

Problem 3.

MA 101

3.a) For $x \in \mathbb{R}$,

$$\text{let } f(x) = \int_{-\pi/4}^{\pi/4} \tan^2(u) \, du.$$

$f(x)$ is well defined because $u \mapsto \tan^2(u)$ is continuous (and bounded) on $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

But $f(x)$ is a constant (with respect to x), so directly differentiable,

and we have: $\forall x \in \mathbb{R}, f'(x) = 0.$

3.b) For $x \in \mathbb{R}$, let $g(x) = \int_0^x e^{-t^2} \, dt$

$$\text{and } f(x) = \int_0^{g(x)} \sin(u) \, du$$

$$= \int_0^{\left(\int_0^x e^{-t^2} \, dt\right)} \sin(u) \, du.$$

* $t \mapsto e^{-t^2}$ is continuous on \mathbb{R} and Riemann-integrable on $[0, x]$, so g is well defined, continuous and differentiable on \mathbb{R} .

3.6) (..)

And \sin is continuous and integrable on \mathbb{R} ,
so $x \mapsto \int_0^x \sin(u) du$ is well defined,
and continuous and differentiable \therefore on \mathbb{R} .

By composition, $f(x) = h(g(x))$
is also well defined, continuous and
differentiable on \mathbb{R} .

And so, for any $x \in \mathbb{R}$,

$$f'(x) = g'(x) \cdot h'(g(x))$$

thanks to the chain rule

ie $f'(x) = g'(x) \cdot \sin(g(x))$

But $g'(x) = e^{-x^2}$

Conclusion.

$$f'(x) = e^{-x^2} \cdot \sin\left(\int_0^x e^{-t^2} dt\right)$$

for any $x \in \mathbb{R}$.

3.c') For $x \in \mathbb{R}$, let $g(x) = \int_0^x \frac{1}{1+v^2} dv$.

$v \mapsto \frac{1}{1+v^2}$ is of class C^0 on \mathbb{R} (ie, well defined and continuous),

so $g(x)$ is well defined, continuous and differentiable on \mathbb{R} ,

Remark: In fact: $g'(x) = \left[\arctan(v) \right]_0^x$

$$= \arctan(x)$$

(the reciprocal function of \tan)

And now, for $x \in \mathbb{R}$, let $f(x) = \int_0^x g(u) du$.

g is continuous on \mathbb{R} so Riemann integrable on $[0, x]$ (or $[x, 0]$, regarding the sign of x) for any $x \in \mathbb{R}$,

so f is well defined, continuous and differentiable on \mathbb{R} (ie f is of class \mathcal{C}^2)

And finally, we know that, for $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= g(x) && \left(= \frac{d}{dx}(x) \cdot g(x) - \frac{d}{dx}(0) \cdot g(0) \right) \\ &= \int_0^x \frac{1}{1+v^2} dv = \underline{\underline{\arctan(x)}} \end{aligned}$$

B.d') Similarly, for $x \in \mathbb{R}$, $g(x) = e^{2x}$

g is of class \mathcal{C}^1 on \mathbb{R} ,

and so if $f(x) = \int_0^{g(x)} \ln(u^2) du$,

f is well defined because for any $t \in \mathbb{R}$,

$u \mapsto \ln(u^2)$ is well defined and integrable
on $(0, t]$ (or $[t, 0)$...)

f is also continuous, and differentiable because
 g is continuous on \mathbb{R} .

Finally, for $x \in \mathbb{R}$,

$$f'(x) = g'(x) \cdot \ln(g(x)^2)$$

$$= e^{2x} \ln(e^{2x})$$

So $\forall x \in \mathbb{R}, f'(x) = 2x \cdot e^{2x}$