

# MA 101: Calculus and Introduction to Analysis

Mid Term Examination II, October 18, 2014

Duration: 120 min; Each problem carries 20 marks

**Problem 1.** For  $a, b \in \mathbb{R}, a < b$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be such that for every  $x \in [a, b]$ ,  $A(x) = \int_a^x f(t)dt$  exists. Let  $c \in (a, b)$ . Consider the following 10 statements:

- |   |   |
|---|---|
| (a) $f$ is continuous at $c$ ,                            | ( $\alpha$ ) $A$ is continuous at $c$ ,         |
| (b) $f$ is discontinuous at $c$ ,                         | ( $\beta$ ) $A$ is discontinuous at $c$ ,       |
| (c) $f$ is continuous and <i>increasing</i> on $(a, b)$ , | ( $\gamma$ ) $A$ is <i>convex</i> on $(a, b)$ , |
| (d) $f'(c)$ exists,                                       | ( $\delta$ ) $A'(c)$ exists,                    |
| (e) $f'$ is continuous at $c$ ,                           | ( $\epsilon$ ) $A'$ is continuous at $c$ .      |

**Hint:** For any  $g : [x_1, x_2] \rightarrow \mathbb{R}$ , if  $g$  is differentiable on  $(x_1, x_2)$  and  $g'$  is increasing then  $g$  is convex on  $(x_1, x_2)$ .

In the table shown below (that should be copied on your answer script), mark a T(True) whenever a statement labeled with a Latin letter (the rows) implies a statement labeled with a Greek letter (the columns). For every 'T', put in a few words of logical explanation, outside the table.

	$\alpha$	$\beta$	$\gamma$	$\delta$	$\epsilon$
a					
b					
c					
d					
e					

**Note:** There are 10 T's in total, each carries 2 marks, 1 for identification and 1 for logical justification. Warning: a 'T' in a wrong place gives -0.5 marks.

**Problem 2.** (2.a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable at  $a \in \mathbb{R}$ . Evaluate the following limit if it exists:

$$\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}.$$

(2.b) For every  $x \in \mathbb{R}$ , use Taylor's theorem (with an explicit remainder) to show that  $1 - \cos(2x) = 2x^2 - \frac{2}{3} \cos(2c)x^4$  for some  $c \in (0, x)$ . Note that this  $c$  can depend on  $x$ . Now, using the above expression evaluate the limit

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{3x^2}.$$

Finally, verify your answer using L'Hôpital's rule or any other technique you are familiar with.

**Problem 3.** Using the Fundamental Theorem of Calculus and the chain rule for differentiation, compute the first derivatives of the following functions of  $x$  (written in form of integrals):

$$(3.a) \int_{-\pi/4}^{\pi/4} \tan^2(u) du,$$

$$(3.c) \int_0^x \left( \int_0^u \frac{1}{1+v^2} dv \right) du,$$

$$(3.b) \int_0^{\left(\int_0^x e^{-t^2} dt\right)} \sin(u) du,$$

$$(3.d) \int_1^{e^x} \ln(u^2) du.$$

**Problem 4.** (4.a) Using Rolle's theorem, show that between every pair of real solutions to the equation  $\cos(x) = e^{-x}$  there is a real solution to the equation  $\sin(x) = -e^{-x}$ .

(4.b) Using Rolle's theorem, prove that a cubic polynomial can have *at most* three real roots (knowing that a quadratic polynomial can have at most two real roots). Extend this idea and using mathematical induction, show that a polynomial of degree  $n$  can have *at most*  $n$  real roots.

**Problem 5.** We would like to show that for every  $x \in (0, \frac{\pi}{2})$ ,  $x - \frac{x^3}{6} \leq \sin(x) \leq x$ . We will use two different approaches for these two inequalities:

1. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function continuous on  $[a, b]$ , and differentiable on  $(a, b)$ . First, by using Taylor's theorem prove the standard result : "if  $f$  has a non-negative first derivative, then  $f$  is non-decreasing on  $[a, b]$ ". Then use it (with the good  $f$ ) to show the *right* inequality :  $\sin(x) \leq x$  on  $(0, \frac{\pi}{2})$ .
2. To prove the *left* inequality,  $x - \frac{x^3}{6} \leq \sin(x)$  on  $(0, \frac{\pi}{2})$ , use the Taylor's theorem (with an explicit remainder), to write  $\sin(x)$  in terms of a polynomial and a remainder.
3. **Bonus (for +3 marks):** How about the case where  $x \geq \frac{\pi}{2}$ ?