

# 1 Field Axioms

Let  $\mathbb{F}$  denote a set on which two binary operators  $\oplus$  and  $\odot$  are defined, that is,  $\oplus : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$  and  $\odot : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ . Let  $x, y \in \mathbb{F}$  then we denote the result of the  $\oplus$  operator as  $x \oplus y$  and the  $\odot$  operator as  $x \odot y$ . Unless it is ambiguous we denote  $x \odot y$  as simply  $xy$ . For the following axioms, let  $x, y, z \in \mathbb{F}$ .

AXIOM 1.1 COMMUTATIVE LAWS:  $x \oplus y = y \oplus x$  and  $xy = yx$ .

AXIOM 1.2 ASSOCIATIVE LAWS:  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$  and  $x(yz) = (xy)z$ .

AXIOM 1.3 DISTRIBUTIVE LAWS:  $x(y \oplus z) = xy \oplus xz$ .

AXIOM 1.4 IDENTITY ELEMENTS: There exist two (distinct) real numbers 0 and 1 such that for every  $x \in \mathbb{F}$ ,  $x \oplus 0 = x$  and  $x \odot 1 = x$ .

AXIOM 1.5 NEGATIVES: For every  $x \in \mathbb{F}$  there exists  $y \in \mathbb{F}$  such that  $x \oplus y = 0$ .

AXIOM 1.6 RECIPROCAL: For every  $x \in \mathbb{F} \setminus \{0\}$ , there exists  $y \in \mathbb{F} \setminus \{0\}$  such that  $xy = 1$ . We denote such  $y$  by  $x^{-1}$ , the *reciprocal* of  $x$ .

**Theorem 1.1** *Let  $a, b, c, d \in \mathbb{F}$ .*

PROPERTY 1.1: *If  $a \oplus b = a \oplus c$  then  $b = c$ .*

PROPERTY 1.2: *There exists a unique  $x \in \mathbb{F}$  such that  $a \oplus x = b$ . (We denote  $x$  by  $b - a$ . Specifically,  $0 - a$  is  $-a$ , the negative of  $a$ ).*

PROPERTY 1.3:  $b - a = b \oplus (-a)$ .

PROPERTY 1.4:  $-(-a) = a$ .

PROPERTY 1.5:  $a(b - c) = ab - ac$ .

PROPERTY 1.6:  $0 \odot a = a \odot 0 = 0$ .

PROPERTY 1.7: *If  $ab = ac$  and  $a \neq 0$  then  $b = c$ .*

PROPERTY 1.8: *If  $a \neq 0$  then there exists a unique  $x \in \mathbb{F}$  such that  $ax = b$  (we denote this  $x$  by  $b/a$  or  $\frac{b}{a}$ . Specifically,  $1/a$  is  $a^{-1}$ , the reciprocal of  $a$ ).*

PROPERTY 1.9: *If  $a \neq 0$  then  $b/a = b \odot a^{-1}$ .*

PROPERTY 1.10: If  $a \neq 0$  then  $(a^{-1})^{-1} = a$ .

PROPERTY 1.11: If  $ab = 0$  then  $a = 0$  or  $b = 0$ .

PROPERTY 1.12:  $(-a)b = -(ab)$  and  $(-a)(-b) = ab$ .

PROPERTY 1.13:  $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad \oplus bc}{bd}$  if  $b \neq 0$  and  $d \neq 0$ .

PROPERTY 1.14:  $\frac{a}{b} \frac{c}{d} = \frac{ac}{bd}$  if  $b \neq 0$  and  $d \neq 0$ .

PROPERTY 1.15:  $\frac{a/b}{c/d} = \frac{ac}{bd}$  if  $b \neq 0$ ,  $c \neq 0$ , and  $d \neq 0$ .

## 2 Problems

**Problem 2.1** Prove Theorem 1.1.

**Problem 2.2** Verify whether all the field axioms and all the properties of Theorem 1.1 hold for:

- (i)  $\mathbb{F} = \mathbb{Z}$ , the set of integers,  $\oplus = +$ , and  $\odot = \cdot$ , the usual addition and multiplication.
- (ii)  $\mathbb{F} = \mathbb{Q}$ , the set of rational numbers,  $\oplus = +$ , and  $\odot = \cdot$ , the usual addition and multiplication.
- (iii) Let  $n$  be a positive integer greater than 1 and let  $\mathbb{F} = \{0, 1, 2, \dots, n-1\}$ . For  $x, y \in \mathbb{F}$ , defined  $x \oplus y = (x + y) \bmod n$  and  $x \odot y = (xy) \bmod n$ . Furthermore, what is  $-1$  in this case and check if  $(-1)(-1) = 1$ .
- (iv) Let  $S$  be a nonempty set and let  $\mathbb{F}$  be such that  $x \in \mathbb{F}$  if and only if  $x \subseteq S$ , that is,  $\mathbb{F}$  contains all the subsets of  $S$  including  $\emptyset$  and  $S$ . Define  $\oplus = \cup$  and  $\odot = \cap$ , the union and the intersection operators, respectively.