

Problem 1

Remark: For this problem, *you can admit and use the result of one question in the next questions* (e.g. A.2 and A.3 can be used for A.4), but not in the other sense: do not use question A.4 to prove question A.3.

The problem is long, and some questions (at the end) are hard, but most of the questions can be done easily, by following step by step the hints.

This problem is about the sum $z(2) = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \right)$

Part A : Convergence of the sequence $S(p)$

Let $S_n(p) = \sum_{k=1}^n \frac{1}{k^p} = \sum_{k=1}^n \left(\frac{1}{k} \right)^p$, where $p, n \in \mathbb{N}^*$ (two positive integers).

Qu.A.1) Let $k \geq 1$. Quickly justify why the function $f : x \mapsto \frac{1}{x^p}$ is decreasing on the interval $[k, k+1]$ ($p \geq 1$ can help you). Then use the **Min-Max inequality** for the monotonic function f on $[k, k+1]$ to prove that:

$$\frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{1}{x^p} dx \leq \frac{1}{k^p}.$$

Qu.A.2) By using the previous question, the definition of that sum $S_n(p)$, and the additivity of the integral on intervals, show that:

$$\forall n \geq 2, S_n(p) - 1 \leq \int_1^n \frac{1}{x^p} dx \leq S_{n-1}(p).$$

Qu.A.3) By computing $\int_1^A \frac{1}{x^p} dx$ (for some real number $A > 1$), and trying to find its limit for $A \rightarrow +\infty$, prove that the function $x \mapsto \frac{1}{x^p}$ is *integrable* on $[1, +\infty)$ *if and only if* $p \geq 2$ (ie. Riemann integrable iff $p \geq 2$).

Qu.A.4) Thanks to the two previous questions (A.2 and A.3), and the squeezing theorem, conclude that the sequence $S(p) = (S_n(p))_{n \in \mathbb{N}^*}$ is convergent¹ *if and only if* $p \geq 2$.

From now on, thanks to the A.4, we can define the function z by $z(p) = \lim_{n \rightarrow +\infty} S_n(p)$ for every $p \geq 2$.

Part B: Calculus of $z(2) = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \right)$

We now define two more functions h and φ_0 :

- the function h is defined on \mathbb{R} , by $h(t) = \frac{t^2}{2\pi} - t$ (for $t \in \mathbb{R}$),
- and the function φ_0 is defined on $[0, \pi]$, by $\varphi_0(t) = \frac{h(t)}{2 \sin\left(\frac{t}{2}\right)}$ (for $t \in (0, \pi]$) and $\varphi_0(0) = -1$.

¹Remember that we say that a sequence $u = (u_n)_{n \in \mathbb{N}^*}$ is *convergent* if it has a *finite* limit when $n \rightarrow +\infty$.

Qu.B.1) Justify that φ_0 is continuous on $[0, \pi]$, differentiable on $(0, \pi]$, and of derivative φ'_0 continuous on $(0, \pi]$ (ie, of class \mathcal{C}^1 from $[0, \pi]$ to \mathbb{R}).

Hint: Be cautious of what is going on at the limits of that interval (ie. the points 0 and π).

Qu.B.2) Let $k \in \mathbb{N}^*$. Thanks to two successive integrations by parts, show that

$$\int_0^\pi h(t) \cos(kt) dt = -\frac{1}{k} \int_0^\pi \left(\frac{t}{\pi} - 1 \right) \sin(kt) dt = \frac{1}{k^2}.$$

Hint: For the first step, writing $f'(t) = \cos(kt)$ and $g(t) = h(t)$ is interesting.

For the second step, writing $f'(t) = \sin(kt)$ and $g(t) = \left(\frac{t}{\pi} - 1 \right)$ is also interesting.

Qu.B.3) This question is subdivided in intermediate steps:

Qu.B.3.a) Prove that $\sum_{k=1}^n \cos(kt) = \frac{1}{2} \left(\frac{e^{i(n+1)t} - 1}{e^{it} - 1} + \frac{e^{-i(n+1)t} - 1}{e^{-it} - 1} \right)$ for all $t \in (0, \pi]$.

Hint: You can follow this way: first use Euler's identity (for \cos) to write $\cos(kt)$ as $\frac{e^{ikt} + e^{-ikt}}{2}$, then write this sum as half of two sums, and finally use twice the formula for a sum of a geometrical progression to conclude the computation.

Qu.B.3.b) Then prove that $\frac{1}{2} \left(\frac{e^{i(n+1)t} - e^{int}}{e^{it} - 1} \right) = \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{t}{2}\right)}$ for all $t \in (0, \pi]$, thanks to Euler's identity for \sin .

Qu.B.3.c) Then, use both B.3.a) and B.3.b) to find a constant $\lambda \in \mathbb{R}$ so that:

$$\forall t \in (0, \pi], \quad \sum_{k=1}^n \cos(kt) = \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{t}{2}\right)} - \lambda.$$

Can you give a *numerical approximation* of that constant λ ? Is it positive?

Qu.B.4) Let φ be a function of class \mathcal{C}^1 (ie $\varphi \in \mathcal{C}^1([0, \pi], \mathbb{R})$).

By an integration by parts, prove that result²:

$$\lim_{n \rightarrow +\infty} \left(\int_0^\pi \varphi(t) \sin\left(\left(n + \frac{1}{2}\right)t\right) dt \right) = 0.$$

Qu.B.5) (**Harder**) Use the definition of $z(2)$, the two results of B.2 and B.3.c, and then take the limit by using B.4 in order to finally prove that $z(2) = \frac{\pi^2}{6}$.

Qu.B.6) Give a *numerical approximation* of that final value $z(2)$.
Is it bigger than 1? Bigger than 2? Smaller than 5?

Qu.B.7) **Bonus:** (+1 mark) Can you guess if this number $z(2)$ is rational or irrational?

²This is *very similar* to a result called LEBESGUE's lemma, as seen in the Exercise 2.3 tutorial sheet on integration by parts, on the Integration Methods chapter...