

Problem 2

Remark: For this problem, you can admit and use the result of one question in the next questions (e.g. A.2 and A.3 can be used for A.4), but not in the other sense: do not use question A.4 to prove question A.3.

This problem is about Wallis integrals.

For any integer $n \in \mathbb{N}$, we define $I_n = \int_0^{\pi/2} \sin^n(t) dt = \int_0^{\pi/2} (\sin(t))^n dt$.

Qu.1) Compute I_0 and I_1 .

Qu.2) Justify quickly that I_n is well defined for any integer n .

Qu.3) Prove that the sequence $I = (I_n)_{n \in \mathbb{N}}$ is decreasing.

Hint: for $t \in (0, \pi/2)$, remark that $0 < \sin(t) < 1$, and then you could use the monotonicity of the integral.

Qu.4) Thanks to an integration by parts, find a recurrence relation between I_n and I_{n+2} (for all $n \in \mathbb{N}$). A recurrence relation means that you have to write I_{n+2} as a function of I_n .

Qu.5) Prove that $\lim_{n \rightarrow +\infty} \frac{I_{n+1}}{I_n} = 1$. First, you will have to quickly justify that I_n is never equal to 0 (before dividing by I_n).

Qu.6) (**Harder!**) Show that $\sqrt{\frac{2n}{\pi}} I_n \xrightarrow{n \rightarrow +\infty} 1$.

Qu.7) For $p \in \mathbb{N}$, compute I_{2p} and I_{2p+1} , thanks to the recurrence relation found on question 4, and the initial values found on question 1. You can use $\binom{2p}{p}$ in your result, if you want.

Qu.8) Check that TODO.

Qu.9) En fait, nous n'avons pas parlé d'équivalents en cours, donc ce problème risque de ne pas être utilisable !