

1 Exercises to (try to) do before tutorials.

1. Find¹ a necessary and sufficient condition (NSC) for $z \in \mathbb{C}$ to satisfy the identity:
 $|z + i| = |z - i|$.

2. Resolve (in \mathbb{C}) the following equations:

$$(1) \quad z^2 + (1 - i)z - 1 + \frac{1}{4} = 0,$$

$$(2) \quad z^2 + (2 - 3i)z - 6i = 0,$$

$$(3) \quad 3z^2 + (1 + i)z + 2 = 0.$$

3. Write on *polar form* the following complex numbers (*i.e.* find the modulus and **one** argument): $1 + i$, $\frac{\sqrt{3}}{2} - \frac{i}{2}$, $1 + e^{i\theta}$ for $\theta \in \mathbb{R}$, and $e^{ia} + e^{ib}$ for $a, b \in \mathbb{R}$.

4. Compute the product of all the n^{th} roots of unity in \mathbb{C} (for n a natural integer, bigger than 2), *i.e.* the product $\prod_{z \in \mathbb{U}_n} z$, when $\mathbb{U}_n = \{z \in \mathbb{C} : z^n = 1\}$.

5. Let $z, z' \in \mathbb{C}$.

(1) Prove the following identity: $|z + z'|^2 + |z - z'|^2 = 2(|z|^2 + |z'|^2)$ (which is called the *parallelogram identity*).

It could be a good idea to illustrate the situation with a drawing, and try to justify graphically the name of this identity.

(2) Conclude that one could have: $|z| + |z'| = |a + \frac{z + z'}{2}| + |a - \frac{z + z'}{2}|$ with a being **one** square root of zz' .

2 Exercises to (try to) do during or after tutorials.

1. Let $u \in \mathbb{C} \setminus \{1\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. Prove the following equivalence: $|u| = 1 \Leftrightarrow \frac{z - u\bar{z}}{1 - u} \in \mathbb{R}$.

2. Simplify the following expressions: $\tan(3\pi + \theta)$, $\cos\left(\frac{\pi}{2} - \theta\right)$, where θ is a real such that these expressions make sense.

3. Simplify the expressions $\cos\left(\frac{\pi}{8}\right)$ and $\sin\left(\frac{\pi}{8}\right)$, first by using a trigonometric method and then an algebraic² one.

¹One could try to raise to the square and use the formula $|z|^2 = z\bar{z}$.

²For the second method, if $z = e^{i\frac{\pi}{8}}$ then $z^2 = e^{i\frac{\pi}{4}}$ and then we know a method to compute **one** square root of a complex number.

4. Resolve in \mathbb{C} the following equation (z being the unknown): $z^4 + z^3 + z^2 + z + 1 = 0$.
One could introduce the new variable $Z := z + \frac{1}{z}$. Then show that the solutions are all 5th roots of unity (*i.e.* in the set \mathbb{U}_5), and use this to compute the value of $\cos\left(\frac{2\pi}{5}\right)$.
5. Resolve in \mathbb{C} the following equation (in z): $(z + 1)^6 + (z - 1)^6 = 0$.
6. Let a be a complex number of modulus 1 (*i.e.* $a \in \mathbb{U}$) and z_1, \dots, z_n the complex solutions of the equation $z^n = a$, for some $n \in \mathbb{N}^*$ (*i.e.* the n^{th} roots of a). Show that the points of complex affixes $(1 + z_1)^n, \dots, (1 + z_n)^n$ are aligned in an euclidian plan³.

3 Bonus exercises.

1. Simplify and compute the following integrals: $\int_0^{\frac{\pi}{2}} \sin^4(x) dx$, and $\int_0^{\frac{\pi}{2}} \cos^5(x) dx$.
2. Resolve (in \mathbb{R}) the following equations:
 - (1) $\cos(x) + \sin(x) = 0$,
 - (2) $\sin(4x) - \sqrt{3}\sin(3x) + \sin(2x) = 0$.
3. Let $\xi = e^{\frac{2i\pi}{n}}$ with $n \in \mathbb{N}^*$. Compute the sum $\sum_{k=0}^n |\xi^k - 1|$.
4. Let us consider a recursive sequence, defined by its first two terms u_0 and u_1 , two complex numbers, and by its recurrence equation: $u_{n+2} - u_{n+1} = a(u_{n+1} - u_n)$, for **one** $a \in \mathbb{C}$ and for any $n \in \mathbb{N}$. Find a necessary **and** sufficient condition on this parameter a for the sequence $(u_n)_{n \in \mathbb{N}}$ to be *periodical*⁴.
5. Let a and b be two real numbers (*i.e.* $a, b \in \mathbb{R}$). Resolve, in \mathbb{C} , the following equation:

$$z^4 - 4 \cos(a) \cos(b) z^3 + 2(1 + \cos(2a) + \cos(2b)) z^2 - 4 \cos(a) \cos(b) z + 1 = 0.$$

³One could try to see if there is a line, coming from the origin, and passing by each of these points. This could be expressed with a criteria over their arguments.

⁴A sequence $(u_n)_{n \in \mathbb{N}}$ is said to be *periodical* when there is an integer $T \geq 1$, called a *period*, such that $\forall k \in \mathbb{N}, u_{k+T} = u_k$.