

1 Exercises to (try to) do before tutorials.

1.1 Prove the following classical inequalities:

- (1) $\forall a, b \in \mathbb{R}, (a + b)^2 \leq 2(a^2 + b^2),$ (3) $\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx,$
 (2) $\forall a, b \in \mathbb{R}, |ab| \leq \frac{a^2 + b^2}{2},$ (4) $\forall x \in \mathbb{R}, \forall a > 1, \exists n \in \mathbb{N}, a^n \geq x.$

1.2 Set upper and lower bounds:

- (1) Bound, for $x \in [0, 3]$, the value of this expression¹: $\frac{x - 1}{e^x + 1},$
 (2) Show that $\left\{ \frac{\sin(x) - 2 \cos(x)}{e^{\sin x}} : x \in \mathbb{R} \right\}$ is a bounded set,
 (3) Show² that $\left\{ \sqrt{n^3 + n + 2} - \sqrt{n^3 + 1} : n \in \mathbb{N} \right\}$ is a bounded set,

1.3 Determine the biggest (resp. smallest) element, and an upper bound (resp. lower) of the following sets E (whenever they exist):

- (1) $E = \left\{ \frac{1}{n}, n \in \mathbb{N}^* \right\}$ (where $\mathbb{N} \stackrel{\text{def}}{=} \mathbb{Z} \cap \mathbb{R}^+$ is the set of non-negative integers number, and $\mathbb{N}^* \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\} = \{n : n \geq 1\}$),
 (2) $E = A + B$, with³ A and B two close intervals of \mathbb{R} (where $A + B \stackrel{\text{def}}{=} \{x + y, (x, y) \in A \times B\}$),
 (3) $E = A - B$, with A and B two close intervals of \mathbb{R} (where $A - B \stackrel{\text{def}}{=} \{x - y, (x, y) \in A \times B\}$),
 (4) (extbfBonus:)What can we say if A and B are both non-empty, bounded and open intervals? And what if they are just bounded and non-empty subsets of \mathbb{R} ?
 (5) $E = A \cup B$, with A and B two bounded sets of \mathbb{R} ?
 (6) (extbfBonus:) $E = \left\{ (-1)^n a + \frac{b}{n}, n \in \mathbb{N}^* \right\}$ with two parameters $a, b \in \mathbb{R}_+^*$ (where $\mathbb{R}_+^* \stackrel{\text{def}}{=} (\mathbb{R} \setminus \{0\}) \cap [0, +\infty) = \mathbb{R} \cap (0, +\infty)$ is the set of positive real numbers).

1.4 Let A be a non-empty and upper bounded subset of \mathbb{R} , with $\sup A > 0$. Show that A contains a non-negative element.

1.5 Let $a, b \in \mathbb{R}^+$ (i.e. $a, b \geq 0$), with $a \geq b$. Simplify the expression: $\sqrt{a + 2\sqrt{a - b}\sqrt{b}} + \sqrt{a - 2\sqrt{a - b}\sqrt{b}}.$

1.6 Let $a, b \in \mathbb{R}$, here and now on, \mathbf{E} will be the **integer part function**⁴ (also written $\mathbf{E}(x) = \lfloor x \rfloor$). Show the following statements:

¹Be sure to verify that it is well defined before using it!

²Hint: conjugate quantities like $x - y = \frac{(x - y)(x + y)}{x + y} = \frac{x^2 - y^2}{x + y}$, and $x + y = \frac{x^2 - y^2}{x - y}$ (if $x \pm y \neq 0$).

³One could show that, if $A = [a, b]$ and $B = [c, d]$ are two close intervals, then $A + B = [a + c, b + d]$.

⁴ $\forall x \in \mathbb{R}, \mathbf{E}(x)$ is given by $\mathbf{E}(x) \in \mathbb{Z}$ and $\mathbf{E}(x) \leq x < \mathbf{E}(x) + 1$.

- (1) $a \leq b \stackrel{\text{def}}{=} \mathbf{E}(a) \leq \mathbf{E}(b)$ (\mathbf{E} is a non-decreasing function),
 (2) $\mathbf{E}(a) + \mathbf{E}(b) \leq \mathbf{E}(a + b) \leq \mathbf{E}(a) + \mathbf{E}(b) + 1$.

2 Exercises to (try to) do during or after tutorials.

- 2.1 Let B a bounded and non-empty subset of \mathbb{R} , and $A \subset B$ is non-empty. Show that $\sup A \leq \sup B$.
- 2.2 Solve, in \mathbb{R} , the following equation: $\sqrt{2 - 2x} + \sqrt{3 + x} = 1$.
- 2.3 Let f, g be bounded⁵ functions from \mathbb{R} to \mathbb{R} . Prove the following inequality:

$$\sup_{x \in \mathbb{R}} |(f + g)(x)| \leq \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \in \mathbb{R}} |g(x)|.$$

- 2.4 For $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$, show this property: $\mathbf{E}\left(\frac{\mathbf{E}(nx)}{n}\right) = \mathbf{E}(x)$.

3 Bonus exercises.

You can try to solve them after the tutorials, or before exams to practice.

- 3.1 Prove these classical inequalities:

$$(1) \quad \forall x \in \mathbb{R}, |\sin(x)| \leq |x|, \quad (2) \quad \forall x \in [0, \frac{\pi}{2}], \frac{2}{\pi}x \leq \sin(x) \leq x,$$

- 3.2 Characterize the set of the functions $f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \stackrel{\text{def}}{=} \mathbb{R}^{\mathbb{R}}$ (*i.e.* the set of all functions $f : \mathbb{R} \longleftrightarrow \mathbb{R}$) verifying: $\forall x, y \in \mathbb{R}, |f(x) - f(y)| = |x - y|$ (such function is called an *isometry*).

- 3.3 Let $n \in \mathbb{N}^*$, and $(x_i)_{i \in [1, \dots, n]}$ be n non-negative real numbers. Prove the following:

$$\left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n \frac{1}{x_i} \right) \geq n^2.$$

- 3.4 (1) Prove: $\forall n \in \mathbb{N}^*, \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n-1}$,

(2) Compute the integer part (*i.e.* \mathbf{E} applied to) of $\frac{1}{2} \sum_{k=1}^{n^2} \frac{1}{\sqrt{k}}$, pour $n \in \mathbb{N}^*$.

⁵ $f : \mathbb{R} \longleftrightarrow \mathbb{R}$ is *bounded* if there is a $M \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, |f(x)| \leq M$.