

MAHINDRA ÉCOLE  
CENTRALE

Mid Term Exam N° = 2 (18/10/14)

MA 101 : "Introduction to  
calculus and analysis".

Problem 1

19h

For  $a, b \in \mathbb{R}$ , with  $a < b$ , and  $f: [a, b] \rightarrow \mathbb{R}$   
such that  $A: [a, b] \rightarrow \mathbb{R}$ ,  $x \mapsto \int_a^x f(t) dt$  exists.  
And let  $c \in (a, b)$ .

(α) Whatever the  $f$ , if  $A$  exists then it is  
always continuous on  $[a, b]$  (in this case).

Indeed, for  $h > 0$  such that  $c+h \in (a, b)$

$$A(c+h) - A(c) = \int_c^{c+h} f(x) dx \xrightarrow{h \rightarrow 0} \int_c^c f(x) dx$$

$$\text{(and similarly for } A(c) - A(c-h)) = 0$$

So  $A$  is always continuous at  $c$ .

The entire table can be found on the  
next page.

$\Rightarrow$	$\alpha)$	$\beta)$	$\gamma)$	$\delta)$	$\epsilon)$
a)	True			True	
b)	True				
c)	True		True	True	
d)	True			True	
e)	True			True	True

\* a)  $\Rightarrow$   $\delta)$  If  $f$  is continuous at  $c$ ,

$$\frac{A(x+h) - A(c)}{h} = \frac{1}{h} \int_c^{c+h} f(x) dx \quad \text{for } h > 0 \text{ small enough}$$

$$\text{But so } \left| \frac{A(x+h) - A(c)}{h} - f(c) \right| = \frac{1}{h} \left| \int_c^{c+h} f(x) - f(c) dx \right|$$

$$\leq \frac{1}{h} \int_c^{c+h} |f(x) - f(c)| dx$$

Let  $\epsilon > 0$   $f$  is continuous at  $c$ , so there is a  $\delta > 0$  such that for any  $x \in [c - \delta, c + \delta]$ ,

$$|f(x) - f(c)| \leq \epsilon.$$

So if  $|h| < \delta$ , we have  $\frac{1}{h} \int_c^{c+h} |f(x) - f(c)| dx \leq \frac{h}{h} \epsilon = \epsilon$

because for all  $x \in [c, c+h]$ ,  $x \in [c - \delta, c + \delta]$ .

So  $\frac{A(c+h) - A(c)}{h} \rightarrow f(c) \quad \stackrel{\text{def}}{=} \left\{ \begin{array}{l} A \text{ is differentiable} \\ \text{at } c. \end{array} \right.$

\* Note For this  $\alpha) \Rightarrow \delta)$ , one could justify more quickly by just saying that we know that from the lectures...

\*  $c) \Rightarrow \delta)$  If  $f$  is increasing on  $(a, b)$ ,  
and continuous on  $[a, b]$ ,  
then  $A$  is differentiable,  $A' = f$  on  $[a, b]$ ,  
and so  $A'$  is increasing.

By using the hint, we conclude that  
 $A$  is convex on  $(a, b)$

\*  $d) \Rightarrow \delta)$  If  $f'(c)$  exists, we know that  
 $f$  is continuous at  $c$  so  $A$  is differentiable at  $c$   
(cf.  $\alpha) \Rightarrow \delta)$  proof) so  $A'(c)$  exists.

\*  $e) \Rightarrow \delta)$  similarly, saying that  $f'$  is  
continuous at  $c$  implies that  $f'(c)$  exists, so  
 $e) \Rightarrow d)$  and  $d) \Rightarrow \delta)$ ,  
so  $e) \Rightarrow \delta)$ .

\*  $e) \Rightarrow \gamma)$  Because  $A'(c) = f(c)$  when  
 $f$  is continuous at  $c$ ,  
if  $f'$  is continuous at  $c$ , this means that  
 $f'$  is defined on a small interval  $[c-h, c+h]$   
for some  $h > 0$ ,  
so  $f$  is differentiable on  $[c-h, c+h]$  iff  $f$  is continuous  
on  $[c-h, c+h]$

\*  $\epsilon) \Rightarrow \delta)$  - So  $f$  is continuous on a small interval centered on  $c$ , and  $A'$  is then well defined on this interval  $[c-h, c+h]$ , and  $A'$  is continuous on  $[c-h, c+h]$  because  $A'(x) = f(x)$  for  $x \in I$ .

Remark. that last  $\epsilon) \Rightarrow \delta)$  is not easy.

\*  $\epsilon) \Rightarrow \delta)$  Indeed if  $f$  is continuous on  $(a, b)$ , it is continuous at  $c$ , so  $A$  is differentiable at  $c$ , ( $\epsilon) \Rightarrow \delta)$ .

Sorry for forgetting this one!