

1 Homework

- 1.1. Show that sum of two functions which are individually continuous at a point is continuous at that point.
- 1.2. Compute the following left-hand side limits, right-hand side limits, and limits (if they exist):

$$(1) \lim_{x \rightarrow 0} \frac{|x|}{x}$$

$$(4) \lim_{h \rightarrow 0} \frac{(t+h)^2 - t^2}{h}$$

$$(2) \lim_{x \rightarrow 0} \frac{\sqrt{x^2}}{x}$$

$$(5) \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a}$$

$$(3) \lim_{x \rightarrow 1} \frac{2x^2 - 3x + 1}{x - 1}$$

$$(6) \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin x}$$

- 1.3. Use the fact that $|\sin x| < |x|$ for all $0 < |x| < \frac{\pi}{2}$ to show that $\sin(\cdot)$ is continuous at 0. Next, use the identity $\cos 2x = 1 - 2\sin^2 x$ to show that $\cos(\cdot)$ is continuous at 0. Finally, use the standard formulae for $\sin(x+h)$ and $\cos(x+h)$ to show that $\sin(\cdot)$ and $\cos(\cdot)$ are continuous everywhere in \mathbb{R} .
- 1.4. Let f be a polynomial of degree n such that the last and first coefficients have opposite signs. Prove that there exists a positive x such that $f(x) = 0$.

2 Tutorial Problems

- 2.1. Let f and g be given below and let $h = f \circ g$. In each case, determine the domains of f , g , and h and given direct formulae for h . Finally, investigate the continuity of h :
- (1) $f(x) = x^2 - 2x$, $g(x) = x + 1$, (3) $f(x) = \sqrt{x}$, $g(x) = x + \sqrt{x}$,
(2) $f(x) = \sqrt{x}$, $g(x) = x^2$, (4) $f(x) = \sqrt{x}$, $g(x) = \sin x$.
- 2.2. If n is an odd positive integer and $a < 0$ prove that there is exactly one negative b such that $b^n = a$.
- 2.3. Let $f(x) = \tan x$. Although $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, there is no x in $[\pi/4, 3\pi/4]$ such that $f(x) = 0$. Why does this not contradict the intermediate value theorem?
- 2.4. Let $f(x) = x \sin(\frac{1}{x})$, $x \neq 0$. What should be $f(0)$ so that f is continuous at 0.

- 2.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $|f(x) - f(y)| \leq L|x - y|$ for some $L \geq 0$. Show that f is continuous everywhere.
- 2.6. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be well-defined. Show that f is continuous everywhere on \mathbb{Z} .

3 Bonus Problems

- 3.1. Let $f(x) = \sin \frac{1}{x}$, $x \neq 0$. Can we choose $f(0)$ such that f is continuous at 0? What about limit?
- 3.2. Show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent (this shows that e^x is well-defined for all $x \in \mathbb{R}$). Next, show that there exists $L > 0$ such that $e^x \leq 1 + Lx$ for all $x \in [0, 1]$. Finally, show that e^x is continuous at 0 and subsequently everywhere in \mathbb{R} .
- 3.3. Use the fact that e^x is continuous to show that $\log x$ is continuous in its domain.
- 3.4. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous everywhere in $[0, 1]$. Assume that $0 \leq f(x) \leq 1$, $x \in [0, 1]$. Show that there exists $c \in [0, 1]$ such that $f(c) = c$.