

Problem 4

(TIA 101)

(4.a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto \cos(x) - e^{-x}$

* f is continuous and differentiable on \mathbb{R} .

* Let α, β be two ^(real) solutions of the equation
 $f(x) = 0$, with $\alpha < \beta$.

Then, on $[\alpha, \beta]$ f is continuous,
 on (α, β) f is differentiable,
 and $f(\alpha) = 0 = f(\beta)$

So thanks to Rolle's theorem, we can
 conclude that there is a $c \in (\alpha, \beta)$
 such that $f'(c) = 0$

* But $f'(c) = -\sin(x) + e^{-x}$ so
 $f'(c) = 0 \Rightarrow -\sin(c) + e^{-c} = 0$
 $\Rightarrow \sin(c) = e^{-c}$ and $c \in (\alpha, \beta)$

Conclusion

*and Error in the question
 the solution, sorry.*

Between every pair of real solutions to the equation
 $\cos(x) = e^{-x}$, there is a real solution to $\sin(x) = e^{-x}$.

4.b')

*First case Let $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto ax^3 + bx^2 + cx + d$
a cubic polynomial.

Let α, β be two real roots for f .

We have:

* $f(\alpha) = f(\beta) = 0$

* f is continuous on $[\alpha, \beta]$ (because on \mathbb{R})

* f is differentiable on (α, β) .

So thanks to Rolle's Theorem there is a
 $c \in (\alpha, \beta) / f'(c) = 0$.

But $f'(x) = 3ax^2 + 2bx + c$ can have at most
two real roots on \mathbb{R} .

If f has more than 3 roots, it has at least
 x_1, x_2, x_3, x_4 , ordered as $x_1 < x_2 < x_3 < x_4$

so we have

$$c_1 \in (x_1, x_2),$$

$$c_2 \in (x_2, x_3),$$

$$c_3 \in (x_3, x_4)$$

such that $f'(c_1) = 0 = f'(c_2) = f'(c_3)$,
which is absurd.

Conclusion

A polynomial of degree 3
can have at most three
real roots.

* General case

For $n \in \mathbb{N}^*$ (i.e. n is a integer, $n > 0$), let P_n be that statement:

"A polynomial of degree n can have at most n real roots".

By induction, let prove that P_n is true for all $n \in \mathbb{N}^*$.

* For $n=1$. A polynomial of degree 1 is a function

$f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto ax + b$, with $a \neq 0$

As $a \neq 0$, $f(x) = 0 \Leftrightarrow ax + b \Leftrightarrow ax = -b$

$\Leftrightarrow x = -b/a$

so f has exactly one real root.

P_1 is true.

* For $n \geq 1$, assume that P_n is true.

And let f be a polynomial of degree $n+1$.
 f' is of degree n , so f' cannot have more than n real roots (thanks to P_n).

If f has more than $n+1$ real roots,
at least it has $x_1, \dots, x_{n+1}, x_{n+2}, \dots$

But on $[x_i, x_{i+1}]$ - * f is continuous on $[x_i, x_{i+1}]$

* $f(x_i) = 0 = f(x_{i+1})$

* f is differentiable on (x_i, x_{i+1})

so there is a $c_i \in (x_i, x_{i+1})$ such that $f'(c_i) = 0$

This gives c_1, \dots, c_{n+1} : $n+1$ (distinct) real roots

for f' : this contradicts P_n , so there is an error

Conclusion: f cannot have more than $n+1$ real roots.

Cl. $\forall n \geq 1, P_n$ is true. \square if $P_n \Rightarrow P_{n+1}$ and P_0