

MA 101: Calculus and Introduction to Analysis

Final Examination

Duration: 3 hours; Total 100 marks: 10 for Pb 2, 30 for Pbs 1, 3 and 4

December the 3rd, 2014

Problem 1. Multiple choice questions, for 30% of the exam total mark (30/100).

Each of the 15 question carries 2 marks, **no need to justify your choice**, and **only one** (A, B, C or D) is correct. You win +2 marks for a correct choice, but loose -0.5 for a wrong choice (minimum is 0/30 for the whole problem, not $-7.5/30!$).

Qu.I.1) Which of these numbers can be compared (with the usual order $<$) to the number 2014:

- A) $7i$ B) -1 C) $e^{i\theta}$ for any $\theta \in \mathbb{R}$ D) $1 + i$.

Qu.I.2) For real numbers, does the intrinsic notions of a “big number” or a “small number” have any conceptual meaning?

- A) Yes, thanks to the density of rationals number,
B) No, these notions only have a concrete meaning when we compare one number with another : $10^{-12} \ll 1$, or $1 \ll \exp(2014)$.
C) Yes, obviously by the archimedian property,
D) Yes, thanks to the “ ε, δ ” definition of converge of real sequences,

Qu.I.3) If $z = x + iy \in \mathbb{C}$, what is $\arg(z)$:

- A) $+90$ if $x = 0$, C) 0 if $z = 0$,
B) any real number of the form $\theta + 2k\pi$
with $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, D) none of the above.

Qu.I.4) Let $(a_n)_{n \in \mathbb{N}}$ a sequence of non-negative real numbers ($\forall n \in \mathbb{N}, a_n \geq 0$). Which of one of

these statements about the series $S = \sum_{k=0}^{+\infty} a_k$ is true:

- A) if $a_n \xrightarrow{n \rightarrow +\infty} 0$ then the series S converges,
B) if the series S diverges then a_n cannot converge to 0,
C) if a_n is never zero and $\frac{a_{n+1}}{a_n} \xrightarrow{n \rightarrow +\infty} 0$ then the series S converges,
D) if $a_n \xrightarrow{n \rightarrow +\infty} l$ for one $0 < l < 1$ then the series S obviously converges to $\frac{1}{1+l}$.

Qu.I.5) Let $s \in \mathbb{R}_+^*$ (ie. $s > 0$), and define the sequence $(x_n)_{n \in \mathbb{N}}$ by $x_0 = 0$ and the recurrence relation $x_{n+1} = s + x_n^2$ for all $n \geq 0$. A **necessary** condition on that number s so that the sequence converges is:

- A) $s \leq \frac{1}{4}$ B) $s > \frac{1}{4}$ C) $s = 0$ D) $s < 1$

Qu.I.6) If $\beta > 0$, what is the value of this limit $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-\beta}}$?

- A) that limit does not even exist! C) $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-\beta}} = 1$,
 B) $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-\beta}} = -\infty$, D) $\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-\beta}} = 0$.

Qu.I.7) **We need one more question on differentiation.**

Qu.I.8) Let $f : \mathbb{R} \rightarrow \mathbb{R}$. f is **not** Riemann integrable if:

- A) f is continuous, C) f is unbounded,
 B) f is differentiable, D) f is bounded.

Qu.I.9) Let f be the function defined by $f(x) = \frac{1}{\sin(x/2)}$. Then f is convex on:

- A) $(\pi, 2\pi)$, C) $\left(-\frac{3\pi}{2}, -\pi\right)$,
 B) $(-\pi, 0)$, D) $(0, \pi)$.

Qu.I.10) We define f on $\left(\ln\left(-\frac{6}{13}\right), +\infty\right)$ by $f(x) = \ln(13e^x - 6)$. Then we can say that:

- A) $y = f(x) \iff x = f^{-1}(y) = \ln(e^y + 6) - \ln(13)$,
 B) f is not correctly defined!
 C) $y = f(x) \iff x = f^{-1}(y) = \ln(e^y + 13) + \ln(6)$,
 D) f is not bijective on its domain, so it does not have any inverse function.

Qu.I.11) **We need one more question on change of variable.**

Qu.I.12) **We need one more question on integration by parts.**

Qu.I.13) Consider the differential equation $y' + \frac{1}{x}y = \frac{1}{x \ln(x)}$ with $y(2) = 0$.

An unique solution to this Initial Value Problem of the 1st order exists on:

- A) $(-10, 3)$, C) $(2, +\infty)$,
 B) $(0, 2)$, D) None of the above.

Qu.I.14) **We need one more question on curves.**

Qu.I.15) **We need one more question on vector calculus ?**

Problem 2. Fill in the blanks, for 10% of the exam total mark (10/100). Your job is to fill in the blanks in order for that theorem and its proof to make sense.

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differential function and f has a local minimum at $x \in \mathbb{R}$.

Then $f'(x) = \underline{\hspace{2cm}}$ (and $f''(x) \underline{\hspace{2cm}}$). (*fill these two blanks*)

Proof: We will follow the following steps.

1. Since x is a local minimum there exists a $\underline{\hspace{2cm}}$ number $\delta \in \mathbb{R}$ such that:

$$\forall h \in (\underline{\hspace{1cm}}, \underline{\hspace{1cm}}) \quad f(x) \leq f(x + h). \quad (\text{step 1})$$

2. Hence, it follows from step 1 these two inequalities:

$$\forall h \in (0, \underline{\hspace{1cm}}), \quad \frac{f(x + h) - f(x)}{h} \geq 0, \quad (\text{step 2.a})$$

$$\forall h \in (\underline{\hspace{1cm}}, 0), \quad \frac{f(x + h) - f(x)}{h} \leq 0. \quad (\text{step 2.b})$$

3. Now, step 2.a and step 2.b directly imply that:

$$\lim_{h \rightarrow 0^+} \frac{f(x + h) - f(x)}{h} \underline{\hspace{1cm}} 0, \quad (\text{step 3.a})$$

$$\lim_{h \rightarrow 0^-} \frac{f(x + h) - f(x)}{h} \underline{\hspace{1cm}} 0. \quad (\text{step 3.b})$$

4. Now, since f is differentiable everywhere in \mathbb{R} , it follows from step 3.a and step 3.b that

$$\lim_{h \rightarrow 0^-} \frac{f(x + h) - f(x)}{h} \underline{\hspace{1cm}} \lim_{h \rightarrow 0^+} \frac{f(x + h) - f(x)}{h} \underline{\hspace{1cm}} 0, \quad (\text{step 4})$$

which further implies that $f'(x) \underline{\hspace{2cm}}$.

5. Finally, can you give the **name** of the result you just proved?

Problem 3. About differential equations, for 30% of the exam total mark (30/100).

You will of course be cautious to **justify every step**, and explain **carefully** the methods you chose to use. At the end, verifying quickly that the solution(s) you are giving is/are indeed a correct solution(s) could be a good idea!

Part A: First Order Linear Differential Equation.

For the following Initial Value Problem (IVP) of order 1:

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}, \quad y(\pi) = 0.$$

Qu.III.A.1) Find the largest interval in which the solution of the IVP is certain to exist and to be unique.

Qu.III.A.2) Find the general solutions of the homogeneous equation $y' + \frac{2}{x}y = 0$ on *that* interval.

Qu.III.A.3) Then, by using the method of variation of parameter, find the solutions of the complete equation $y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}$ on the same interval.

Qu.III.A.4) Quickly *but properly* check that the functions you give as solutions are indeed solutions, by the method you like.

Part B: Second Order Linear Differential Equations.

Qu.III.B.1) Find the general solution to this differential equation:

$$y'' - 3y' + 2y = x^2 3^x$$

Qu.III.B.2) Find a particular solution to these two differential equations:

$$(E_1) \quad y'' - 4y' + 5y = x.e^{2x}.\sin(x)$$

$$(E_2) \quad y'' - 3y' = 5x^2 + 6x + 1$$

Problem 4. Last problem, about integral calculus, functions, sequences and series, and complex numbers; for 30% of the exam total mark (**30/100**). The problem is long, and some questions (at the end) are hard, but most of the questions can be done easily, by following step by step the hints.

We are going to compute the value of the sum $z(2) = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots + \frac{1}{n^2} \right) = \sum_{k=1}^{+\infty} \frac{1}{k^2}$.

Remark: For this problem, *you can admit and use the result of one question in the next questions* (e.g. A.2 and A.3 can be used for A.4), but not in the other sense: do not use question A.4 to prove question A.3.

Part A: Convergence of the sequence $S(p)$.

Let $S_n(p) = \sum_{k=1}^n \frac{1}{k^p} = \sum_{k=1}^n \left(\frac{1}{k} \right)^p$, where $p, n \in \mathbb{N}^*$ (two positive integers).

Qu.IV.A.1) Let $k \geq 1$. Quickly justify why the function $f : x \mapsto \frac{1}{x^p}$ is decreasing on the interval $[k, k+1]$ ($p \geq 1$ can help you). Then use the **Min-Max inequality** for the monotonic function f on $[k, k+1]$ to prove that:

$$\frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{1}{x^p} dx \leq \frac{1}{k^p}.$$

Qu.IV.A.2) By using the previous question, the definition of that sum $S_n(p)$, and the additivity of the integral on intervals, show that:

$$\forall n \geq 2, \quad S_n(p) - 1 \leq \int_1^n \frac{1}{x^p} dx \leq S_{n-1}(p).$$

Qu.IV.A.3) By computing $\int_1^A \frac{1}{x^p} dx$ (for some real number $A > 1$), and trying to find its limit for $A \rightarrow +\infty$, prove that the function $x \mapsto \frac{1}{x^p}$ is integrable on $[1, +\infty)$ if and only if $p \geq 2$ (ie. Riemann integrable iff $p \geq 2$).

Qu.IV.A.4) Thanks to the two previous questions (A.2 and A.3), and the squeezing theorem, conclude that the sequence $S(p) = (S_n(p))_{n \in \mathbb{N}^*}$ is convergent¹ if and only if $p \geq 2$.

From now on, thanks to A.4), we can define the function z by $z(p) = \lim_{n \rightarrow +\infty} S_n(p)$ for every $p \geq 2$.

¹Remember that we say that a sequence $u = (u_n)_{n \in \mathbb{N}^*}$ is convergent if it has a finite limit when $n \rightarrow +\infty$.

Part B: Calculus of $z(2) = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \right)$.

We now define two more functions h and φ_0 :

- the function h is defined on \mathbb{R} , by $h(t) = \frac{t^2}{2\pi} - t$ (for $t \in \mathbb{R}$),
- and φ_0 is defined on $[0, \pi]$, by $\varphi_0(t) = \frac{h(t)}{2 \sin(\frac{t}{2})}$ (for $t \in (0, \pi]$) and $\varphi_0(0) = -1$.

Qu.IV.B.1) Justify that φ_0 is continuous on $[0, \pi]$, differentiable on $(0, \pi]$, and of derivative φ_0' continuous on $(0, \pi]$ (ie, φ_0 is of class \mathcal{C}^1 from $[0, \pi]$ to \mathbb{R}).

Hint: Be cautious about the extreme points of that interval (ie. 0 and π).

Qu.IV.B.2) Let $k \in \mathbb{N}^*$. Thanks to two successive integrations by parts, show that

$$\int_0^\pi h(t) \cos(kt) dt = -\frac{1}{k} \int_0^\pi \left(\frac{t}{\pi} - 1 \right) \sin(kt) dt = \frac{1}{k^2}.$$

Hint: For the first step, writing $f'(t) = \cos(kt)$ and $g(t) = h(t)$ is interesting.

For the second step, writing $f'(t) = \sin(kt)$ and $g(t) = \left(\frac{t}{\pi} - 1 \right)$ is also interesting.

Qu.IV.B.3) This question is subdivided in intermediate steps $a)$, $b)$, $c)$:

Qu.IV.B.3.a) Prove that $\sum_{k=1}^n \cos(kt) = \frac{1}{2} \left(\frac{e^{i(n+1)t} - 1}{e^{it} - 1} + \frac{e^{-i(n+1)t} - 1}{e^{-it} - 1} \right)$ for all $t \in (0, \pi]$.

Hint: You can follow this way: first use Euler's identity (for \cos) to write $\cos(kt)$ as $\frac{e^{ikt} + e^{-ikt}}{2}$, then write this sum as half of two sums, and finally use twice the formula for a sum of a geometrical progression to conclude the computation.

Qu.IV.B.3.b) Then prove that $\frac{1}{2} \left(\frac{e^{i(n+1)t} - e^{int}}{e^{it} - 1} \right) = \frac{\sin((n + \frac{1}{2})t)}{2 \sin(\frac{t}{2})}$ for all $t \in (0, \pi]$, thanks to Euler's identity for \sin .

Qu.IV.B.3.c) Then, use both B.3.a) and B.3.b) to find a constant $\lambda \in \mathbb{R}$ so that:

$$\forall t \in (0, \pi], \quad \sum_{k=1}^n \cos(kt) = \frac{\sin((n + \frac{1}{2})t)}{2 \sin(\frac{t}{2})} - \lambda.$$

Can you give a *numerical approximation* of that constant λ ? Is it smaller, bigger or *equal* to 1?

Qu.IV.B.4) Let φ be a function of class \mathcal{C}^1 (ie $\varphi \in \mathcal{C}^1([0, \pi], \mathbb{R})$).

By using one integration by parts and *then* taking the limit, prove that result²:

$$\lim_{n \rightarrow +\infty} \left(\int_0^\pi \varphi(t) \sin((n + 1/2)t) dt \right) = 0.$$

²This is *very similar* to a result called LEBESGUE's lemma, as seen in the Exercise 2.3 in the tutorial sheet on integration by parts, for the Integration Methods chapter.

Qu.IV.B.5) (**Harder**) Use the definition of $z(2)$, the two results of B.2 and B.3.c, and then take the limit by using B.4 in order to finally prove that $z(2) = \frac{\pi^2}{6}$.

Qu.IV.B.6) Give a *numerical approximation* of that final value $z(2)$.
Is it bigger than 1? Bigger than 2? Smaller than 5?

Qu.IV.B.7) **Bonus:** (+2 marks) Can you guess if this number $z(2)$ is rational ($\in \mathbb{Q}$) or irrational ($\in \mathbb{R} \setminus \mathbb{Q}$)? (we do not ask any justification or proof here, just guess!)