

1 Homework

- 1.1. Show that the sequence $(\sqrt{n})_{n \in \mathbb{N}}$ diverges to $+\infty$.
- 1.2. Consider sequences where terms are given by following terms. Check if each of these sequences are convergent or divergent. If they are convergent then compute the limits.

$$(1) u_n \triangleq \frac{n^2+3n-4}{4n^2+5},$$

$$(3) u_n \triangleq \frac{n^4+7n}{5n^4+\cos(n^2)-\frac{1}{n}},$$

$$(2) u_n \triangleq \frac{\sin(n^2)}{n},$$

$$(4) u_n \triangleq \frac{a^n-b^n}{a^n+b^n}, \text{ with } a, b \in \mathbb{R}_+^*,$$

- 1.3. Show the following equalities:

$$(1) \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1,$$

$$(2) \sum_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \frac{3}{2},$$

$$(3) \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2},$$

$$(4) \sum_{n=2}^{\infty} \frac{\log[(1+\frac{1}{n})^n(1+n)]}{(\log n^n)[\log(n+1)^{n+1}]} = \log_2 \sqrt{e}.$$

- 1.4. Use convergence tests to explore the convergence/divergence of the following series:

$$(1) \sum_{n=1}^{\infty} \frac{n^2}{2^n},$$

$$(4) \sum_{n=1}^{\infty} \frac{1}{1000n+1},$$

$$(2) \sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2},$$

$$(5) \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!},$$

$$(3) \sum_{n=2}^{\infty} \frac{n!}{(n+2)!},$$

$$(6) \sum_{n=1}^{\infty} \frac{1000^n}{n!}.$$

2 Tutorial Problems

- 2.1. Is the product sequence of two lower-bounded sequences still lower-bounded?
- 2.2. Show that a non-decreasing sequence is lower-bounded.
- 2.3. Let u be an integer sequence. Prove that u converges *if and only if*¹ u is constant after a certain number of terms (that is, $\exists n_0$ such that $\forall n \geq n_0, u_n = u_{n_0}$).

¹If and only if will be abbreviated **iff**.

- 2.4. Let u be a bounded sequence, satisfying: $\forall n \in \mathbb{N}^*, 2u_n \leq u_{n-1} + u_{n+1}$. Let $v = (v_n)_{n \in \mathbb{N}}$ be given by $\forall n \in \mathbb{N}, v_n \triangleq u_{n+1} - u_n$. Show² that v converges, and compute its limit.
- 2.5. Let u be a convergent sequence. Let v be a sub-sequence of u . Show that then v converges (to the same limit).
- 2.6. Consider $u \in \mathbb{R}^{\mathbb{N}}$ a real sequence (that is, a sequence with values being reals), and assume that the three sequences $(u_{2n+1})_{n \in \mathbb{N}}$, $(u_{2n})_{n \in \mathbb{N}}$, $(u_{3n})_{n \in \mathbb{N}}$ are all converging. Then show that u converges (to the same limit, obviously).

3 Bonus Problems

- 3.1. Let u be a bounded sequence and v be a sequence diverging to $+\infty$. Show that $u + v$ diverges to $+\infty$.
- 3.2. Let u be a convergent sequence and v be a divergent sequence. Show that $u + v$ diverges. What can you say about the product sequence $(u.v)_n \triangleq u_n.v_n, \forall n \in \mathbb{N}$?
- 3.3. Let u and v be two sequences converging to l et l' , respectively. We consider the two sequences wi and ws defined by: $\forall n \in \mathbb{N}, wi_n \triangleq \min\{u_n, v_n\}, ws_n \triangleq \max\{u_n, v_n\}$. Show that both the sequences wi and ws converge and compute their limits.
- 3.4. Let u be a non-decreasing real sequence. If we assume that u has a sub-sequence which converges, then show that u also converges.

²First prove that v is monotonic and bounded, then prove by contradiction that its limit is zero.