

1 Exercises to do before tutorials.

Ex.1.1) Show that sum $(f + g)$ of two functions (f, g) which are individually continuous at a point (x_0) is continuous at that point.

Answer: Let $\varepsilon > 0$.

- f is continuous at x_0 , so we have a $\delta_{f,\varepsilon} > 0$ such that $\forall x \in D_f, |x - x_0| < \delta_{f,\varepsilon} \implies |f(x) - f(x_0)| < \frac{\varepsilon}{2}$.
- g is continuous at x_0 , so we have a $\delta_{g,\varepsilon} > 0$ such that $\forall x \in D_g, |x - x_0| < \delta_{g,\varepsilon} \implies |g(x) - g(x_0)| < \frac{\varepsilon}{2}$.

Let now define $\delta \stackrel{\text{def}}{=} \min(\delta_{f,\varepsilon}, \delta_{g,\varepsilon}) > 0$. If $x \in D_f \cap D_g = D_{f+g}$, and if $|x - x_0| < \delta$, then we have both $|x - x_0| < \delta_{f,\varepsilon}$ and $|x - x_0| < \delta_{g,\varepsilon}$, so we have $|(f+g)(x) - (f+g)(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Therefore, $(f + g)$ is continuous at x_0 .

Ex.1.2) Compute the following left-hand side limits, right-hand side limits, and limits (if they exist):

- (a) $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1, \lim_{x \rightarrow 0^+} \frac{|x|}{x} = +1$, so no limit at 0.
- (b) Exactly the same, as $\sqrt{x^2} = |x|$. Hence, $\lim_{x \rightarrow 0^-} \frac{\sqrt{x^2}}{x} = -1, \lim_{x \rightarrow 0^+} \frac{\sqrt{x^2}}{x} = +1$, so no limit at 0.
- (c) $\lim_{x \rightarrow 1^-} \frac{2x^2 - 3x + 1}{x - 1} = \lim_{x \rightarrow 1^+} \frac{2x^2 - 3x + 1}{x - 1} = \lim_{x \rightarrow 1} \frac{2x^2 - 3x + 1}{x - 1} = 1$.
- (d) $\lim_{\substack{x \rightarrow 0^- \\ 2t + h \rightarrow 2t}} \frac{(t+h)^2 - t^2}{h} = \lim_{x \rightarrow 0^+} \frac{(t+h)^2 - t^2}{h} = \lim_{x \rightarrow 0} \frac{(t+h)^2 - t^2}{h} = 2t$, because $\frac{(t+h)^2 - t^2}{h} = \frac{2th + h^2}{h} = 2t + h$.
- (e) $\lim_{x \rightarrow a^-} \frac{\sin x - \sin a}{x - a} = \lim_{x \rightarrow a^+} \frac{\sin x - \sin a}{x - a} = \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a} = 1$,
- (f) $\sin(\theta) \underset{\theta \rightarrow 0}{\simeq} \theta$, so $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin x} = 5$.

Ex.1.3) Use the fact¹ that $|\sin(x)| < |x|$, true for all $0 < |x| < \frac{\pi}{2}$ to show that \sin is continuous at 0. Next, use the identity $\cos(2x) = 1 - 2\sin^2(x)$ ($\forall x \in \mathbb{R}$) to show that \cos is also continuous at 0. Finally, use the standard formula for $\sin(x+h)$ and $\cos(x+h)$ to show that \sin and \cos are continuous everywhere on \mathbb{R} .

Answer: We have $\sin(0) = 0$. Let $\varepsilon > 0$, and define $\delta = \varepsilon$. If $0 < |x - 0| < \delta$, then $|\sin(x) - \sin(0)| = |\sin(x)| < |x| < \varepsilon$. So, by definition, \sin is continuous at 0.

Let call $f : y \mapsto 1 - 2y^2$. f is a polynomial function, so it is of course continuous everywhere. Thanks to the composition property, $f \circ \sin$ is still continuous on 0. But the given identity say $\cos(x) = (f \circ \sin)(\frac{x}{2})$, and $g : x \mapsto \frac{x}{2}$ is also continuous at 0. So $\cos = (f \circ \sin) \circ g = f \circ \sin \circ g$ is continuous at 0.

For $x, h \in \mathbb{R}$, we have $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$. So if $h \rightarrow 0$, $\sin(x+h) \rightarrow \sin(x)\cos(0) + \cos(x)\sin(0)$, but $\cos(0) = 1, \sin(0) = 0$, so $\sin(x+h) \rightarrow \sin(x)$, which is saying that \sin is continuous at that point x .

And similarly, for $x, h \in \mathbb{R}$, we have $\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$. So if $h \rightarrow 0$, $\cos(x+h) \rightarrow \cos(x)\cos(0) - \sin(x)\sin(0)$, so $\cos(x+h) \rightarrow \cos(x)$, which is saying that \cos is continuous at that point x .

Ex.1.4) Let f be a real polynomial function of degree $n > 0$ such that the last and first coefficients have opposite signs. Prove² that there exists a positive x (i.e. $x \in \mathbb{R}_+^*$) such that $f(x) = 0$.

¹You can also try to prove that fact.

²You should start by naming all these coefficients $a_0, \dots, a_n \in \mathbb{R}$, and use contradiction ("if f has no zero in \mathbb{R}_+^* , then ...")

Answer: Let start by naming these coefficients: for $x \in \mathbb{R}$, $f(x)$ can be written as $a_0 + a_1x + \dots + a_{n-1}x^{n-1} + a_nx^n = \sum_{k=0}^n a_kx^k$, with $a_0, a_1, \dots, a_n \in \mathbb{R}$ being the (real) coefficients of this polynomial function, and $a_n \neq 0$ (because f is of degree exactly n).

The hypothesis is $a_0a_n < 0$ (they have a different sign). If $f(0) = a_0 = 0$, then $x = 0$ is already good to have $f(x) = 0$.

Assume now that $a_0 \neq 0$. If $a_n < 0$, we have $f(x) \xrightarrow{x \rightarrow +\infty} -\infty$, and if $a_n > 0$, we have $f(x) \xrightarrow{x \rightarrow +\infty} +\infty$.

- First case is if $a_0 < 0$. Then by the hypothesis of a_0 and a_n being of different signs, we have $a_n > 0$. So $f(x) \xrightarrow{x \rightarrow +\infty} +\infty$. That is, we can find a $A > 0$ such that $f(A) > 1$ (direct application of the definition of $\xrightarrow{x \rightarrow +\infty} +\infty$). Then we can use BOLZANO's theorem in $[0, A]$, because $f(0) < 0, f(A) > 0$. There we have a $x_1 \in (0, A)$ such that $f(x_1) = 0$.
- The other case is if $a_0 > 0$, and we do the same. We have $a_n > 0$. So $f(x) \xrightarrow{x \rightarrow +\infty} -\infty$. That is, we can find a $B > 0$ such that $f(B) < -1$ (direct application of the definition of $\xrightarrow{x \rightarrow +\infty} +\infty$). Then we can use BOLZANO's theorem in $[0, B]$, because $f(0) > 0, f(B) < 0$. There we have a $x_2 \in (0, B)$ such that $f(x_2) = 0$.

In both cases, we have at least on $x \geq 0$ such that $f(x) = 0$.

2 Exercises to do during or after tutorials.

Ex.2.1) Let f and g be given below and let $h = f \circ g$. In each case, determine the domains of f , g , and h and given direct formula for h . Finally, investigate the continuity of h :

- (i) $f(x) = x^2 - 2x, g(x) = x + 1,$ (iii) $f(x) = \sqrt{x}, g(x) = x + \sqrt{x},$
(ii) $f(x) = \sqrt{x}, g(x) = x^2,$ (iv) $f(x) = \sqrt{x}, g(x) = \sin(x).$

Answer:

- (i) f and g are both defined and continuous on \mathbb{R} . So h is also continuous on \mathbb{R} . For $x \in \mathbb{R}, h(x) = (f \circ g)(x) = f(x + 1) = (x + 1)^2 - 2(x + 1) = x^2 + 2x + 1 - 2x - 2 = x^2 - 1$.
- (ii) $D_f = \mathbb{R}_+,$ and $D_g = \mathbb{R},$ on which f and g are continuous. But $g(x) \geq 0$ for all x , so $h = f \circ g$ is well defined for all $x \in \mathbb{R}$. Hence h is also continuous on \mathbb{R} . For $x \in \mathbb{R}, h(x) = (f \circ g)(x) = f(x^2) = \sqrt{x^2} = |x|$.
- (iii) $D_f = D_g = \mathbb{R}_+$ because of the \sqrt{x} in their expression, and f and g are continuous on their domains. But $g(x) \geq 0$ for all $x \geq 0$, so $h = f \circ g$ is well defined for all $x \in \mathbb{R}_+$. Hence h is also continuous on \mathbb{R}_+ . For $x \in \mathbb{R}_+, h(x) = (f \circ g)(x) = f(x + \sqrt{x}) = \sqrt{x + \sqrt{x}}$.
- (iv) $D_f = \mathbb{R}_+,$ and $D_g = \mathbb{R},$ on which f and g are continuous. But $g(x) \geq 0 \Leftrightarrow x \in [0, \pi] + 2\pi\mathbb{Z},$ so $h = f \circ g$ is well defined on $[0, \pi] + 2\pi\mathbb{Z},$ and it is obviously continuous on this domain. For $x \in [0, \pi] + 2\pi\mathbb{Z}, h(x) = (f \circ g)(x) = f(\sin(x)) = \sqrt{\sin(x)}$.

Ex.2.2) If n is an odd positive integer ($n \in 2\mathbb{N} + 1$) and $a < 0$ prove that there is exactly one negative $b < 0$ such that $b^n = a$. And what happens if n is even?

Quick answer: If $n \in 2\mathbb{N} + 1$, write n as $2k + 1$, with $k \in \mathbb{N}$. Let $x \stackrel{\text{def}}{=} \sqrt[n]{|a|}$, and $b \stackrel{\text{def}}{=} -x$, then $b^n = (-1)^n x^n = (-1)^{2k+1} a = -a$. So that was for *existence*. And for *uniqueness*, it comes³ from the fact that $t \mapsto t^n$ is strictly increasing in that case (where n is odd).

If n is even, there is no possible solution, as $n = 2k$ gives $b^n = (b^k)^2 > 0$ if $b \neq 0$.

³We should justify this step more precisely, if possible. Do it yourself to practice.

Ex.2.3) Let $f = \tan$ (on a certain domain you should precise). Although $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, there is no x in $[\pi/4, 3\pi/4]$ such that $f(x) = 0$.

Why does this not contradict the *intermediate value theorem*?

Answer: \tan is not continuous on $[\pi/4, 3\pi/4]$ because it is not defined on $[\pi/2]$: $\lim_{x \rightarrow \pi/2^-} \tan(x) = +\infty$, but $\lim_{x \rightarrow \pi/2^+} \tan(x) = -\infty$. And the *intermediate value theorem* is only for continuous function on an interval $[a, b]$.

Ex.2.4) Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x \sin\left(\frac{1}{x}\right)$ if $x \neq 0$.

What should be the value of $f(0)$ so that f is continuous at 0.

Short answer: $f(0)$ have to be 0. Indeed, for $x \in \mathbb{R}^*, -x \leq x \sin\left(\frac{1}{x}\right) \leq x$ so $f(x) \xrightarrow{x \neq 0, x \rightarrow 0} 0$.

Ex.2.5) For $a, b \in \mathbb{R}, a < b$, let $f : [a, b] \rightarrow \mathbb{R}$ be such that⁴ $\forall x, y \in [a, b], |f(x) - f(y)| \leq L \times |x - y|$, for some $L \geq 0$. Show that f is continuous on $[a, b]$.

Answer: If $L = 0$, then f is constant so obviously continuous.

Assume $L > 0$. Let $x_0 \in [a, b]$ and $\varepsilon > 0$. We choose $\delta(\varepsilon) = \frac{\varepsilon}{L} > 0$. For $y \in [a, b]$, if $|x_0 - y| < \delta(\varepsilon)$, then $|f(y) - f(x_0)| \leq L|x_0 - y| \leq L\delta(\varepsilon) = L\frac{\varepsilon}{L} = \varepsilon$. So by definition, f is continuous at x_0 . And x_0 was any $x_0 \in [a, b]$, so f is continuous on all $[a, b]$.

Ex.2.6) Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be well-defined. Show that f is continuous everywhere on \mathbb{Z} .

Quick answer: We just use the “ $\varepsilon, \delta(\varepsilon)$ ” definition, because if $\delta(\varepsilon) < 1, x, y \in \mathbb{Z}$, then $(|x - y| < \delta(\varepsilon) \implies x = y)$.

So, for one $x_0 \in \mathbb{Z}$, let take $\varepsilon > 0$. If we choose $\delta(\varepsilon) = \frac{1}{2} > 0$, then we observe that we have, for every $y \in \mathbb{Z}, |x_0 - y| < \delta(\varepsilon) \implies x_0 = y \implies |f(x_0) - f(y)| = 0 < \varepsilon$. So f is continuous at this x_0 .

3 Bonus exercises.

Ex.3.1) Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \sin\left(\frac{1}{x}\right)$ if $x \neq 0$. Can we choose $f(0)$ such that f is continuous at 0? What about limit?

Answer: The answer is no.

Let prove it, by defining $u_n = \frac{1}{\pi/2 + 2n\pi}$ and $v_n = \frac{1}{3\pi/2 + 2n\pi}$ for $n \in \mathbb{N}$. This define two sequences u and v . Observe that $u_n > 0, v_n > 0$ for every n .

For $n \in \mathbb{N}, f(u_n) = \sin(\pi/2 + 2n\pi) = \sin(\pi/2) = 1$, and $f(v_n) = \sin(3\pi/2 + 2n\pi) = \sin(3\pi/2) = -1$. But $u_n \xrightarrow{n \rightarrow +\infty} 0$, and $v_n \xrightarrow{n \rightarrow +\infty} 0$.

So if we were able to give a value to $f(0)$ for f to be continuous at 0, then we would have both $1 = \lim_{n \rightarrow +\infty} f(u_n) = f(\lim_{n \rightarrow +\infty} u_n) = f(0)$ and $-1 = \lim_{n \rightarrow +\infty} f(v_n) = f(\lim_{n \rightarrow +\infty} v_n) = f(0)$. That is absurd, as $f(0)$ cannot be 1 and -1 .

Conclusion, we cannot choose a value for $f(0)$ to f to be continuous at 0.

⁴ f is said to be L -LIPSCHITZ, or LIPSCHITZ continuous of ratio L on $[a, b]$.

Ex.3.2) For $x \in \mathbb{R}$, show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent (this shows that e^x is well-defined for all $x \in \mathbb{R}$).
 Next, show that there exists $\lambda > 0$ such that $e^x \leq 1 + \lambda x$ for all $x \in [0, 1]$. Finally, show that $\exp : x \mapsto e^x$ is continuous at 0, and subsequently everywhere in \mathbb{R} .

Answer: By ratio test, it is obvious, because if $a_n \stackrel{\text{def}}{=} \frac{x^n}{n!}$, then $a_n \neq 0$ for any $n \in \mathbb{N}$, and $\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \rightarrow 0 < 1$. So that series is convergent for any $x \in \mathbb{R}$.

We know that $\exp : x \mapsto e^x$ is strictly convex on $[0, 1]$, so a good λ can be such that $e^0 + \lambda \times 1 = e^0 + e^1$, i.e. $\lambda = e^1 - e^0 = e - 1$. Indeed, thanks to convexity, we know that “the curve is always below a string”. The equation of the *curve* is $y = e^x$, the equation of the *string* (from $x = 0$ to $x = 1$) is $y = \exp(0) + \frac{\exp(1) - \exp(0)}{1 - 0}x = 1 + e \times x$.

Therefore, with $\lambda \stackrel{\text{def}}{=} e - 1$, we have $|e^x - 1| \leq \lambda x$ for all $x \in [0, 1]$. Let $\varepsilon > 0$. Now choose $\delta(\varepsilon) \stackrel{\text{def}}{=} \frac{\varepsilon}{\lambda} > 0$, then if $|x - 0| = |x| = x < \delta(\varepsilon)$, then $|e^x - e^0| = |e^x - 1| \leq \lambda x < \lambda \delta(\varepsilon) = \lambda \frac{\varepsilon}{\lambda} = \varepsilon$. So by definition of being continuous at $x_0 = 0$, $\exp : x \mapsto e^x$ is continuous at 0.

Let $x_0 \in \mathbb{R}$. For $h \neq 0$, we have $\exp(x_0 + h) = e^{x_0+h} = e^{x_0} \times e^h$. Thanks to continuity at 0, proved previously, we know that $e^h \xrightarrow{h \rightarrow 0} 1$, we have $\exp(x_0 + h) \xrightarrow{h \rightarrow 0} e^{x_0} \times 1 = e^{x_0} = \exp(x_0)$. So \exp is continuous in x_0 . Hence, \exp is continuous on \mathbb{R} .

Ex.3.3) Use the fact that $\exp : x \mapsto e^x$ is continuous to show that $\ln : x \mapsto \ln x$ is continuous in its domain.

Answer: We know that the reciprocal of a continuous function is continuous on the correct domain. We could prove it, but hey work by yourself from time to time.

Ex.3.4) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous on $[0, 1]$. Assume that $\forall x \in [0, 1], 0 \leq f(x) \leq 1$. Show that there exists $c \in [0, 1]$ such that $f(c) = c$.

Answer: Let $g : [0, 1] \rightarrow \mathbb{R}, x \mapsto f(x) - x$. g is continuous on $[0, 1]$ because f is. And $g(0) = f(0) - 0 = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$.

If $g(0) = 0$, then $c_1 \stackrel{\text{def}}{=} 0$ is good to have $f(c_1) = c_1$.

If $g(1) = 0$, then here $c_2 \stackrel{\text{def}}{=} 1$ is also good to have $f(c_2) = c_2$.

Now assume that $g(0) \neq 0$, and $g(1) \neq 0$. So $g(0) > 0$ and $g(1) < 0$ (by the hypothesis on $0 \leq f(x) \leq 1$). By using BOLZANO's theorem to g on $[0, 1]$, we conclude that there is $c_3 \in (0, 1)$ such that $g(c_3) = 0$ i.e. $f(c_3) = c_3$.

Hence, in every case, we have one $c \in [0, 1]$ such that $f(c) = c$, which is what we wanted. **Victory!**

Ex.3.5) Is it possible to have a function $f : \mathbb{R} \rightarrow \mathbb{R}$, well-defined everywhere, but continuous **nowhere**?

Remember that \mathbb{Q} is the set of rational numbers, dense in \mathbb{R} .

Show that $\mathbb{1}_{\mathbb{Q}} (\mathbb{R} \rightarrow \mathbb{R}, x \mapsto 1 \text{ if } x \in \mathbb{Q}, 0 \text{ otherwise})$ is continuous nowhere, by using a sequence $(r_n)_{n \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}, r_n \xrightarrow{n \rightarrow +\infty} \alpha$ for a $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, to show that $\mathbb{1}_{\mathbb{Q}}$ is not continuous at α , (and conversely $(\beta_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \mathbb{Q})^{\mathbb{N}}, \beta_n \xrightarrow{n \rightarrow +\infty} q$ for a $q \in \mathbb{Q}$, for non-continuity at q).

Quick answer: First of all, for $q \in \mathbb{Q}, \alpha \in \mathbb{R} \setminus \mathbb{Q}$, we can justify the existence of these two sequences r and β thanks to density of \mathbb{Q} in \mathbb{R} (and $\mathbb{R} \setminus \mathbb{Q}$ in \mathbb{R}).

By using all these notations, if f is continuous at q , then $f(\beta_n) \xrightarrow{n \rightarrow +\infty} f(q)$, but $\forall n \in \mathbb{N}, f(\beta_n) = 0$ because $\beta_n \notin \mathbb{Q}$, and $f(q) = 1$. As $0 \neq 1$, f cannot be continuous at q . The same stands for continuity at α .