

Remark 1. *This document have to be proofread!*

1 Problem on Convexity.

We need interesting problem here !

- **Ex.1.1)** (**Hard**) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and twice differentiable on $(0, 1)$. Prove that if $f''(x) \geq 0$ for every $x \in (0, 1)$, then f is **convex**.
- **Ex.1.2)** Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and differentiable on (a, b) . Prove that if $f'(x) \geq 0$ for every $x \in (a, b)$, then f is **non-increasing** (ie $\forall x, y \in [a, b], x > y \implies f(x) \geq f(y)$).
And prove that if $f'(x) > 0$ for every $x \in (a, b)$, then f is **increasing** (ie $\forall x, y \in [a, b], x > y \implies f(x) > f(y)$).
- **Ex.1.3)** MORE !
- **Ex.1.4)** MORE !

2 Problem on Differential Calculus.

- **Ex.2.1)** (*From SATYA's suggestion*) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for all x and y in \mathbb{R} ,

$$|f(x) - f(y)| \leq |x - y|^3.$$

Prove that f is a constant function.

- **Ex.2.2)** (*From SATYA's suggestion*) Show that between any two solutions x_1 and x_2 of the equation $e^x \cos(x) = 1$ there exists at least one root y of the equation $e^x \sin(x) - 1 = 0$ (ie $y \in (x_1, x_2)$).
- **Ex.2.3)** Find all the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable everywhere on \mathbb{R} such that

$$\forall x, y \in \mathbb{R}, f(x + y) = f(x) + f(y),$$

- **Ex.2.4)** On which domain (of \mathbb{R}), the following functions are continuous and on which domain are they differentiable?

$$\text{a) } f_1 : x \mapsto x|x| \quad \text{b) } f_2 : x \mapsto \frac{x}{|x| + 1}.$$

- **Ex.2.5)** Study the differentiability of these functions:

$$\text{a) } g_1 : x \mapsto \sqrt{x^2 - x^3} \quad \text{b) } g_2 : x \mapsto (x^2 - 1) \arccos(x^2).$$

- **Ex.2.6)** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function, differentiable at the point $a \in \mathbb{R}$. Study

$$\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}.$$

- **Ex.2.7) (Hard)** Let $f : [a, b] \rightarrow \mathbb{R}$ differentiable, which satisfy

$$f(a) = f(b) = 0 \text{ and } f'(a) > 0, f'(b) > 0.$$

Prove that there exists $c_1, c_2, c_3 \in (a, b)$ such that $c_1 < c_2 < c_3$ and

$$f'(c_1) = f'(c_2) = f'(c_3) = 0.$$

One could think of using the intermediate value theorem, and the ROLLE's theorem.

- **Ex.2.8)** Domain of definition and domain of differentiability of the function $x \mapsto \ln(x + 1 + x^2)$?
- **Ex.2.9)** Bound the error when we approximate $\sqrt{10001}$ by 100.

Hint: one could think of using TAYLOR expansion of order 1 at the point $x = 0$, also called a linearisation, for the function $f : x \rightarrow \sqrt{10000}\sqrt{1+x}$, by observing that $x = 1/10000$ is small regarding to 1, so close to 0.

3 Problem on Integration.

- **Ex.3.1)** (Require integration by part) Let $I_n = \int_0^{\pi/2} \cos^n(t)dt$. Compute I_0 , et I_1 .
Show that for all $n \geq 2, I_n = I_{n-2}$.
Conclude that the sequence $u = (u_n)_{n \in \mathbb{N}} = (n+1)I_n I_{n+1})_{n \geq 0}$ is constant.
- **Ex.3.2) (Hard)** Domain of definition, domain of differentiability and derivative of the function $f : x \mapsto f(x) = \int_x^{x^2} \frac{e^{-t^2}}{1+t} dt$.
- **Ex.3.3)** Let f be a continuous function on $[0, 1]$, satisfying $\int_0^1 f(t)dt = \frac{1}{2}$.
Show the existence of a $c \in [0, 1]$ such that $f(c) = c$.

Hint: What can you say about a continuous function which does not cancel on $[0, 1]$? Study $\int_0^1 (f(t) - t)dt$ to conclude.

- **Ex.3.4)** Show that $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{x} dx \geq \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{x} dx$.
- **Ex.3.5)** By using simple Riemann sums, compute $\int_0^1 x^2 dx$.

Hint: Remind that $\sum_1^n k^2 = \frac{n(n+1)(n+2)}{6}$ is true for any $n \in \mathbb{N}$ (bonus point of you can explain how to prove this formula!).

- **Ex.3.6)** MORE !

4 Problem on Special Functions.

- **Ex.4.1)** For $a, b, c \in \mathbb{R}$, which of the following relationships are true?

$$\begin{array}{llll} 1) (a^b)^c = a^{bc} & 2) a^b a^c = a^{bc} & 3) a^{2b} = (a^b)^2 & \\ 5) (ab)^c = a^{c/2} b^{c/2} & 4) (a^b)^c = a^{(bc)} & 6) (a^b)^c = (a^c)^b & \end{array}$$

- **Ex.4.2) (Hard)** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, such that $\forall x \in \mathbb{R}, f(\sin(x)) = f(x)$. Show that f is constant.
- **Ex.4.3) (Hard)** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Assume that we have a **polynomial function** P such that $\forall x \in \mathbb{R}, P(f(x)) = 0$. Show that f is constant.
- **Ex.4.4)** Find all the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which are :
 - Non-increasing and non-decreasing at the same time,
 - Monotone and periodic.
- **Ex.4.5) (Vicious)** Let $f :]0, 1] \rightarrow \mathbb{R}$, a non-decreasing function. Assume that there is a constant a , such that $0 < a < 1$ and satisfying : $\forall x \in]0, 1], f(x) = f(ax)$. Show that f is constant on $]0, 1]$.
Hint : One could consider $f(a^n x)$, and for all $x \in]0, 1]$, justify that there exists (at least) one $n \in \mathbb{N}$ such that $a^n \leq x$ to conclude.
- **Ex.4.6) (Hard)** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{3}\right)$. Solve $f(x) = -2$. Show that f is periodic. Find the *smalles* possible period of f .