Solutions to Exercises on Sequences and Series

1.1. Let $a_n \triangleq \sqrt{n}$. Need to show that for every $M>0$ there exists $N(M)$ such that $a_n \geq M$ for all $n \geq N(M)$.

Let $M>0$ and $N(M) \triangleq M^2$. Then for all $n \geq N(M) = M^2$ it follows that $\sqrt{n} \geq M$.

1.2. (1) $$u_n = \frac{n^2+3n-4}{4n^2+5} = \frac{1+\frac{3}{n}-\frac{4}{n^2}}{4+\frac{5}{n^2}}.$$ Define

$$f(x) = \frac{1+3x-4x^2}{4+5x^2}$$. Note $f(\cdot)$ is continuous at 0 and $f(0)=\frac{1}{4}$. $\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} f(\frac{1}{n})$

$$= f(0) = \frac{1}{4}.$$

(2) $$u_n = \frac{\sin(n^2)}{n}.$$ Now, $-\frac{1}{n} \leq u_n \leq \frac{1}{n}$.

Let $\varepsilon>0$.

Since $\frac{1}{n} \to 0$ as $n\to\infty$ it follows that $\exists\, N(\varepsilon)$ such that $\frac{1}{n} < \varepsilon \;\forall\, n \geq N(\varepsilon)$ [infact $N(\varepsilon) \geq \lceil \frac{1}{\varepsilon} \rceil$].

Hence,

$$|u_n| \leq \frac{1}{n} < \varepsilon, \;\forall\, n \geq N(\varepsilon).$$

$$\Rightarrow u_n \to 0.$$

(3) $$u_n = \frac{n^4+7n}{5n^4+\cos(n^2)-\frac{1}{n}} = \frac{1+\frac{7}{n^3}}{5+\frac{\cos(n^2)}{n^4}-\frac{1}{n^5}}.$$

First, show that $\frac{7}{n^3} \to 0$, $\frac{\cos(n^2)}{n^4} \to 0$, $\frac{1}{n^5} \to 0$ as $n\to\infty$.

Hence $u_n \to \frac{1}{5}$ using properties $\lim(a_n+b_n) = \lim a_n + \lim b_n$ and $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$ if $b_n \neq 0$ and $\lim b_n \neq 0$.

(4) $$u_n = \frac{a^n - b^n}{a^n + b^n} = \frac{1-\left(\frac{b}{a}\right)^n}{1+\left(\frac{b}{a}\right)^n} = \frac{\left(\frac{a}{b}\right)^n - 1}{\left(\frac{a}{b}\right)^n + 1}.$$

If $a=b$ then $u_n = 0 \Rightarrow u_n \to 0$ as $n \to \infty$.

If $a<b$ then $\left(\frac{a}{b}\right)^n \to 0 \Rightarrow u_n \to -1$ as $n \to \infty$.

If $b>0$ then $\left(\frac{b}{a}\right)^n \to 0 \Rightarrow u_n \to 1$ as $n \to \infty$.

1.3. (1) $$a_n = \frac{2n+1}{n^2(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \underbrace{\frac{1}{n^2}}_{b_n} - \underbrace{\frac{1}{(n+1)^2}}_{b_{n+1}}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = b_1 - \underbrace{\lim_{n\to\infty} b_n}_{\to 0} = b_1 = 1.$$

(2) $$a_n = \frac{2^n + 3^n}{6^n} = \frac{1}{3^n} + \frac{1}{2^n} = \frac{1}{2}\cdot\frac{3-1}{3^n} + \frac{2-1}{2^n}$$

$$= \frac{1}{2}\left[\frac{1}{3^{n-1}} - \frac{1}{3^n}\right] + \left[\frac{1}{2^{n-1}} - \frac{1}{2^n}\right]$$

$$= \underbrace{\left(\frac{1}{2}\cdot\frac{1}{3^{n-1}} + \frac{1}{2^{n-1}}\right)}_{b_n} - \underbrace{\left(\frac{1}{2}\cdot\frac{1}{3^n} + \frac{1}{2^n}\right)}_{b_{n+1}}.$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = b_1 - \underbrace{\lim_{n\to\infty} b_n}_{\to 0} = \frac{1}{2} + 1 = \frac{3}{2}.$$

(3) $$a_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

Let $b_n = \frac{1}{2}\cdot\frac{1}{(2n-1)} \Rightarrow \sum a_n = b_1 - \underbrace{\lim b_n}_{\to 0}$

$$\Rightarrow \sum a_n = \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

(4) $$a_n = \frac{\log\left(\left(1+\frac{1}{n}\right)^n (1+n)\right)}{\log n^n \, \log (n+1)^{n+1}} = \frac{\log\left(\frac{(n+1)^{n+1}}{n^n}\right)}{\log n^n \, \log (n+1)^{n+1}}$$

$$= \frac{1}{\log n^n} - \frac{1}{\log (n+1)^{n+1}} \Rightarrow \sum_{n=2}^{\infty} a_n = \frac{1}{\log 2^2} = \log_2 \sqrt{e}.$$

1.4.

(1) $a_n = \frac{n^2}{2^n} \Rightarrow a_{n+1} = \frac{(n+1)^2}{2^{n+1}} \Rightarrow \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^2 \frac{2^n}{2^{n+1}}$.

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} \Rightarrow$ $\sum a_n$ is convergent from the ratio test.

(2) $a_n = \frac{|\sin nx|}{n^2}$. $0 \leq a_n \leq \frac{1}{n^2}$. Comparison test implies that $\sum a_n$ is convergent since $\sum \frac{1}{n^2}$ is.

(3) $a_n = \frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)} \leq \frac{1}{n^2} \Rightarrow \sum a_n$ is convergent.

(4) $\sum_{n=1}^{\infty} \frac{1}{1000n+1} = \frac{1}{1000} \sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{1000}}$

Let $a_n = \frac{1}{n+\frac{1}{1000}}$ and $b_n = \frac{1}{n} \Rightarrow \frac{a_n}{b_n} = \frac{n}{n+\frac{1}{1000}}$.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum b_n$ diverges it follows from the limit comparison test that

$\sum_{n=1}^{\infty} \frac{1}{1000n+1}$ diverges.

(5) $a_n = \frac{(n!)^2}{(2n)!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}$

$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2(n+1)(2n+1)} = \frac{n+1}{4(n+\frac{1}{2})} \Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4} < 1$.

$\Rightarrow \sum \frac{(n!)^2}{(2n)!}$ is convergent from the ratio test.

(6) Use ratio test.

2.1. No. Let $a_n = n$ and $b_n = -1$, both lower bounded but $c_n = a_n b_n = -n$ is not lower-bounded.

2.2. Trivial.

2.3. i) If $\exists\, n_0$ such that $u_n = u_{n_0}$, $n \geq n_0$

then for every $\varepsilon > 0$,
$$|u_n - u_{n_0}| = 0 < \varepsilon \quad \forall\ n \geq n_0.$$

$$\Rightarrow \quad u_n \to u_{n_0} \text{ as } n \to \infty.$$

ii) If the integer sequence u_n converges then there exists $l \in \mathbb{R}$ (since u_n are also reals) such that ~~that~~ for every $\varepsilon > 0$ $\exists\ N(\varepsilon)$ s.t.
$$|u_n - l| < \varepsilon, \quad n \geq N(\varepsilon).$$

If l is not an integer then for every integer $|l - u_n| > 0 \Rightarrow$ contradiction.

Hence, l is an integer. If $u_n \neq l$ then $|u_n - l| \geq 1 \Rightarrow l = u_n \ \forall\ n \geq N(\varepsilon)$.

The result follows by setting $n_0 = N(\varepsilon)$.

2.4. Note $2u_n \leq u_{n-1} + u_{n+1} \Leftrightarrow u_n - u_{n-1} \leq u_{n+1} - u_n$.

Hence, $v_{n-1} \leq v_n \Rightarrow v_n$ is a monotic sequence.

Since, ~~u_n is~~ u is bounded $\exists$ B such that $|u_n| \leq B \ \forall\ n$.

$$\Rightarrow \quad |v_n| \leq |u_{n+1}| + |u_n| \leq 2B \ \forall\ n.$$

$$\Rightarrow \quad v \text{ is bounded} \Rightarrow \text{convergent.}$$

~~$\Rightarrow$ it is also~~

Now, suppose $\lim_{n\to\infty} v_n = l \neq 0$. ~~Let~~ Let $\varepsilon < |l|$.

$\exists$ $N(\varepsilon)$ such that

$$|v_n - l| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$$

$$\Rightarrow \quad l-\varepsilon < v_n < l+\varepsilon, \quad \forall\ n \geq N(\varepsilon)$$

$$\Rightarrow u_n + l - \varepsilon < u_{n+1} < u_n + l + \varepsilon, \ \forall\ n \geq N(\varepsilon).$$

$$\Rightarrow u_n + m(l-\varepsilon) < u_{n+m} < u_n + m(l+\varepsilon), \ \forall\ n \geq N(\varepsilon).$$

If $l > 0 \Rightarrow l - \varepsilon > 0 \Rightarrow u_{n+m} > u_n + m(l-\varepsilon)$

$\Rightarrow \lim_{n\to\infty} u_{n+3} = \infty$. Similarly, $u_{n+m} \to -\infty$ if $l < 0$.

$\Rightarrow$ contradicting boundedness of u.

Hence, $l = 0$.

2.5. ~~Since~~ Let $u \to l$ as $n \to \infty$. Let $\varepsilon > 0$. There exists $N(\varepsilon)$ such that

$$|u_n - l| < \varepsilon \quad \forall\ n \geq N(\varepsilon).$$

Since v is a subsequence $v_n = u_{k(n)}$ for some $k : \mathbb{N}^* \to \mathbb{N}^*$ such $k(n) \geq n$.

$$\Rightarrow \quad |v_n - l| = |u_{k(n)} - l| < \varepsilon, \ \forall\ n \geq N(\varepsilon)$$

since $k(n) \geq n$.

$$\Rightarrow \quad v_n \to l \ \text{ as } n \to \infty.$$

2.6. Let v be a subsequence defined by

$$v_n = u_{6n+3}$$

Note v_n is a subsequence of both (u_{2n+1}) and (u_{3n}) $\Rightarrow$ v_n converges to a limit l (which is unique). From Problem 2.5 it follows that (u_{2n+1}) and (u_{3n}) converge to l.
Similarly (u_{2n}) and (u_{3n}) converge to l
[By considering $v_n = u_{6n}$].
Hence, (u_{2n+1}) and (u_{2n}) [as well as (u_{3n})] converge to limit l. Now, the result can be easily proved by constructing $N(\varepsilon)$ (for a given $\varepsilon > 0$) such that $|u_n - l| < \varepsilon$, $\forall n \geq N(\varepsilon)$.

3.2. i) Suppose $u+v$ is convergent. Then
$v = (u+v) - u$ $\Rightarrow$ v is convergent (since linear combination of convergent sequences is convergent).
$\Rightarrow$ contradiction. $\Rightarrow$ $u+v$ is divergent.

ii) Nothing can be said.

<u>Ex 1</u>: $u_n = \frac{1}{n}$, $v_n = n$ $\Rightarrow$ $(u_n \cdot v_n) = 1$ (convergent).

<u>Ex 2</u>: $u_n = \frac{1}{n}$, $v_n = n^2$ $\Rightarrow$ $(u_n \cdot v_n) = n$ (divergent).

3.1. ~~From~~ $|u_n| \leq B$ for some $B \geq 0$.

Next, note

$$\cancel{|u_n+v_n| \geq |v_n| - |u_n|} \quad \cancel{\geq |v_n| - B}$$

$$v_n + u_n \geq v_n - B.$$

$\Rightarrow$ If v_n diverges to $+\infty$ then u_n+v_n diverges to $+\infty$.

3.3. Suppose $l < l'$ and let $\varepsilon > 0$ be such that $\varepsilon < l'-l$. ~~Let $\varepsilon > 0$~~ $\exists\, N(\varepsilon)$ such that

$$|u_n - l| < \varepsilon/2 \text{ and } |v_n - l'| < \varepsilon/2 \quad \forall\, n \geq N(\varepsilon)$$

$$[\text{ Take } N(\varepsilon) = \max\{N_l(\varepsilon), N_{l'}(\varepsilon)\}].$$

For all $n \geq N(\varepsilon)$:

~~If $l = l'$ then the result is trivial.~~

~~Suppose $l < l'$.~~ $\Rightarrow$ $l - \varepsilon/2 < u_n < l + \varepsilon/2$ and

$$l' - \varepsilon/2 < v_n < l' + \varepsilon/2$$

$$\Rightarrow \quad u_n < l + \varepsilon/2 \leq l' - \varepsilon/2 < v_n.$$

$$\Rightarrow \quad w_{in} = u_n \Rightarrow |w_{in} - l| < \varepsilon/2 < \varepsilon \ \forall\, n \geq N(\varepsilon).$$

$$\Rightarrow \quad \boxed{\lim_{n\to\infty} w_{in} = l}.$$

Also

$$w_{sn} = v_n \quad \Rightarrow \quad |w_{sn} - l'| < \varepsilon \ \forall\, n \geq N(\varepsilon).$$

$$\Rightarrow \quad \boxed{\lim_{n\to\infty} w_{sn} = l'}.$$

~~The result~~ Hence, for this case, $\lim_{n\to\infty} w_{in} = \min(l, l')$

and $\lim_{n\to\infty} w_{sn} = \max(l, l')$.

The cases $l = l'$ and $l > l'$ can be shown similarly.

3.4. Let v be the convergent subsequence of u. Since all convergent sequences are bounded v is also bounded. Specifically, v is upperbounded, that is,

$v_n \leq B$ for some $B \geq 0$. where

$v_n = u_{k(n)}$ for some $k : \mathbb{N} \to \mathbb{N}$ such that $k(n) \geq n$.

Hence, $u_n \leq u_{k(n)} = v_n \leq B$.

$\Rightarrow$ u is bounded.

$\Rightarrow$ u is converging (since it is monotonic).