pm\infty$ , but this limit has to be

positive because that function  $x \mapsto \frac{f(x)}{x}$  is decreasing. So in fact  $f(x)$  needs to converges to a positive value when  $x \rightarrow 0$ , so  $f(0) \geq 0$ , hence  $f(0) \leq f(0) + f(0)$ , and so finally we have  $f(0 + x_2) \leq f(0) + f(x_2)$ , as wanted.

And  $x_1, x_2$  are any number in  $[0, +\infty)$ , with just the condition  $x_1 \leq x_2$ . Obviously, if we swap them, we still have the same result, so we conclude that  $f$  is indeed subadditive.

(b) Assume that  $f$  is convex and subadditive<sup>10</sup> on  $[0, +\infty)$ , then we have to prove that  $x \mapsto \frac{f(x)}{x}$  is a decreasing function (on that interval  $[0, +\infty)$ ).

I have not been able to do it, any suggestion is welcome.

(c) We give here two examples of  $f$  such that  $x \mapsto \frac{f(x)}{x}$  is decreasing on  $[0, +\infty)$ ,

- one with  $f$  being convex:  $f = 0$  works well, any constant works also!
- one with  $f$  not convex (harder): TODO.

7. Show that each of the equations  $\sin(\cos(x)) = x$  and  $\cos(\sin(x)) = x$  has *exactly* one root in  $[0, \pi/2]$ . Moreover, show that if  $x_1$  and  $x_2$  are the roots of the former and the latter equation, respectively, then  $x_1 < x_2$ .

Let  $f(x) = \sin(\cos(x)) - x$  and  $g(x) = \cos(\sin(x)) - x$  on  $[0, \pi/2]$ . Both are continuous and differentiable, and we have  $f(0) = \sin(1) > 0$  and  $f(\pi/2) = \sin(0) - \pi/2 = -\pi/2 < 0$ .

So  $f$  is continuous on a closed interval, positive at the left limit of the interval (start value) and negative at the right limit of the interval (end value). By the intermediate value theorem, we can conclude that  $f$  has *at least* one root in  $[0, \pi/2]$ . That is a start, but it is not enough!

But  $f'(x) = -\sin(x) \cos(\cos(x)) - 1$ , and  $-1 < -\sin(x) \cos(\cos(x)) < 1$  on  $[0, \pi/2]$ , so  $-2 < f'(x) < 0$ , ie  $f' \leq 0$  so  $f$  is (strictly) decreasing. Then<sup>11</sup>,  $f$  is in fact a **bijection** (one-to-one

<sup>10</sup>The initial exercise asked to work on  $f$  defined only on  $(0, +\infty)$ , but it was not enough to prove this part. We need to be able to compute  $f(0)$ , which satisfies  $f(0) \leq f(0) + f(0)$ , so ...

<sup>11</sup>We use here the **theorem of the monotonic bijection** : if  $f : [a, b] \rightarrow [\alpha, \beta]$  is continuous and monotonic (ie, either strictly increasing or decreasing) then it is a bijection of inverse  $f^{-1} : [\alpha, \beta] \rightarrow [a, b]$  (this is a consequence of the Intermediate Value Theorem). Please go read about it on the Thomas Calculus book if you want more details about this (important) result.

and one-by-one) between  $[0, \pi/2]$  and  $[f(\pi/2), f(0)] = [-\pi/2, \sin(1)]$ . So there is only one antecedent of 0, ie only one solution of the equation  $\sin(\cos(x)) = x$ .

We can do the same for  $g : g'(x) = -\cos(x)\sin(\sin(x)) - 1 < 0$  so  $g$  is decreasing, and  $g(0) = 1 > 0$  and  $g(\pi/2) = \cos(1) - \pi/2 < 0$  because  $\cos(1) \leq 1 < \pi/2$ .

Let  $x_1$  be the root of the first equation, and  $x_2$  of the second equation.

**TODO: to conclude.**

### 3 Solutions for “Bonus exercises”.

1. Show that the following functions  $f_i$  are *bijective* (one-to-one and one-by-one), on the correct sets, and try to find an expression for their inverse ( $f_i^{-1}$ ) (precise their domain):

- (1)  $f_1 : x \mapsto 3x - 5$  is well defined on  $D_{f_1} = \mathbb{R}$ .

$$\text{For } x \in D_{f_1}, y \in \mathbb{R}, y = f_1(x) \text{ iff } x = \frac{y + 5}{3}.$$

So the function  $f_1$  is indeed a bijection, from  $\mathbb{R}$  to  $\mathbb{R}$ , of inverse  $f_1^{-1} : x \mapsto \frac{x + 5}{3}$ .

- (2)  $f_2 : x \mapsto \sqrt{3} - x$  is well defined on  $D_{f_2} = \mathbb{R}$ .

$$\text{For } x \in D_{f_2}, y \in \mathbb{R}, y = f_2(x) \text{ iff } y = \sqrt{3} - x.$$

So the function  $f_2$  is indeed a bijection, from  $\mathbb{R}$  to  $\mathbb{R}$ , of inverse itself  $f_2^{-1} = f_2$ .

- (3)  $f_3 : x \mapsto x^2 - 1$  is well defined on  $D_{f_3} = \mathbb{R}_-$ .

For  $x \in D_{f_3}, y \in \mathbb{R}, y = f_3(x) \text{ iff } y = x^2 - 1 \text{ iff } x^2 = y + 1$  so  $y \geq -1$ , and so as  $x \leq 0$ , we conclude that  $y = f_3(x) \text{ iff } x = -\sqrt{y + 1}$ .

So the function  $f_3$  is indeed a bijection, from  $[-1, +\infty)$  to  $\mathbb{R}_-$ , of inverse  $f_3^{-1} : x \mapsto -\sqrt{x + 1}$ .

- (4)  $f_4 : x \mapsto \frac{1}{3x - 2}$  is well defined on  $D_{f_4} = \mathbb{R} \setminus \left\{ \frac{2}{3} \right\}$ .

For  $x \in D_{f_4}, y \in \mathbb{R}, y = f_4(x) \text{ iff } y = \frac{1}{3x - 2}$  so  $y \neq 0$ , and so  $y = f_4(x) \text{ iff } y \neq 0$  and  $x = \left( \frac{1}{y} + 2 \right) / 3$ .

So the function  $f_4$  is indeed a bijection, from  $\mathbb{R} \setminus \{0\}$  to  $\mathbb{R} \setminus \left\{ \frac{2}{3} \right\}$ , of inverse  $f_4^{-1} : x \mapsto$

$$\left( \frac{1}{x} + 2 \right) / 3.$$

- (5)  $f_5 : x \mapsto \frac{3x + 2}{2x - 1}$  is well defined on  $D_{f_5} = \mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$ .

For  $x \in D_{f_5}, y \in \mathbb{R}, y = f_5(x) \text{ iff } 3x + 2 = y(2x - 1) \text{ iff } y \neq 2/3$  and  $x = -\frac{2 + y}{3 - 2y}$ .

So the function  $f_5$  is indeed a bijection, from  $\mathbb{R} \setminus \left\{ \frac{2}{3} \right\}$  to  $\mathbb{R} \setminus \left\{ \frac{3}{2} \right\}$ , of inverse  $f_5^{-1} : x \mapsto$

$$-\frac{2 + x}{3 - 2x}.$$

- (6)  $f_6 : x \mapsto \ln(2e^x - 1)$  is well defined on  $D_{f_6} = (-\ln(2), +\infty)$ .

For  $x \in D_{f_6}, y \in \mathbb{R}, y = f_6(x) \text{ iff } 2e^x - 1 = e^y \text{ iff } x = \ln(e^y + 1) - \ln(2)$ .

So the function  $f_6$  is indeed a bijection, from  $(-\ln(2), +\infty)$  to  $\mathbb{R}$ , of inverse  $f_6^{-1} : x \mapsto \ln(e^x + 1) - \ln(2)$ .

2. Same functions as the previous exercise, but now study the derivability of the functions and their inverses, then compute their derivatives. You can also try to look for some particular points where the functions and their derivatives are canceled.

- (1)  $f_1 : x \mapsto 3x - 5$  is well differentiable on its domain ( $D_{f_1} = \mathbb{R}$ ).  
 For  $x \in D_{f_1}$ ,  $f_1'(x) = 3$ .  
 And for any  $x \in \mathbb{R}$ ,  $(f_1^{-1})'(x) = \frac{1}{3}$ .
- (2)  $f_2 : x \mapsto \sqrt{3} - x$  is well differentiable on its domain ( $D_{f_2} = \mathbb{R}$ ).  
 For  $x \in D_{f_2}$ ,  $f_2'(x) = -1 = (f_2^{-1})'(x)$ .
- (3)  $f_3 : x \mapsto x^2 - 1$  is well differentiable on its domain ( $D_{f_3} = \mathbb{R}_-$ ).  
 For  $x \in D_{f_3}$ ,  $f_3'(x) = 2x$ .  
 And for the inverse, for  $x \geq -1$ , we have  $(f_3^{-1})'(x) = -\frac{1}{2\sqrt{x+1}}$ .
- (4)  $f_4 : x \mapsto \frac{1}{3x-2}$  is well differentiable on its domain ( $D_{f_4} = \mathbb{R} \setminus \left\{ \frac{2}{3} \right\}$ ).  
 For  $x \in D_{f_4}$ ,  $f_4'(x) = \frac{-3}{(3x-2)^2}$ .  
 And for the inverse, for  $x \neq 0$ , we have  $(f_4^{-1})'(x) = \frac{-1}{3x^2}$ .
- (5)  $f_5 : x \mapsto \frac{3x+2}{2x-1}$  is well differentiable on its domain ( $D_{f_5} = \mathbb{R} \setminus \left\{ \frac{1}{2} \right\}$ ).  
 For  $x \in D_{f_5}$ ,  $f_5'(x) = \frac{-7}{(2x-1)^2}$ .  
 And for the inverse, for  $x \neq 3/2$ , we have  $(f_5^{-1})'(x) = -\frac{7}{(2x-3)^2}$ .
- (6)  $f_6 : x \mapsto \ln(2e^x - 1)$  is well differentiable on its domain ( $D_{f_6} = (-\ln(2), +\infty)$ ).  
 For  $x \in D_{f_6}$ ,  $f_6'(x) = \frac{2e^x}{2e^x - 1}$ .  
 And for the inverse, for  $x > 0$ , we have  $(f_6^{-1})'(x) = \frac{e^x}{e^x - 1}$ .

3. Solve the equation  $\sinh(x) = 3$ , **without** using a calculator nor the formula expressing  $\operatorname{arcsinh}$  as a logarithm.

For  $x \in \mathbb{R}$ ,  $\sinh(x) = 3$  iff  $e^x - e^{-x} = 6$  iff  $e^{2x} - 6e^x - 1 = 0$  iff  $X^2 - 6X - 1 = 0$  with  $X = e^x$ . We solve this quadratic equation in  $X$ , which has two solutions  $X_{1,2} = 3 \pm \sqrt{10}$ , but  $X > 0$ , so  $x = \ln(X_2) = \ln(3 + \sqrt{10})$  is the only solution of the equation  $\sinh(x) = 3$ .

**Remark 8.** With the formula we know for  $\operatorname{arcsinh}$  (inverse hyperbolic sine), we have  $\sinh(x) = 3$  iff  $x = \operatorname{arcsinh}(3) = \ln(3 + \sqrt{1+3^2}) = \ln(3 + \sqrt{10})$ , as we found directly.

4. Solve<sup>12</sup> the equation  $\sum_{k=0}^n \sinh(2+kx) = 0$  for  $n \in \mathbb{N}$ .

Let  $S_n(x) = \sum_{k=0}^n \sinh(2 + kx)$  for  $x \in \mathbb{R}$ .

If  $x = 0$ ,  $S_n(0) = (n + 1) \sinh(2) \neq 0$  because  $n + 1 > 0$  and  $\sinh(2) > 0$  ( $\sinh(2) \simeq 3.6268$ ), so  $x = 0$  is not a solution.

Assume now that  $x \neq 0$ .  $2S_n(x) = \left( \sum_{k=0}^n \exp(2 + kx) \right) - \left( \sum_{k=0}^n \exp(-2 - kx) \right)$ . But  $e^x \neq 1$  because  $x \neq 0$ .

So  $2S_n(x) = e^2 \left( \frac{1 - e^{(n+1)x}}{1 - e^x} \right) - e^{-2} \left( \frac{1 - e^{-(n+1)x}}{1 - e^{-x}} \right)$  by using twice the formula of the sum of a geometric progression.

Hence  $2S_n(x) = \left( \frac{1 - e^{(n+1)x}}{1 - e^x} \right) (e^{-2} - e^{-nx-2})$ , and then we see that  $S_n(x) = 0$  iff  $e^{-2} - e^{-nx-2} = 0$  so iff  $-2 = -nx - 2$  iff  $x = -\frac{4}{n}$ . **Finally:**  $S_n(x) = 0 \iff x = -\frac{4}{n}$ .

5. For two numbers  $x, y \in \mathbb{R}_+^*$ , they satisfies this system:

$$\begin{cases} \operatorname{arcsinh}(x) &= 2 \operatorname{arcsinh}(y) \\ 3 \ln(x) &= 2 \ln(y) \end{cases}$$

iff  $x = \sinh(2 \operatorname{arcsinh}(y)) = 2y \cosh(\operatorname{arcsinh}(y)) = 2y\sqrt{1 + y^2}$  iff  $x^3 = y^2$ .

6. Let  $a, b \in \mathbb{R}$ , solve the following equation ( $x$  being the unknown):

$$(\cosh(x) + \sinh(x))^{\operatorname{arcsinh}(x-a)} = (\cosh(x) - \sinh(x))^{\operatorname{arcsinh}(x-b)}.$$

For one  $x \in \mathbb{R}$ ,  $x$  is the solution of this equation iff  $\exp(x \operatorname{arcsinh}(x-a)) = \exp(-x \operatorname{arcsinh}(x-b))$ , so iff  $x (\operatorname{arcsinh}(x-a) + \operatorname{arcsinh}(x-b)) = 0$ , so iff  $x = 0$  or  $x = \frac{a+b}{2}$ .

7. Let  $x, y \in \mathbb{R}$ , show that, if  $x = \ln\left(\tan\left(\frac{\pi}{4} + \frac{y}{2}\right)\right)$ , then:  $\tanh\left(\frac{x}{2}\right) = \tan\left(\frac{y}{2}\right)$ ,  $\tanh(x) = \sin(y)$  and  $\cosh(x) = \frac{1}{\cos(y)}$ .

- For the first one,  $\tanh\left(\frac{x}{2}\right) = \frac{e^x - 1}{e^x + 1} = \frac{\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) + 1}{\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) - 1} = \tan\left(\frac{y}{2}\right)$  as wanted.
- For the second one,  $\tanh(x) = \sin^2\left(\frac{\pi}{4} + \frac{y}{2}\right) - \cos^2\left(\frac{\pi}{4} + \frac{y}{2}\right) = -\cos\left(\frac{\pi}{2} + y\right) = \sin(y)$ .
- For the last one,  $\cosh(x) = \frac{1}{2} \left( \tan\left(\frac{\pi}{4} + \frac{y}{2}\right) + \arctan\left(\frac{\pi}{4} + \frac{y}{2}\right) \right)$  by definition of  $x$  in function of  $y$ , and so  $\cosh(x) = \frac{1}{\sin\left(\frac{\pi}{2} + y\right)} = \frac{1}{\cos(y)}$ .

8. Simplify the following expressions:

- (i)  $\arctan\left(\tan\left(\frac{5\pi}{6}\right)\right) = -\pi/6$ .
- (ii)  $\arctan\left(\sin\left(\frac{\pi}{6}\right)\right) = \arctan(1/2)$  which cannot be simplified I think.  
A numerical approximation is  $\arctan(1/2) \simeq 0.463647$ .
- (iii)  $\arctan\left(\cos\left(-\frac{\pi}{6}\right)\right) = \arctan(\sqrt{3}/2)$  which cannot be simplified I think.  
A numerical approximation is  $\arctan(\sqrt{3}/2) \simeq 0.71372$ .

9. Study these expressions (one could try to look for some trick):

- (1) For  $\arccos\left(\frac{1-t^2}{1+t^2}\right) + \arcsin\left(\frac{2t}{1+t^2}\right)$  (where  $t \in \mathbb{R}$ ), we can use the change of variable  $t = \tan\left(\frac{u}{2}\right)$ , for  $u \in (-\pi, \pi)$ .  
Then  $\left(\frac{1-t^2}{1+t^2}\right) = \cos(u)$  and  $\left(\frac{2t}{1+t^2}\right) = \sin(u)$ .  
So the sum is  $\arccos\left(\frac{1-t^2}{1+t^2}\right) + \arcsin\left(\frac{2t}{1+t^2}\right) = \arccos(\cos(u)) + \arcsin(\sin(u))$ .  
But  $-\pi < u < \pi$ , so  $\arccos(\cos(u)) = u$  if  $u \geq 0$  and  $= u + 2\pi$  if  $u < 0$  (remember that  $\arccos : [-1, 1] \rightarrow [0, 2\pi]$  by convention).  
And  $\arcsin(\sin(u)) = u$ , so finally the sum is  $\arccos\left(\frac{1-t^2}{1+t^2}\right) + \arcsin\left(\frac{2t}{1+t^2}\right) = 2u = 4 \arctan t$ .
- (2) For  $x, y \in \mathbb{R}$ , with  $xy \neq 1$ ,  $\arctan\left(\frac{x+y}{1-xy}\right) - \arctan(x) - \arctan(y) = k\pi$ , with  $k = 0$  if  $xy < 1$ ,  $k = 1$  if  $xy > 1$ ,  $x > 0$  and  $k = -1$  if  $xy < 1$  and  $x < 0$ .  
Let  $a = \arctan(x)$ ,  $b = \arctan(y)$ . So  $\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)} = \frac{x+y}{1-xy}$ .  
Finally, to conclude, we just need to observe that :
- $a + b \in (-\pi/2, \pi/2)$  if  $xy < 1$ ,
  - $a + b \in (\pi/2, \pi)$  if  $xy > 1$ , and  $x < 0$ ,
  - $a + b \in (-\pi, -\pi/2)$  if  $xy > 1$ , and  $x > 0$ .

For more details on this formula, which will be used in other exercices just below, please go read [https://en.wikipedia.org/wiki/Inverse\\_trigonometric\\_functions#Arctangent\\_addition](https://en.wikipedia.org/wiki/Inverse_trigonometric_functions#Arctangent_addition)

10. Solve these equations.

**Remark:** here we write  $x = y \pmod z$  (for three real numbers  $x, y, z$ ), which means that there is an integer  $k \in \mathbb{Z}$  such that  $x = y + kz$ .

- (1)  $\tan(3 \arcsin(x)) = 1$  for  $x \in [-1, 1]$ .

First, study which values are *not allowed*:  $x$  can be a solution if  $3 \arcsin(x) \neq \frac{\pi}{2} \pmod{\pi}$ . But  $3 \arcsin(x) = \frac{\pi}{2} \pmod{\pi}$  iff  $\arcsin(x) \in \left\{ -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \right\}$  and so iff  $x \in \left\{ -1, -\frac{1}{2}, \frac{1}{2}, 1 \right\}$ . So let  $x \in (-1, 1) \setminus \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$ . Then  $\tan(3 \arcsin(x)) = 1$  iff  $3 \arcsin(x) = \frac{\pi}{4} \pmod{\pi}$  iff  $\arcsin(x) = \frac{\pi}{12} \pmod{\pi}$  iff  $\arcsin(x) \in \left\{ -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{2} \right\}$ . Finally, we conclude by  $\tan(3 \arcsin(x)) = 1$  iff  $x \in \left\{ -\frac{\sqrt{2}}{2}, \sin\left(\frac{\pi}{12}\right), \sin\left(\frac{5\pi}{12}\right) \right\}$ , which cannot be simplified more I think.

(2)  $\arctan(x) + \arctan(2x) = \frac{\pi}{5}$  iff  $\tan(\arctan(x) + \arctan(2x)) = 1$ . But we know that  $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$  (for any  $a, b \in \mathbb{R}$  such that  $ab \neq 1$ ), so in this case,  $x$  is a solution iff  $\frac{3x}{1-2x^2} = 1$  so by solving this quadratic equation, and as  $x > 0$ , we conclude that the *only possible* solution is  $x = \frac{\sqrt{17}-1}{4}$ .

(3) In order to solve  $\arctan\left(\frac{p}{q}\right) + \arctan\left(\frac{q-p}{q+p}\right) = \frac{\pi}{5}$  (with  $p, q \in \mathbb{N}$  verifying  $0 < p < q$ ), we will use the formula for a sum like  $\arctan(a) + \arctan(b)$ .

So let  $a = \frac{p}{q}$  and  $b = \frac{q-p}{q+p}$ . We have  $a, b > 0$ . And  $ab < 1$  because  $a < 1, b < 1$ . So:

$$\begin{aligned} \arctan\left(\frac{p}{q}\right) + \arctan\left(\frac{q-p}{q+p}\right) &= \arctan\left(\frac{\frac{p}{q} + \frac{q-p}{q+p}}{1 - \left(\frac{p}{q}\right)\left(\frac{q-p}{q+p}\right)}\right) \\ &= \arctan\left(\frac{p(q+p) + q(q-p)}{q(q+p) - p(q-p)}\right) \\ &= \arctan\left(\frac{p^2 + q^2}{p^2 + q^2}\right) = \arctan(1) = \frac{\pi}{2} \\ \arctan\left(\frac{p}{q}\right) + \arctan\left(\frac{q-p}{q+p}\right) &= \frac{\pi}{2} \end{aligned}$$

So the equation has no solution, because  $\arctan\left(\frac{p}{q}\right) + \arctan\left(\frac{q-p}{q+p}\right)$  is always  $\pi/2$ , so cannot be  $\pi/5$ .

11. Compute these sums:

(1) Quickly:  $\arctan(1) + \arctan(2) + \arctan(3) = \arctan(1) + \arctan\left(\frac{2+3}{1-2 \times 3}\right) + \pi = \frac{\pi}{4} + \arctan\left(\frac{5}{-5}\right) + \pi = \frac{\pi}{2} - \frac{\pi}{2} + \pi = \pi$ ,

(2) With more details:

$$\arctan(2) + (\arctan(5) + \arctan(8)) = \arctan(2) + \left(\arctan\left(\frac{5+8}{1-5 \times 8}\right) + \pi\right)$$

Because if  $x = 5$ ,  $y = 8$ ,  $x, y > 0$ , but  $xy > 1$ .

$$\begin{aligned} \arctan(2) + \arctan(5) + \arctan(8) &= \arctan(2) + \arctan\left(-\frac{13}{39}\right) + \pi \\ &= (\arctan(2) + \arctan\left(-\frac{1}{3}\right)) + \pi \\ &= + \arctan\left(\frac{2 \times 3 - 1}{3 + 2}\right) + \pi \\ &= \frac{\pi}{4} + \pi \\ \arctan(2) + \arctan(5) + \arctan(8) &= \frac{5\pi}{4}. \end{aligned}$$

**Remark 9.** We used twice the formula proved previously:  $\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right) + k\pi$ , true for any  $a, b \in \mathbb{R}$  such that  $ab \neq 1$ , with  $k = 0$  if  $ab < 1$ ,  $k = 1$  if  $ab > 1$ ,  $a > 0$  and  $k = -1$  if  $ab < 1$  and  $a < 0$ .

12. For  $n \in \mathbb{N}^*$ , and  $x_1, \dots, x_n \in (0, \pi)$ , let  $x \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^n x_k$ .

(a) To prove  $\prod_{k=1}^n \sin(x_k) \leq (\sin(x))^n$ , we apply log on both sides, and prove that  $\ln\left(\prod_{k=1}^n \sin(x_k)\right) \leq \ln((\sin(x))^n)$ .

Let  $g : (0, \pi) \rightarrow \mathbb{R}, t \mapsto \ln(\sin(t))$ .  $g$  is twice differentiable, and  $g'(t) = \frac{\cos(t)}{\sin(t)}$  and  $g''(t) = -1 - \frac{\cos^2(x)}{\sin^2(x)} < 0$  on  $(0, \pi)$ . So  $g'' < 0$  implies that  $g$  is (strictly) concave on its domain.

And thanks to the generalized concavity inequality (with  $n$  points), we have then  $g\left(\frac{x_1 + \dots + x_n}{n}\right) \geq \frac{g(x_1) + \dots + g(x_n)}{n}$ , so with the definition of this special point  $x$ , we conclude that  $\ln(\sin(x)) \geq \frac{\ln(\sin(x_1)) + \dots + \ln(\sin(x_n))}{n}$ .

Hence, by applying the *increasing* function exp on both sides, we conclude that  $\prod_{k=1}^n \sin(x_k) \leq (\sin(x))^n$ , as wanted.

(b) To prove  $\prod_{k=1}^n \frac{\sin(x_k)}{x_k} \leq \left(\frac{\sin(x)}{x}\right)^n$ , we do exactly the same, but with the function  $g_2 : (0, \pi) \rightarrow \mathbb{R}, t \mapsto \ln\left(\frac{\sin(t)}{t}\right)$ , which is also concave.

Indeed,  $g_2'(t) = \cot(t) - \frac{1}{t}$  and  $g_2''(t) = \frac{1}{t^2} - \cot^2(t) - 1$  so  $g_2''(t) < 0$  on  $(0, \pi)$ .

13. For  $x \in \mathbb{R}$  and  $y \in \mathbb{R}_+^*$ , show that  $xy \leq e^x + y(\ln(y) - 1)$ .

Prove also that the inequality is an equality (ie, the equality holds) if and only if  $y = e^x$ .

Let  $y > 0$  be a fixed real number, and  $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^x + y(\ln(y) + 1) - xy$ .  $f$  is of class  $\mathcal{C}^1$  on  $\mathbb{R}$ , and  $f'(x) = e^x - y$  on  $\mathbb{R}$ .

So  $f'$  is negative on  $(-\infty, \ln(y))$ , positive on  $(\ln(y), +\infty)$ , and zero at  $\ln(y)$ .

Hence  $f$  is decreasing on  $(-\infty, \ln(y))$ , increasing on  $(\ln(y), +\infty)$ , and zero<sup>13</sup> at  $\ln(y)$ .

So we can conclude that  $\forall x \in \mathbb{R}, f(x) \geq 0$ , and  $f(x) = 0$  iff  $x = \ln(y)$  (ie iff  $y = e^x$ ).

Therefore we conclude that the inequality is true for every  $x \in \mathbb{R}, y > 0$ , and the equality holds iff  $y = e^x$ .

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<sup>13</sup>Indeed,  $f(\ln(y)) = y + y \ln(y) - y - \ln(y)y = 0$ . In fact,  $f$  has a global minimum at that point  $x = \ln(y)$ , and this fact directly implies the inequality we wanted!