

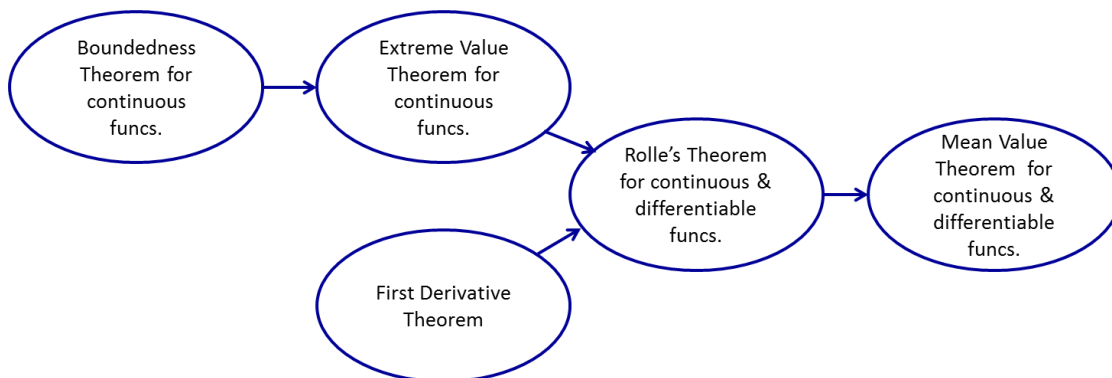
## Summary Sheet of Important Items of Part –II Lectures

This part of the course deals only with the Real number space in one dimension, hence all references to variables and functions below should be interpreted accordingly.

1. Basic definition of derivatives in terms of limits:  $\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ , or  $\frac{df(x)}{dx} = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$
2. Theorem of Continuity of Differentiable functions: A function differentiable at some value of its argument is continuous around that value.
- 3a. Intermediate Value Theorem for Continuous functions: Let  $f$  be continuous at each point of a closed interval  $[a, b]$ . Choose two arbitrary points  $x_1 < x_2$  in  $[a, b]$  such that  $f(x_1) \neq f(x_2)$ . Then  $f$  takes on every value between  $f(x_1)$  and  $f(x_2)$  somewhere in the interval  $(x_1, x_2)$ . [Note: It does not say that  $f$  does not take on values outside the range covered by  $f(x_1)$  and  $f(x_2)$  within the interval  $(x_1, x_2)$ .]
- 3b. Intermediate Value Theorem for Derivatives: Let  $f$  be differentiable at each point of a closed interval  $[a, b]$ . Choose two arbitrary points  $x_1 < x_2$  in  $[a, b]$  such that  $f'(x_1) \neq f'(x_2)$ . Then  $f'$  takes on every value between  $f'(x_1)$  and  $f'(x_2)$  somewhere in the interval  $(x_1, x_2)$ . [Note: It does not say that  $f'$  does not take on values outside the range covered by  $f'(x_1)$  and  $f'(x_2)$  within the interval  $(x_1, x_2)$ .]
4. Leibniz rule for higher order derivatives of a product of two functions: If  $u$  and  $v$  are 2 functions differentiable upto  $n^{\text{th}}$  order w.r.t. the independent variable, then
 
$$(uv)^{(n)} = u^{(n)}v^{(0)} + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2}u^{(n-2)}v^{(2)} + \dots + {}^nC_m u^{(n-m)}v^{(m)} + \dots + nu^{(1)}v^{(n-1)} + u^{(0)}v^{(n)}; \quad \text{where } {}^nC_m = \frac{(n)!}{(n-m)!(m)!}$$
5. Chain rule of differentiation: Let  $f$  be the composition of two functions  $u$  and  $v$ , say  $f = u \circ v$ , and suppose that both derivatives  $v'(x)$  and  $u'(y)$  exist, where  $y = v(x)$ . Then the derivative  $f'(x)$  also exists and is given by the formula  $f'(x) = u'(y) \cdot v'(x)$ .
6. Linearization: Let  $f(x)$  represent a continuous function in an interval ' $I$ ' and ' $a$ ' be a point in  $I$  at which  $f$  is differentiable. Then  $L(x) = f(a) + f'(a)(x - a)$  represents the equation of the tangent to the curve  $f(x)$  at  $a$  and is the Linearization of  $f(x)$  in the neighbourhood of  $a$ .
7. Boundedness Theorem for Continuous Functions: Let  $f$  be continuous on a closed interval  $[a, b]$ . Then  $f$  is bounded on  $[a, b]$ . That is, there is a number  $C \geq 0$  such that  $|f(x)| \leq C$  for all  $x$  in  $[a, b]$ .
8. Extreme Value Theorem for Continuous functions: Assume  $f$  is continuous on a closed interval  $[a, b]$ . Then there exist points  $c$  and  $d$  in  $[a, b]$  such that  $f(c) = \sup f$  and  $f(d) = \inf f$ , that is,  $f$  attains both an absolute maximum value and an absolute minimum value in  $[a, b]$ .
9. First Derivative Theorem for Local Extremum: If  $f(x)$  has a local maximum or minimum value at an interior point ' $c$ ' of its domain, and if  $f'(x)$  is defined at  $c$ , then  $f'(c) = 0$ .
10. Second Derivative Test for Local Extrema: Suppose  $f''(x)$  is continuous on an open interval that contains the point ' $c$ '. Then: (a) if  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local *maximum* at  $c$ ; (b) if  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local *minimum* at  $c$ , and (c) if  $f'(c) = 0$  and  $f''(c) = 0$ , the test is inconclusive, i.e.  $f$  may have a local maximum, or minimum, or neither.
11. Rolle's Theorem: Suppose  $f(x)$  is continuous at every point of a closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ . Then, if  $f(a) = f(b)$ , there will be at least one point ' $c$ ' in  $(a, b)$  at which  $f'(c) = 0$ .
12. Mean Value Theorem for Derivatives: Suppose  $f(x)$  is continuous at every point of a closed interval  $[a, b]$  and differentiable at every point of its interior  $(a, b)$ . Then there will be at least one point ' $c$ ' in  $(a, b)$  at which
 
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
- 12a. Physical interpretation of Mean Value Theorem for Derivatives: For a function that is continuous and differentiable within an interval, the value of its average gradient or mean slope over the interval is exactly attained by the instantaneous derivative of the function at at least one interior point.

12b. An important corollary of the Mean Value Theorem of derivatives: If two different functions have the same derivative with respect to a common independent variable within an interval, then the functions differ only by a constant within that interval.

13. Logical sequence of Theorems and Definitions leading to the Mean Value Theorem of derivatives:



14. L'Hopital's Rule – First and Second forms: First form: Suppose that within an interval 'I',  $f(a) = g(a) = 0$ , and that  $f'(a)$  and  $g'(a)$  exist, and also  $g'(a) \neq 0$ . Then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$ .

Second form: If  $f(x)$  and  $g(x)$  approach  $\infty$  as  $x \Rightarrow a$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

provided the limit on the right exists.

15. Definition of Power Series, its center and coefficients: A power series about  $x = 0$  is a series of the form

$$c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots = \sum_{k=0}^{\infty} c_k x^k;$$

a power series about  $x = a$  is of form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots = \sum_{k=0}^{\infty} c_k (x-a)^k;$$

in which the 'center'  $a$  and the coefficients  $c_0, c_1, \dots, c_n$  are constants.

16. Convergence Theorem for Power Series: If the power series  $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ ; converges for  $x = c \neq 0$ , then it converges absolutely for all  $x$  with  $|x| < |c|$ . If the series diverges for  $x = d$ , then it diverges for all  $x$  with  $|x| > |d|$ .

17. Term by Term Differentiation Theorem for Power Series: If  $\sum_{n=0}^{\infty} c_n (x-a)^n$  converges for  $a - R < x < a + R$  for some  $R >$

0, it defines a function  $f: f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ ,  $a - R < x < a + R$ . Such a function  $f$  has derivatives of all orders inside the interval of convergence. The function derivatives can be obtained by differentiating the original series term by term:  $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ ,  $f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$ , and so on. Each of these derived series converges at every interior point of the original series.

18. Taylor Series: Let  $f$  be a function with derivatives of all orders throughout some interval containing 'a' as an interior point. Then the Taylor Series generated by  $f$  at  $x = a$  is:  $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k =$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \dots$$

19. Taylor Polynomial of order  $n$ : Let  $f$  be a function with derivatives of order  $k$  for  $k = 1, 2, \dots, N$  in some interval containing ' $a$ ' as an interior point. Then for any integer  $n$  from 0 through  $N$ , the Taylor polynomial of order  $n$  generated by  $f$  at  $x = a$  is the polynomial  $P_n(x)$  expressed as:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

20. Taylor's Theorem: If  $f$  and its first  $n$  derivatives  $f', f'', \dots, f^{(n)}$  are continuous on the closed interval  $[a, b]$ , and  $f^{(n)}$  is differentiable on the open interval  $(a, b)$ , then there exists a  $c \in (a, b)$  such that

$$f(b) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

21. Taylor's formula with Remainder: If  $f$  has derivatives of all orders in an open interval  $I$  containing point  $a$ , then for each  $n \in \mathbb{N}^*$  and for each  $x \in I$ ,  $f(x) = P_n(x) + R_n(x)$ , where  $P_n(x)$  is the Taylor Polynomial of order  $n$  and  $R_n(x)$  is the

Remainder term for order  $n$  given by  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ , for some  $c$  lying between  $x$  and  $a$ .

22. Remainder Estimation Theorem: If there is a positive constant  $M$  such that  $|f^{(n+1)}(t)| \leq M$  for all  $t$  between  $a$  and  $x$  (inclusive), then the remainder term  $R_n(x)$  in Taylor's Theorem (and Taylor's formula) satisfies the inequality

$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$ . If this condition holds for every  $n$  and other conditions of Taylor's Theorem are satisfied by  $f$ , then the series converges to  $f(x)$ .

23. Anti-derivative of a function: The function  $F(x)$  is the anti-derivative of the function  $f(x)$  in the latter's domain if  $F'(x) = f(x)$ .

24. Riemann Partition, Norm and Sum: For a function  $f(x)$  defined over a closed interval  $[a, b]$ , and for an  $n \in \mathbb{N}^*$ , a set of  $(n-1)$  points  $x_i, i \in \{1, \dots, n-1\}$ , such that  $x_0 = a < x_1 < x_2 < \dots < x_i < \dots < x_{n-1} < b = x_n$ , define a partition  $P$  over  $[a, b]$ . The largest value of  $\Delta x_i = x_i - x_{i-1}$ , is defined as the norm of the partition, and denoted as  $\|P\|$ . Then, for any possible  $c_i$  satisfying  $x_{i-1} \leq c_i \leq x_i$  for all  $i$ , the value  $S_p = \sum_{i=1}^n f(c_i) \Delta x_i$  is called the Riemann Sum of the function  $f$  over the interval  $[a, b]$  for a partition  $P$  with norm  $\|P\|$ .

25. Definition of Definite Integral as Limit of Riemann Sums: Let  $f(x)$  be a function defined over a closed interval  $[a, b]$ . We say that a number  $I$  is the Definite Integral of  $f$  over  $[a, b]$  and that  $I$  is the limit of the Riemann sums  $\sum_{k=1}^n f(c_k) \Delta x_k$

if the following condition is satisfied: Given any number  $\varepsilon > 0$  there is a corresponding number  $\delta > 0$  such that for every partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  with  $\|P\| < \delta$  and any choice of  $c_k$  in  $[x_{k-1}, x_k]$ , we have

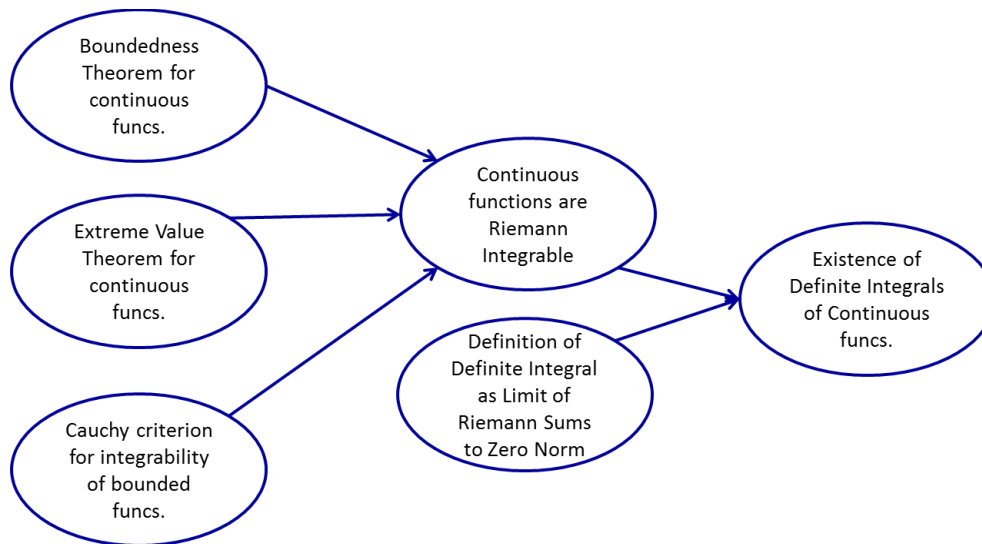
$$\left| \sum_{k=1}^n f(c_k) \Delta x_k - I \right| < \varepsilon.$$

26. Cauchy criterion for integrability: A bounded function  $f$  defined on the interval  $[a, b]$  is Riemann integrable iff for every  $\varepsilon > 0$  there exists a partition  $P$  of  $[a, b]$  such that  $U(f; P) - L(f; P) < \varepsilon$ , where  $U$  and  $L$  represent the upper and lower Riemann sums over  $[a, b]$ .

*From a physical viewpoint, this implies that a bounded function is integrable iff its upper and lower sums get arbitrarily close together for all sufficiently refined partitions.*

27. Theorem of Existence of Definite Integral of a continuous function: A continuous function  $f(x)$  defined over a closed interval  $[a, b]$  is Riemann Integrable, and its corollary is that a Definite Integral exists for a continuous function defined over a closed interval  $[a, b]$ .

28. Logical sequence of Theorems and Definitions leading to Existence Theorem of a Definite Integral:



29. Mean Value Theorem of Integrals: If  $f(x)$  is continuous on  $[a, b]$ , then at some point  $c$  in  $[a, b]$

$$f(c) = \frac{1}{(b-a)} \int_a^b f(x) dx$$

29a. Physical Interpretation of the Mean Value Theorem of Integrals: A function value in an interval precisely acquires its average value over the interval at some point within the interval. (Translate this interpretation to that situation where the 'function' is a derivative).

30. Linkages between continuity, differentiability and integrability: For a function defined on an interval,

DIFFERENTIABILITY  $\Rightarrow$  CONTINUITY, while  
 CONTINUITY  $\xrightarrow{X}$  DIFFERENTIABILITY; ('X' below arrow denotes 'does not imply')  
 further  
 INTEGRABILITY  $\xrightarrow{X}$  CONTINUITY; while  
 CONTINUITY  $\Rightarrow$  INTEGRABILITY.

A corollary of the above is that an anti-derivative of a function is, by definition, differentiable and continuous within the domain of interest. But the derivative of the anti-derivative, i.e. the original function, need not be continuous.

31. Fundamental Theorem of Calculus Part-1 (The Concept): If  $f(x)$  is continuous on  $[a, b]$  then  $F(x) = \int_a^x f(t) dt$  is

continuous on  $[a, b]$  and differentiable on  $(a, b)$  and its derivative is  $f(x)$ , i.e.  $F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$ .

32. Similarity of anti-derivative and integral: It should be conceptually clear that the anti-derivative of a function with a constant 'C' to insert generality (or indefiniteness), corresponds to the integral where the lower limit of integration 'a' inserts generality; the effect of differentiation leading to  $F'(x)$ , where F is either an anti-derivative or an integral, will filter out the effect of indefiniteness imposed either through a 'C' or an 'a'.

33. Fundamental Theorem of Calculus Part-2 (The Calculation): If  $f(x)$  is continuous at every point of  $[a, b]$  and  $F$  is any anti-derivative of  $f$  on  $[a, b]$ , then  $\int_a^b f(x) dx = F(b) - F(a)$ .

Note that this lays the basis of actual (2-step) calculation of the definite integral – first evaluate the anti-derivative (usually the complicated part) and then substitution of limits of integration – thus eliminating the need for the alternative approach of taking Riemann sums on partition norms approaching zero.

34. Leibniz Rule for integration with variable limits:  $\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x)$ .

It may be seen that in the standard case when  $u(x) = 1$  and  $v(x) = x$ , this reduces to the conditions of Fundamental Theorem of Calculus Part-1.

35. Differentiation and Integration are Inverses of each other: We have seen that

$$\frac{d}{dx} \int_a^x f(t)dt = \frac{d(F(x))}{dx} = f(x), \text{ i.e. the derivative of the integral of a function gives back that function.}$$

We can also see that  $\int_a^x \frac{dF}{dt} dt = \int_a^x f(t)dt = F(x)$ , i.e. the integral of the derivative of a function gives back that function.

Thus, whichever way we look at integration and differentiation, they are inverses of each other.

36. Application Formulae for FTC-2 for areas and volumes of axisymmetric solids with principal axis aligned along either x- or y- axes: Illustrations and corresponding formulae have been uploaded on Moodle.

37. Rule of composition of one-to-one functions: First, inverses can be properly defined only for one-to-one functions within their respective domains. Second, if  $f(x)$  represents a function and  $f^{-1}(x)$  represents its inverse, then the composition of  $f^{-1}$  and  $f$  applied at a point in the domain of the original function will exactly return that point, i.e.

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x.$$

38. Derivative rule of inverse functions: For  $f^{-1}(x)$  and  $f(x)$  as mutual inverse functions and 'a' a point in the range of the original function,  $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$ .

39. Definition of the function  $\ln(x)$  and its derivative: The function  $\ln(x)$  is defined as  $\ln(x) = \int_1^x \frac{1}{t} dt$ .

Then it follows from the Fundamental Theorem Part-1 that  $\frac{d(\ln x)}{dx} = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x}$ .

40. Definition of  $e^x$  as the inverse function of  $\ln(x)$ : First, the number 'e' is defined as that value of  $x$  at which  $\ln(x)$  equals 1, i.e.  $\ln(e) = 1$ . Then it follows from Power Rule of function  $\ln(x)$  that  $\ln(e^x) = x \ln(e) = x$ , i.e.  $(\ln \circ e)(x) = x \Rightarrow e^x = \ln^{-1}(x) \Rightarrow e^x$  and  $\ln(x)$  are inverse functions.

41. Definition of  $\log_a(x)$  as the inverse of  $a^x$ , and thence demonstration that  $\ln(x)$  is also a log function where  $a = e$ :

It is demonstrated that the constructed function  $\ln(x)$  is a special case of the log function with base as e, which provides it with elegant properties associated with its differentiation and integration.

42. Interpretation of  $\delta$  and  $\epsilon$  for proving various Mathematical Relationships: Very often we will encounter mathematical expressions of the type – for  $|Y| < \epsilon$ , there exists an X such that  $|X| < \delta$ . So as not to get confused, certain tips are provided below:

a) Most of these relationships deal with an independent term (or quantity, or process) and a dependent term (or quantity, or process). The first point of note is that “ $\delta$ ” is always associated with the independent process, and “ $\epsilon$ ” associated with the dependent process.

b) The second point of note is that the dependence between  $\epsilon$  and  $\delta$  always runs in the direction opposite to the dependence of the main process. Thus, if

- X represents the independent process and Y represents the dependent process, then

- always  $\delta$  is associated with conditions on X (i.e. with the independent process) and  $\epsilon$  with Y

- the *direction of dependence* between  $\delta$  and  $\epsilon$  is *opposite to the direction of dependence* between X and Y, thus, we will say that “for every  $\epsilon$  ... (some condition on Y)”, “there will exist a  $\delta$  such that ... (some condition on X)”.