

1 Exercises to do before tutorials.

Ex.1.1) Show that sum $(f + g)$ of two functions (f, g) which are individually continuous at a point (x_0) is continuous at that point.

Ex.1.2) Compute the following left-hand side limits, right-hand side limits, and limits (if they exist):

$$\begin{array}{ll} \text{(a)} \lim_{x \rightarrow 0} \frac{|x|}{x}, & \text{(d)} \lim_{h \rightarrow 0} \frac{(t+h)^2 - t^2}{h} \quad (t \in \mathbb{R} \text{ is fixed}), \\ \text{(b)} \lim_{x \rightarrow 0} \frac{\sqrt{x^2}}{x}, & \text{(e)} \lim_{x \rightarrow a} \frac{\sin x - \sin a}{x - a} \quad (a \in \mathbb{R} \text{ is fixed}), \\ \text{(c)} \lim_{x \rightarrow 1} \frac{2x^2 - 3x + 1}{x - 1}, & \text{(f)} \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin x}. \end{array}$$

Ex.1.3) Use the fact¹ that $|\sin(x)| < |x|$, true for all $0 < |x| < \frac{\pi}{2}$ to show that \sin is continuous at 0. Next, use the identity $\cos 2x = 1 - 2\sin^2 x$ ($\forall x \in \mathbb{R}$) to show that \cos is also continuous at 0. Finally, use the standard formula for $\sin(x+h)$ and $\cos(x+h)$ to show that \sin and \cos are continuous everywhere on \mathbb{R} .

Ex.1.4) Let f be a real polynomial function of degree $n > 0$ such that the last and first coefficients have opposite signs. Prove² that there exists a positive x (i.e. $x \in \mathbb{R}_+^*$) such that $f(x) = 0$.

2 Exercises to do during or after tutorials.

Ex.2.1) Let f and g be given below and let $h = f \circ g$. In each case, determine the domains of f , g , and h and given direct formula for h . Finally, investigate the continuity of h :

$$\begin{array}{ll} \text{(i)} f(x) = x^2 - 2x, g(x) = x + 1, & \text{(iii)} f(x) = \sqrt{x}, g(x) = x + \sqrt{x}, \\ \text{(ii)} f(x) = \sqrt{x}, g(x) = x^2, & \text{(iv)} f(x) = \sqrt{x}, g(x) = \sin(x). \end{array}$$

Ex.2.2) If n is an odd positive integer ($n \in 2\mathbb{N} + 1$) and $a < 0$ prove that there is exactly one negative $b < 0$ such that $b^n = a$. And what happens if n is even?

¹You can also *try to* prove that fact.

²You should start by naming all these coefficients $a_0, \dots, a_n \in \mathbb{R}$, and use contradiction (“if f has no zero in \mathbb{R}_+^* , then ...”)

Ex.2.3) Let $f = \tan$ (on a certain domain you should precise). Although $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, there is no x in $[\pi/4, 3\pi/4]$ such that $f(x) = 0$. Why does this not contradict the *intermediate value theorem*?

Ex.2.4) Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x \sin\left(\frac{1}{x}\right)$ if $x \neq 0$.

What should be the value of $f(0)$ so that f is continuous at 0.

Ex.2.5) For $a, b \in \mathbb{R}, a < b$, let $f : [a, b] \rightarrow \mathbb{R}$ be such that³ $\forall x, y \in [a, b], |f(x) - f(y)| \leq L \times |x - y|$, for some $L \geq 0$. Show that f is continuous on $[a, b]$.

Ex.2.6) Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be well-defined. Show that f is continuous everywhere on \mathbb{Z} .

3 Bonus exercises.

Ex.3.1) Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \sin\left(\frac{1}{x}\right)$ if $x \neq 0$. Can we choose $f(0)$ such that f is continuous at 0? What about limit?

Ex.3.2) For $x \in \mathbb{R}$, show that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent (this shows that e^x is well-defined for all $x \in \mathbb{R}$). Next, show that there exists $\lambda > 0$ such that $e^x \leq 1 + \lambda x$ for all $x \in [0, 1]$. Finally, show that $\exp : x \mapsto e^x$ is continuous at 0, and subsequently everywhere in \mathbb{R} .

Ex.3.3) Use the fact that $\exp : x \mapsto e^x$ is continuous to show that $\ln : x \mapsto \ln x$ is continuous in its domain.

Ex.3.4) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous on $[0, 1]$. Assume that $\forall x \in [0, 1], 0 \leq f(x) \leq 1$. Show that there exists $c \in [0, 1]$ such that $f(c) = c$.

Ex.3.5) Is it possible to have a function $f : \mathbb{R} \rightarrow \mathbb{R}$, well-defined everywhere, but continuous **nowhere**?

Remember that \mathbb{Q} is the set of rational numbers, dense in \mathbb{R} .

Show that $\mathbb{1}_{\mathbb{Q}} (\mathbb{R} \rightarrow \mathbb{R}, x \mapsto 1 \text{ if } x \in \mathbb{Q}, 0 \text{ otherwise})$ is continuous nowhere, by using a sequence $(r_n)_{n \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}, r_n \xrightarrow{n \rightarrow +\infty} \alpha$ for a $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, to show that

$\mathbb{1}_{\mathbb{Q}}$ is not continuous at α , (and conversely $(\beta_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \mathbb{Q})^{\mathbb{N}}, \beta_n \xrightarrow{n \rightarrow +\infty} q$ for a $q \in \mathbb{Q}$, for non-continuity at q).

³ f is said to be L -LIPSCHITZ, or LIPSCHITZ continuous of ratio L on $[a, b]$.