

## Problem 2

MA 101

2.a Let  $a \in \mathbb{R}$ , and  $f: \mathbb{R} \rightarrow \mathbb{R}$  be differentiable at  $a$ .

For  $x \neq a$ , we can write

$$\begin{aligned} \frac{a \cdot f(x) - x \cdot f(a)}{x - a} &= \frac{a \cdot f(x) - a \cdot f(a) + a \cdot f(a) - x \cdot f(a)}{x - a} \\ &= a \cdot \frac{f(x) - f(a)}{x - a} + f(a) \cdot \frac{a - x}{x - a} \\ &= a \cdot \frac{f(x) - f(a)}{x - a} - f(a) \end{aligned}$$

But  $f$  is differentiable at  $a$ , so

$$\frac{f(x) - f(a)}{x - a} \xrightarrow{x \rightarrow a} f'(a)$$

Finally, we conclude that

$$\frac{a \cdot f(x) - x \cdot f(a)}{x - a} \xrightarrow{x \rightarrow a} a \cdot f'(a) - f(a)$$

2.6 . Let  $x \in \mathbb{R}$  and  $g(x) = 1 - \cos(2x)$

$g$  is continuous and differentiable, with first derivative being also continuous and differentiable, and second and third derivative is continuous on  $\mathbb{R}$ .

Remark.  $g$  is said to be of class  $\mathcal{C}^3$  on  $\mathbb{R}$ .

So we can use Taylor's formula of order 3 ( $n=3$ ) with an explicit remainder

So there exists a  $c \in (0, x)$  such that

$$g(x) = g(0) + g'(0) \cdot x + g''(0) \cdot \frac{x^2}{2} + g'''(0) \cdot \frac{x^3}{6} + g^{(4)}(c) \cdot \frac{x^4}{24}$$

But  $g(0) = 1 - \cos(2 \times 0) = 1 - 1 = 0$ ,

$$g'(0) = 2 \sin(2 \times 0) = 0$$

$$g''(0) = 4 \cos(2 \times 0) = 4$$

$$g'''(0) = -8 \sin(2 \times 0) = 0$$

and  $g^{(4)}(c) = -16 \cos(2c)$  but  $\frac{16}{24} = \frac{2}{3}$

So  $1 - \cos(2x) = 2x^2 - \frac{2}{3} \cos(2c)x^4$

for one  $c \in (0, x)$ , as wanted.

i). To estimate that limit, we write

$$\begin{aligned} \frac{1 - \cos(2x)}{3x^2} &= \frac{2x^2 - \frac{2}{3} \cos(2c) x^4}{3x^2} \\ &= \frac{2}{3} - \frac{2}{9} \cos(2c) x^2 \\ &\xrightarrow{x \rightarrow 0} \frac{2}{3} - \frac{2}{9} \cdot \cos(2x \pm) \cdot 0^2 \\ &= \frac{2}{3} \end{aligned}$$

Indeed,  $c$  is depending on  $x$ , but  $c \in (0, x)$  so if  $x \rightarrow 0$ , necessarily,  $c \rightarrow 0$  also.

Conclusion.

$$\frac{1 - \cos(2x)}{3x^2} \rightarrow \frac{2}{3}$$

ii) We can also use L'Hôpital rule to the second order,

because  $\frac{d}{dx} (1 - \cos(2x)) \Big|_{x=0} = 0$

and  $\frac{d}{dx} (3x^2) \Big|_{x=0} = 0$

So  $\begin{cases} \frac{d^2}{dx^2} (1 - \cos(2x)) \Big|_{x=0} = 4 \cos(2x) \Big|_{x=0} = 4 \\ \frac{d^2}{dx^2} 3x^2 \Big|_{x=0} = 6 \Big|_{x=0} = 6 \neq 0 \end{cases}$

so finally

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{3x^2} = \frac{4}{6} = \frac{2}{3}$$