

1 Field Axioms

Let \mathbb{F} denote a set on which two binary operators¹ \oplus and \odot are defined, that is, two functions $\oplus : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ and $\odot : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$.

Let $x, y \in \mathbb{F}$, then we denote the result of the \oplus operator as $x \oplus y$, and the \odot operator as $x \odot y$. Unless it is ambiguous, we denote $x \odot y$ as simply xy .

For the following axioms², let $x, y, z \in \mathbb{F}$.

Axiom 1.1 *Commutative laws:* $x \oplus y = y \oplus x$ and $xy = yx$.

Axiom 1.2 *Associative laws:* $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and $x(yz) = (xy)z$.

Axiom 1.3 *Distributive laws:* $x(y \oplus z) = xy \oplus xz$.

Axiom 1.4 *Identity elements:* There exist two (distinct) real numbers 0 and 1, such that for every $x \in \mathbb{F}$, $x \oplus 0 = x$ and $x \odot 1 = x$.

Axiom 1.5 *Negatives:* For every $x \in \mathbb{F}$ there exists $y \in \mathbb{F}$ such that $x \oplus y = 0$.

Axiom 1.6 *Reciprocals:* For every $x \in \mathbb{F} \setminus \{0\}$ (i.e. $x \neq 0$), there exists $y \in \mathbb{F} \setminus \{0\}$ such that $xy = 1$. We denote such y by x^{-1} , called the *reciprocal* of x .

Theorem 1. *Let $a, b, c, d \in \mathbb{F}$.*

Property 1.1: If $a \oplus b = a \oplus c$ then $b = c$ (i.e. every element is cancellable).

Property 1.2: There exists a unique $x \in \mathbb{F}$ such that $a \oplus x = b$. (We denote x by $b - a$. Specifically, $0 - a$ is $-a$, the negative of a).

Property 1.3: $b - a = b \oplus (-a)$.

Property 1.4: $-(-a) = a$ (i.e. the function $\mathbb{F} \longleftrightarrow \mathbb{F}, x \mapsto -x$ is an involution).

Property 1.5: $0 \odot a = a \odot 0 = 0$ (i.e. 0 is absorbing).

Property 1.6: $a(b - c) = ab - ac$.

Property 1.7: If $ab = ac$ and $a \neq 0$ then $b = c$.

Property 1.8: If $a \neq 0$ then there exists a unique $x \in \mathbb{F}$ such that $ax = b$ (we denote this x by b/a or $\frac{b}{a}$. Specifically, $1/a$ is a^{-1} , the reciprocal of a).

¹Also called *internal composition laws*.

²An axiom is a fact, assumed as true. You do **not** have to try to prove these axioms!

Property 1.9: If $a \neq 0$ then $b/a = b \odot a^{-1}$.

Property 1.10: If $a \neq 0$ then $(a^{-1})^{-1} = a$.

Property 1.11: If $ab = 0$ then $a = 0$ or $b = 0$.

Property 1.12: $(-a)b = -(ab)$ and $(-a)(-b) = ab$.

Property 1.13: $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad \oplus bc}{bd}$ if $b \neq 0$ and $d \neq 0$.

Property 1.14: $\frac{a}{b} \frac{c}{d} = \frac{ac}{bd}$ if $b \neq 0$ and $d \neq 0$.

Property 1.15: $(\frac{a}{b})/(\frac{c}{d}) = \frac{ad}{bc}$ if $b \neq 0$, $c \neq 0$, and $d \neq 0$.

2 Problems

Problem 1. Prove at least 12 properties among the 15 exposed in Theorem 1.

Problem 2. Verify³ whether all the field axioms and all the properties of Theorem 1 hold for:

1. $\mathbb{F} = \mathbb{Z}$, the set of integers, $\oplus = +$, and $\odot = \cdot$, the usual addition and multiplication.
2. $\mathbb{F} = \mathbb{Q}$, the set of rational numbers, $\oplus = +$, and $\odot = \cdot$, the usual addition and multiplication.
3. Let n be a positive integer greater than 1 and let $\mathbb{F} = \{0, 1, 2, \dots, n-1\}$. For $x, y \in \mathbb{F}$, we define $x \oplus y = (x+y) \bmod n$ and $x \odot y = (xy) \bmod n$. Furthermore, what is -1 in this case and check if $(-1)(-1) = 1$.
4. (*extbfBonus:*) Let S be a nonempty set, and let \mathbb{F} be such that $A \in \mathbb{F}$ if and only if $A \subseteq S$, that is, \mathbb{F} contains all the subsets of S including \emptyset and S (\mathbb{F} is called the power set of S , set of all subsets of S). Define $\oplus = \cup : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ and $\odot = \cap : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$, the union and the intersection operators, respectively. (\mathbb{F} can be written as 2^S .)

Example: if $S = \{a, b\}$ with two elements, then $F = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, with four “elements” being subsets of S . If S is finite, then $\text{Card } F = 2^{\text{Card } S}$.

³When you think one of 6 axioms is not true for this particular \mathbb{F} , you can *try* to prove this with a counter-example.