

**Additional exercises.**

1. About dense subset of  $\mathbb{R}$ .
  - (1) Let  $X$  be a dense subset of  $\mathbb{R}$ , prove that  $X$  is unbounded.
  - (2) Prove that even if we remove a finite number of elements in  $X$ , we still have a dense set in  $\mathbb{R}$ . Is this still the case if we remove an infinite<sup>1</sup> number of elements?
  - (3) Now prove<sup>2</sup> that if we remove a countable subset of  $X$ , we still have a dense subset of  $\mathbb{R}$ .
  - (4) Finally, let  $X \subset Y \subset \mathbb{R}$ . Assume that  $X$  is finite, and *dense in*<sup>3</sup>  $Y$ . Prove that  $Y = X$ .

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<sup>1</sup>As the next question will show, it depends on the sense we give to infinite.

<sup>2</sup>One could try to prove that, no matter how small it is, a real non-empty open interval  $(a, b)$  is never countable.

<sup>3</sup>We keep the same definition of density as in  $\mathbb{R}$ , with sequences and limits.