

Real sequences and series : Ex.1.7 (*Babylonian method*)

Let $u_0 = 1$ and define, for $n \in \mathbb{N}$, $u_{n+1} = \frac{u_n + \frac{2}{u_n}}{2}$.

- (i) Show that the sequence $u = (u_n)_{n \in \mathbb{N}}$ is well defined.
- (ii) Compute the first 6 terms (by using your calculator or smartphone). What do you think the limit will be?
- (iii) Show that u is converging¹, and compute its limit.

Solution

Lemma 0.1. $\forall n \in \mathbb{N}, u_n > 0$, so u_{n+1} is well defined.

Proof 0.1. *Obvious, by a quick induction.*

- If $n = 0$, $u_0 = 1 > 0$,
- If $n \geq 0$, and $u_n > 0$, then $\frac{u_n}{2} > 0$ and $\frac{1}{u_n}$ exists and is positive. So $u_{n+1} = \frac{u_n}{2} + \frac{1}{u_n} > 0$. \square

First values

For question (ii), cf. Figure 1 or this array:

n	u_n
0	$u_0 = 1$
1	$u_1 = 1.5$
2	$u_2 \simeq 1.416666666666665186$
3	$u_3 \simeq 1.4142156862745096646$
4	$u_4 \simeq 1.4142135623746898698$
5	$u_5 \simeq 1.4142135623730949234$
6	$u_6 \simeq 1.4142135623730949234 \simeq \sqrt{2}$

Compute the limit

Lemma 0.2. $\forall n \in \mathbb{N}, u_{n+1} - \sqrt{2} = \frac{(u_n - \sqrt{2})^2}{2u_n} (> 0)$.

Proof 0.2. Let $n \in \mathbb{N}$.

$$\begin{aligned}
 u_{n+1} - \sqrt{2} &= \frac{u_n}{2} + \frac{1}{u_n} - \sqrt{2} \\
 &= \frac{u_n^2}{2u_n} + \frac{2}{2u_n} - \frac{2\sqrt{2}u_n}{2u_n} \\
 &= \frac{u_n^2 + 2 - 2\sqrt{2}u_n}{2u_n} \\
 u_{n+1} - \sqrt{2} &= \frac{(u_n - \sqrt{2})^2}{2u_n} > 0. \square
 \end{aligned}$$

Remark 1. So we have $\forall n \in \mathbb{N}, \sqrt{2} \leq u_n$ (with Lemma 0.2 it is obvious, because $\frac{1}{2u_n} > 0$).

Lemma 0.3. $\forall n \geq 1, u_{n+1} \leq u_n \leq u_1$.

Proof 0.3. Let $n \in \mathbb{N}$. By the previous Remark 1, $\sqrt{2} \leq u_n$, so $\frac{2}{u_n} \leq u_n$. Therefore, $u_{n+1} = \frac{u_n + \frac{2}{u_n}}{2} = \frac{u_n}{2} + \frac{1}{u_n} \leq u_n$.

So the sequence $(u_n)_{n \geq 1}$ is non-increasing (in fact, you can prove also that it is decreasing, i.e. $u_{n+1} < u_n$, because $u_n \in \mathbb{Q}, \forall n \in \mathbb{N}$ so Remark 1 gives $\sqrt{2} < u_n$). So $\forall n \in \mathbb{N}, u_n \leq u_1 = \frac{3}{2} = 1.5$. \square

Conclusion 0.1. So that sequence is decreasing, lower-bounded (by $\sqrt{2}$), so it converges to a limit $l > 0$. The only possible value for l is $\sqrt{2}$ (because it verifies $l = \frac{l + \frac{2}{l}}{2} \Leftrightarrow l^2 = 2$).

Remark 2. This method to approximate the numerical value of $\sqrt{2}$ is quite an old one. That Wikipédia page explains a little more about the history of this method, often referred as the Hero's method, or Babylonian method.

Remark 3. It can be generalized to approximate the square root of any $A > 0$. Just take $u_0 = [A](= [A] + 1)$ and $\forall n \in \mathbb{N}, u_{n+1} = \frac{u_n + \frac{A}{u_n}}{2}$.

Bonus: do again the same exercise for the general case, showing that $u_n \xrightarrow{n \rightarrow +\infty} \sqrt{A}^+$ (i.e. by being always greater than \sqrt{A}).

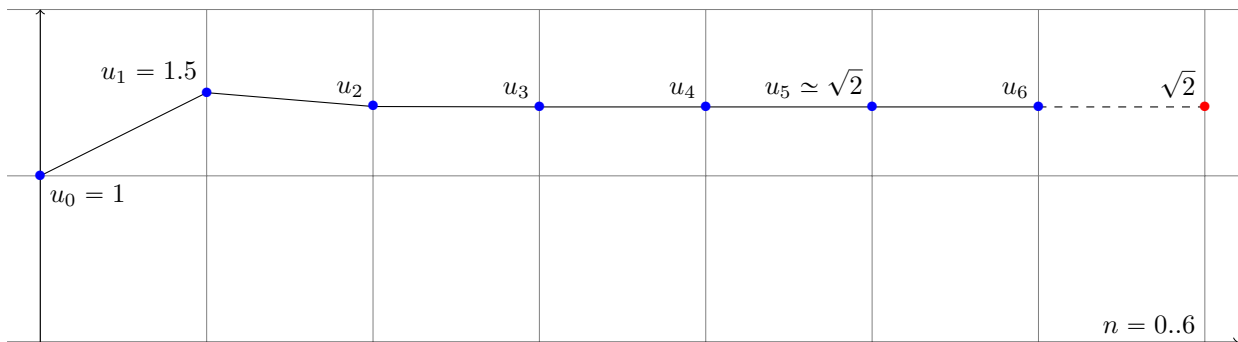


Figure 1: The first values of u_n , from $n = 0$ to $n = 6$.