

## 1 Exercises to do before tutorials.

- Ex.1.1) Show that the sequence  $(\sqrt{n})_{n \in \mathbb{N}}$  diverges to  $+\infty$ .
- Ex.1.2) Consider sequences given below by their terms  $u_n$  ( $\forall n \in \mathbb{N}$ ). Check if each of these sequences are convergent or divergent. If they are convergent then compute the limits.
- (i)  $u_n \stackrel{\text{def}}{=} \frac{n^2 + 3n - 4}{4n^2 + 5}$ ,                      (iii)  $u_n \stackrel{\text{def}}{=} \frac{n^4 + 7n}{5n^4 + \cos(n^2) - \frac{1}{n}}, n \neq 0$ ,
- (ii)  $u_n \stackrel{\text{def}}{=} \frac{\sin(n^2)}{n}, n \neq 0$ ,                      (iv)  $u_n \stackrel{\text{def}}{=} \frac{a^n - b^n}{a^n + b^n}$ , with  $a, b > 0$ .
- Ex.1.3) Is the product sequence of two lower-bounded sequences still lower-bounded?
- Ex.1.4) Show that a non-decreasing sequence is lower-bounded.
- Ex.1.5) Let  $u$  be an integer sequence (*i.e.*  $u \in \mathbb{N}^{\mathbb{N}}$ ). Prove that  $u$  converges *if and only if*<sup>1</sup>  $u$  is constant after a certain number of terms (*i.e.*  $\exists n_0 \in \mathbb{N} / (\forall n \geq n_0, u_n = u_{n_0})$ ).
- Ex.1.6) Use one of the convergence tests saw in lectures to explore the convergence/divergence of the following series:

(i)  $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ ,                      (iv)  $\sum_{n=1}^{\infty} \frac{1}{1000n + 1}$ ,

(ii)  $\sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}$  (for  $x \in \mathbb{R}$ ),                      (v)  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ ,

(iii)  $\sum_{n=2}^{\infty} \frac{n!}{(n+2)!}$ ,                      (vi)  $\sum_{n=1}^{\infty} \frac{1000^n}{n!}$ .

- Ex.1.7) (*Babylonian method*) Let  $u_0 = 1$  and define, for  $n \in \mathbb{N}$ ,  $u_{n+1} = \frac{u_n + \frac{2}{u_n}}{2}$ .
- (i) Show that the sequence  $u = (u_n)_{n \in \mathbb{N}}$  is well defined.
- (ii) Compute the first 6 terms (by using your calculator or smartphone). What do you think the limit will be?
- (iii) Show that  $u$  is converging<sup>2</sup>, and compute its limit.

## 2 Exercises to do during or after tutorials.

- Ex.2.1) Let  $u$  be a bounded sequence, satisfying:  $\forall n \in \mathbb{N}^*, 2u_n \leq u_{n-1} + u_{n+1}$ . Let  $v = (v_n)_{n \in \mathbb{N}}$  be given by  $\forall n \in \mathbb{N}, v_n \stackrel{\text{def}}{=} u_{n+1} - u_n$ . Show<sup>3</sup> that  $v$  converges, and

<sup>1</sup>If *and only if* will be abbreviated “**iff**”.

<sup>2</sup>One could show that  $(u_n)_{n \geq 1}$  is decreasing and lower-bounded by 1.

<sup>3</sup>First prove that  $v$  is monotonic and bounded, then prove by contradiction that its limit is zero.

compute its limit.

Ex.2.2) Let  $u$  be a convergent sequence. Let  $v$  be a sub-sequence of  $u$ . Show that then  $v$  converges (to the same limit).

Ex.2.3) Show the following equalities:

$$(i) \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1,$$

$$(ii) \sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} = \frac{3}{2},$$

$$(iii) \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2},$$

$$(iv) \sum_{n=2}^{\infty} \frac{\log\left(\left(1 + \frac{1}{n}\right)^n (1+n)\right)}{(\log n^n)(\log(n+1)^{n+1})} = \log_2(\sqrt{e}).$$

Ex.2.4) Consider  $u \in \mathbb{R}^{\mathbb{N}}$  a real sequence (*i.e.* a sequence with values being reals), and assume that the three sequences  $(u_{2n+1})_{n \in \mathbb{N}}$ ,  $(u_{2n})_{n \in \mathbb{N}}$ ,  $(u_{3n})_{n \in \mathbb{N}}$  are all converging. Then show that  $u$  converges (to the same limit, obviously).

### 3 Bonus exercises.

Ex.3.1) Let  $u$  be a bounded sequence and  $v$  be a sequence diverging to  $+\infty$ . Show that  $u + v$  diverges to  $+\infty$ .

Ex.3.2) Let  $u$  be a convergent sequence and  $v$  be a divergent sequence. Show that  $u + v$  diverges. What can you say about the product sequence  $(u.v)_n \stackrel{\text{def}}{=} u_n.v_n, \forall n \in \mathbb{N}$ ?

Ex.3.3) Let  $u$  and  $v$  be two sequences converging to  $l$  et  $l'$ , respectively. We consider the two sequences  $w$  and  $w'$  defined by:  $\forall n \in \mathbb{N}, w_n \stackrel{\text{def}}{=} \min(u_n, v_n), w'_n \stackrel{\text{def}}{=} \max(u_n, v_n)$ . Show that both the sequences  $w$  and  $w'$  converge and compute their limits.

Ex.3.4) Let  $u$  be a non-decreasing real sequence. If we assume that  $u$  has a sub-sequence which converges, then show that  $u$  also converges. (A *sub-sequence* is of the form  $u_\phi = (u_{\phi(n)})_{n \in \mathbb{N}}$  if  $\phi : \mathbb{N} \mapsto \mathbb{N}$  is a *increasing* and *injective* function, called an *extractor*.)