

1 Exercises to do before tutorials.

2 Exercises to do during or after tutorials.

3 Bonus exercises.

1. Express, with quantifiers, these statements:

- (1) the sequence u (i.e. $(u_n)_{n \in \mathbb{N}}$) does **not** converge to $l \in \mathbb{R}$,
- (2) the sequence u does not diverge to $+\infty$,
- (3) the sequence u diverges.

2. Show that the sequence $(\sqrt{n})_{n \in \mathbb{N}}$ diverge to $+\infty$.

3. Find a divergent sequence but which does not tend to $\pm\infty$; and then a non-bounded sequence, non-negative, which does not tend to $+\infty$.

4. Study the next sequences (are they well defined? First terms, eventual limit, asymptotic behavior etc):

$$(1) u_n \stackrel{\text{def}}{=} \frac{n^2 + 3n - 4}{4n^2 + 5},$$

$$(3) u_n \stackrel{\text{def}}{=} \frac{n^4 + 7n}{5n^4 + \cos(n^2) - \frac{1}{n}},$$

$$(2) u_n \stackrel{\text{def}}{=} \frac{\sin(n^2)}{n},$$

$$(4) u_n \stackrel{\text{def}}{=} \frac{a^n - b^n}{a^n + b^n}, \text{ with } a, b \in \mathbb{R}_+^*,$$

5. Is a product of two lower-bounded sequences still lower-bounded?

6. Show that a non-decreasing sequence (from a certain rank) is lower-bounded.

7. Let u be a bounded sequence and v a sequence with limit being $+\infty$. Show that $u + v$ also has $+\infty$ for limit.

8. Let u be a convergent sequence and v divergent sequence. Show that $u + v$ diverges. What can you say about the product sequence $((u.v)_n \stackrel{\text{def}}{=} u_n.v_n, \forall n \in \mathbb{N}$, noted w)?

9. Let u and v be two sequences, respectively converging to l et l' . We consider the two sequences w and s defined by: $\forall n \in \mathbb{N}, w_n \stackrel{\text{def}}{=} \min\{u_n, v_n\}, s_n \stackrel{\text{def}}{=} \max\{u_n, v_n\}$. Show that both sequences w and s are converging, then compute their limits.

10. Let u be a real sequence, which all terms are relative integers (i.e. $u \in \mathbb{Z}^{\mathbb{N}}$). Prove that u converges *if and only if*¹ u is constant from a certain rank $n_0 \geq 0$ (i.e. $\exists n_0 / \forall n \geq n_0, u_n = u_{n_0}$).

¹If *and only if* will be abbreviated **iff**.

11. Prove² the convergence for these sequences, given by their general term³ $u_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k2^k}$, et $v_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k^2}$.

12. Let u be a bounded sequence, verifying: $\forall n \in \mathbb{N}^*, 2u_n \leq u_{n-1} + u_{n+1}$. Then we define $v = (v_n)_{n \in \mathbb{N}}$ with $\forall n \in \mathbb{N}, v_n \stackrel{\text{def}}{=} u_{n+1} - u_n$. Show⁴ that v converges, and compute its limit.

13. We define u , and v with their first term $u_0 > 0$ et $v_0 > 0$, and the recurrence relation:

$$\forall n \in \mathbb{N}, u_{n+1} = \sqrt{u_n v_n} \text{ and } v_{n+1} = \frac{u_n + v_n}{2}.$$

Show⁵ that these sequences converge, and show that they have the same limit (which therefore depends only on u_0, v_0). Can you give the name⁶ of this limit?

14. Consider $u \in \mathbb{R}^{\mathbb{N}}$ a real sequence (*i.e.* a sequence with values being reals), and assume that the three sequences $(u_{2n+1})_{n \in \mathbb{N}}, (u_{2n})_{n \in \mathbb{N}}, (u_{3n})_{n \in \mathbb{N}}$ are all converging. Then show that u converges (to the same limit, obviously).

15. Let u be a non-decreasing real sequence. If we assume that u has one extracted sub-sequence which converges, then conclude that u also converges.

16. For a real sequence u , we define v as the arithmetic means of the first terms of u :

$$\forall n \in \mathbb{N}^*, v_n \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^n u_k.$$

- (1) We want to prove the CESÀRO's theorem: "if u converges, then v converges too, and has the same limit". You can try to prove it directly, or you can follow this proof scheme:

(i) Show that we can strengthen the hypothesis and only show the theorem when u converges to 0.

(ii) Now, we can assume that u converges to 0. Take $\varepsilon > 0$ (fixed). According the definition of the convergence, we have a rank N from which $|u_n| \leq \varepsilon$ (*i.e.* $\forall n \geq N$). Split the big sum in two parts, one with indexes before N and one after, then conclude (as the first part tends to 0, for $n \longleftarrow +\infty$).

- (2) What about the other implication⁷? And if you assume u non-decreasing?

²For the second one, the formula $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, true for any $n \in \mathbb{N}$ might help.

³When dealing with sequences, it is common to forget to precise here that $n \in \mathbb{N}$.

⁴One could prove that v is monotone and bounded, then reason by contradiction to prove that its limit is zero.

⁵One could try to prove, that from a certain rank, one of the sequences is non-decreasing, the other is non-increasing, and one is always greater than the other. . .

⁶It is usually called the arithmetico-geometric mean of u_0 and v_0 .

⁷*i.e.* if v converges, then u converges too.

17. If we have $u \in \mathbb{R}^{\mathbb{N}}$, and $l \in \mathbb{R}$, such that: $\lim_{n \rightarrow +\infty} (u_{n+1} - u_n) = l$. Show⁸ that:

$$\lim_{n \rightarrow +\infty} \frac{u_n}{n} = l.$$

18. Study⁹ u , given by its general term $u_n \stackrel{\text{def}}{=} \sum_{k=0}^n \frac{(-1)^k}{k!}$.

19. Find a **simple** equivalent (for $n \rightarrow +\infty$) of these sequences:

$$(1) u_n = \ln \sqrt{\frac{n+1}{n-1}},$$

$$(5) u_n = \frac{\ln n + n!}{n^2 + (n+1)!},$$

$$(2) u_n = \sqrt{n+1} - \sqrt{n},$$

$$(6) u_n = \frac{\sqrt{n + \sqrt{n^2 + 1}}}{\sqrt{n + \sqrt{n^2 - 1}}} -$$

$$(3) u_n = \frac{e^n + n!}{n+1},$$

$$(4) u_n = \frac{e^n + e^{-n} + n}{\sqrt{n^2 + n - n}},$$

$$(7) u_n = e^{n^2 + n! + \frac{1}{n}},$$

20. If you admit this lemma¹⁰: “if u converges to $l \in \mathbb{R}$, then $e^u \stackrel{\text{def}}{=} (e^{u_n})_{n \in \mathbb{N}}$ converges to e^l ”, study these real sequences:

$$(1) u_n = \left(1 + \frac{1}{n}\right)^n,$$

$$(2) u_n = \left(1 - \frac{1}{n}\right)^n,$$

$$(3) u_n = \left(\cos\left(\frac{1}{n}\right)\right)^{n^4 \sin\left(\frac{1}{n^2}\right)}.$$

21. Find an equivalent (for $n \rightarrow +\infty$, as always when dealing with sequences) to every one of these sequences u , and precise their (asymptotic) behavior:

$$(1) u_n = \sin(\tan(\ln(n+1) - \ln(n))),$$

$$(2) u_n = \frac{\sqrt{\cos\left(\frac{1}{n}\right)} - 1}{\ln(n+1) - \ln(n)},$$

$$(3) u_n = \frac{e^{\sin\left(\frac{1}{n^2}\right)} - 1}{1 - \cos(e^{-n})} \ln\left(\cos\left(\frac{1}{n}\right)\right).$$

22. Let u be a real sequence, defined by $u_0 \in \mathbb{R}_+^*$, and $\forall n \in \mathbb{N}, u_{n+1} = u_n^2 + u_n$.

(1) Show that u diverges to $+\infty$,

⁸One could maybe use the Cesàro's theorem.

⁹One could focus separately on the two sequences $(u_{2n})_{n \in \mathbb{N}}$ and $(u_{2n+1})_{n \in \mathbb{N}}$.

¹⁰Which is a direct consequence of the exponential being continuous, and the general definition of continuity in a metric space (you might learn this later).

- (2) If v is defined with $\forall n \in \mathbb{N}, v_n \stackrel{\text{def}}{=} \frac{1}{2^n} \ln(u_n)$, prove that v is non-decreasing.
- (3) Find an upper-bound, for any $n \in \mathbb{N}$, of $v_{n+1} - v_n$. Then use it to prove that v is upper-bounded, and conclude to its convergence. v 's limit will be referred as α ,
- (4) Prove $\forall n, p \in \mathbb{N}^2, 0 \leq v_{n+p+1} - v_n \leq \frac{1}{2^n} \ln\left(1 + \frac{1}{u_n}\right)$,
- (5) Conclude that we have: $u_n \underset{n \rightarrow +\infty}{\sim} e^{2^n \alpha}$.