

Problem 5.

1.) i). With Taylor's theorem up to the order 0 with the explicit formula for the remainder we have:

For every $\alpha \in [a, b]$, $x \in [a, b]$, $x > \alpha$ there is a $c \in (\alpha, x)$ such that $f(x) = f(\alpha) + f'(c) \cdot (x - \alpha)$

So if $f'(c) \geq 0$ for any $c \in \mathbb{R}$,

$$f(x) - f(\alpha) = \underbrace{f'(c)}_{\geq 0} \cdot \underbrace{(x - \alpha)}_{> 0} \geq 0$$

so $f(x) \geq f(\alpha)$ when $x > \alpha$
i.e. f is non-decreasing on $[a, b]$

ii). With $f: [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$

$$x \mapsto x - \sin(x)$$

f is of class \mathcal{C}^1 , and $f'(x) = 1 - \cos(x)$ for any $x \in \mathbb{R}$

but $\forall t \in \mathbb{R}$, $-1 \leq \cos(t) \leq 1$, so $f'(x) \geq 0$

Hence f is non-decreasing on $[0, \frac{\pi}{2}]$.

So $\forall x \in [0, \frac{\pi}{2}]$, $f(x) \geq f(0) = 0 - \sin(0) = 0 - 0 = 0$

Conclusion $\left\{ \begin{array}{l} \text{On } [0, \frac{\pi}{2}], \sin(x) \leq x \end{array} \right.$

2.)

Now, let's use the Taylor's theorem up to the order 3 with the explicit formula for the remainder:

For $x \in (0, \pi/2)$, there is a $c \in (0, x)$ such that

$$\sin(x) = \sin(0) + x \times \sin'(0) + \frac{x^2}{2} \times \sin''(0) + \frac{x^3}{6} \times \sin'''(0) + \frac{x^4}{24} \times \sin^{(4)}(c)$$

But

- $\sin(0) = 0$
 - $\sin'(0) = \cos(0) = 1$
 - $\sin''(0) = -\sin(0) = 0$
 - $\sin'''(0) = -\cos(0) = -1$
- and $\sin^{(4)} = \sin$

So
$$\sin(x) = x - \frac{x^3}{6} + \frac{x^4}{24} \cdot \sin(c)$$

But on $(0, \pi/2)$, $\sin(c) > 0$,

$$\frac{x^4}{24} > 0$$

so
$$\sin(x) = x - \frac{x^3}{6} + \underbrace{\frac{x^4 \sin(c)}{24}}_{> 0}$$

$> x - \frac{x^3}{6}$

Conclusion On $(0, \pi/2)$,

$$x - \frac{x^3}{6} \leq \sin(x)$$

3.) These two inequalities are still true for $x > \pi/2$.

- * $\sin(x) \leq x$ is obvious if $x \geq 1$ because $\sin(x) \leq 1$
- * Bonus: solve yourself the last inequality!