

Problem 4

Part A

4)

Conclusion. The sequence $S(p) = (S_n(p))_p$ is
convergent if $p \geq 2$,
divergent (to $+\infty$) if $p < 2$,

so convergent if and only if $p \geq 2$.

Part B

1.)
$$\varphi_0(t) = \begin{cases} -1 & \text{if } t = 0 \\ \frac{t^2/2\pi - t}{2\sin(t/2)} & \text{if } t \in (0, \pi] \end{cases}$$

Continuity: If $t \in (0, \pi]$, $\varphi_0(t)$ is continuous,
only $t=0$ is a potential problem.

For $t \rightarrow 0^+$,
$$\frac{t^2/2\pi - t}{2\sin(t/2)} = \frac{t/2}{\sin(t/2)} \times \left(\frac{t}{2\pi} - 1\right)$$

and
$$\left\{ \begin{array}{l} \frac{t}{2\pi} - 1 \longrightarrow -1 \\ \frac{t/2}{\sin(t/2)} \longrightarrow 1 \end{array} \right. \quad \text{so}$$

$$\varphi_0(t) \xrightarrow[t \rightarrow 0^+]{} -1 = \varphi_0(0)$$

So φ_0 is indeed continuous on $[0, \pi]$.

Differentiability: For $t \in (0, \pi]$, φ_0 is differentiable at t ,

and
$$\varphi_0'(t) = \frac{\left(\frac{t}{\pi} - 1\right)}{2\sin(t/2)} - \frac{\left(\frac{t^2}{2\pi} - t\right) \cos(t/2)}{4\sin^2(t/2)}$$

$$\longrightarrow \frac{1}{2\pi} \quad \text{if } t \rightarrow 0^+ \quad \text{so } \underline{\varphi_0'(0) \text{ exists.}}$$

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Part B

φ_0' is continuous.

We can compute $\varphi_0'(0)$ by $\lim_{t \rightarrow 0^+} \frac{\varphi_0(t) - \varphi_0(0)}{t - 0}$

$$\begin{aligned} \text{But } \frac{\varphi_0(t) - \varphi_0(0)}{t} &= \frac{t/2\pi - 1}{2 \sin(t/2)} + \frac{1}{t} \\ &= \frac{(t/2\pi - 1) \cdot t + 2 \sin(t/2)}{2t \cdot \sin(t/2)} \\ &\sim \frac{t^2/2\pi - t}{t^2} \rightarrow \frac{1}{2\pi} = \lim_{t \rightarrow 0^+} \varphi_0'(t) \end{aligned}$$

Conclusion φ_0 is well continuous, differentiable and φ_0' is continuous on $[0, \pi]$.

2) we follow the hints. Let $k \in \mathbb{N}^*$.

$$\int_0^\pi h(t) \cos(kt) dt = \left[h(t) \cdot \frac{1}{k} \sin(kt) \right]_0^\pi - \frac{1}{k} \int_0^\pi h'(t) \sin(kt) dt$$

by the integration by parts with

$$\begin{cases} f(t) = h(t) \\ g'(t) = \cos(kt) \end{cases} \quad \begin{cases} f'(t) = h'(t) \\ g(t) = \frac{1}{k} \sin(kt) \end{cases}$$

$$= 0 - \frac{1}{k} \int_0^\pi \left(\frac{t}{\pi} - 1\right) \sin(kt) dt = -\frac{1}{k} \int_0^\pi \left(\frac{t}{\pi} - 1\right) \sin(kt) dt$$

$$= \left[\frac{1}{k^2} \left(\frac{t}{\pi} - 1\right) \cos(kt) \right]_0^\pi - \frac{1}{k^2} \int_0^\pi \cos(kt) dt$$

Problem 4 Part B

2) So

$$\int_0^\pi h(t) \cos(ht) dt = -\frac{1}{h} \int_0^\pi \left(\frac{t}{\pi} - 1\right) \cos(ht) dt$$

$$= +\frac{1}{h^2} - \frac{1}{h^2} \int_0^\pi \cos(ht) dt = \frac{1}{h^2} - \frac{1}{h^3} \left[\sin(ht) \right]_0^\pi$$

$$= \frac{1}{h^2}$$

Conclusion.

$$\int_0^\pi h(t) \cos(ht) dt = -\frac{1}{h} \int_0^\pi \left(\frac{t}{\pi} - 1\right) \cos(ht) dt = \frac{1}{h^2}$$

3) We solve it, step by step.

a) Let $m, k \in \mathbb{N}^*$



There was a small typo on the question paper,

it was "for all $t \in (0, \pi]$."

For $t \in (0, \pi]$,

$$\cos(ht) = \frac{e^{ikt} + e^{-ikt}}{2} = \frac{(e^{it})^k + (e^{-it})^k}{2}$$

by Euler's formula for \cos .

Because $t > 0$, $e^{it} \neq 1$ and $e^{-it} \neq 1$.

$$So \sum_{k=0}^m \cos(ht) = \sum_{k=0}^m \left(\frac{(e^{it})^k}{2} + \frac{(e^{-it})^k}{2} \right) = \frac{1}{2} \sum_{k=1}^m (e^{it})^k + \frac{1}{2} \sum_{k=1}^m (e^{-it})^k$$

$$= \frac{1}{2} \left(\frac{(e^{it})^{m+1} - 1}{e^{it} - 1} + \frac{(e^{-it})^{m+1} - 1}{e^{-it} - 1} \right)$$

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3) b) For $n \in \mathbb{N}^*$, and $t \in (0, \pi]$,

$$e^{it} \neq 1 \text{ so } e^{it} - 1 \neq 0$$

• And so $\frac{1}{2} \left(\frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} \right)$ is well defined.

• Similarly, $\sin(t/2) \neq 0$.

Euler's identity for sin is $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$
for any $x \in \mathbb{R}$.

$$\begin{aligned} \text{• So } \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin(t/2)} &= \frac{e^{i\left(n+\frac{1}{2}\right)t} - e^{-i\left(n+\frac{1}{2}\right)t}}{2i} \cdot \frac{1}{2\left(e^{it/2} - e^{-it/2}\right)} \\ &= \frac{1}{2} \frac{e^{i\left(n+\frac{1}{2}\right)t} - e^{-i\left(n+\frac{1}{2}\right)t}}{e^{it/2} - e^{-it/2}} \times \frac{e^{it/2}}{e^{it/2}} \\ &= \frac{1}{2} \frac{e^{i\left(n+\frac{1}{2}\right)t+it/2} - e^{-i\left(n+\frac{1}{2}\right)t-it/2}}{e^{it/2+it/2} - e^{-it/2+it/2}} = \frac{1}{2} \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} \end{aligned}$$

As wanted.

3) c) We use the two previous result.

For $n, h \in \mathbb{N}^*$, and $t \in (0, \pi]$,

$$\begin{aligned} \sum_{h=1}^n \cos(ht) &= \sum_{h=0}^n \cos(ht) - \cos(0 \cdot t) \\ &= \frac{1}{2} \left(\frac{e^{i(n+1)t} - 1}{e^{it} - 1} + \frac{e^{-i(n+1)t} - 1}{e^{-it} - 1} \right) - 1 \end{aligned}$$

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3) c)

$$\text{So } \sum_{k=1}^n \cos(kt) = \frac{1}{2} \left(\frac{e^{i(n+1)t} - 1}{e^{it} - 1} + \frac{e^{-i(n+1)t} - 1}{1 - e^{-it}} \right) - 1$$

$$= \frac{1}{2} \left(\frac{e^{i(n+1)t} - 1 - e^{-i(n+1)t} + e^{-it}}{e^{it} - 1} \right) - 1$$

$$= \frac{1}{2} \left(\frac{e^{i(n+1)t} - e^{-i(n+1)t}}{e^{it} - 1} \right) + \frac{1}{2} \left(\frac{e^{-it} - 1}{e^{it} - 1} \right) - 1$$

$$\stackrel{(4)}{=} \frac{\sin((n+1/2)t)}{2 \sin(t/2)} + \frac{1}{2} - 1$$

" - 1/2 = -1/2

Conclusion: With $\lambda = 1/2$, we have as wanted

$$\sum_{k=1}^n \cos(kt) = \frac{\sin((n+1/2)t)}{2 \sin(t/2)} - \lambda$$

For all $n, k \in \mathbb{N}^*$ and $t \in (0, \pi]$.

4) Let $n \in \mathbb{N}^*$. We follow the hint: first we compute the integral, then we take the limit.

$$\text{Let } \begin{cases} f(t) = \varphi(t) \\ g'(t) = \sin((n+1/2)t) \end{cases} \quad \text{so} \quad \begin{cases} f'(t) = \varphi'(t) \\ g(t) = \frac{-1}{n+1/2} \cos((n+1/2)t) \end{cases}$$

We use an integration by parts.

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$$i) \text{ So } \int_0^{\pi} \varphi(t) \sin\left(\left(m+\frac{1}{2}\right)t\right) dt = \left[\varphi(t) \left(\frac{-1}{m+1/2}\right) \cos\left(\left(m+\frac{1}{2}\right)t\right) \right]_0^{\pi} + \frac{1}{m+1/2} \int_0^{\pi} \varphi'(t) \cos\left(\left(m+\frac{1}{2}\right)t\right) dt.$$

And now, as seen in tutorial,

$$0 \leq \left| \int_0^{\pi} \varphi(t) \sin\left(\left(m+\frac{1}{2}\right)t\right) dt \right| \stackrel{(*)}{\leq} \frac{1}{m+1/2} (|\varphi(\pi)| + |\varphi(0)|) + \frac{1}{m+1/2} \left(\int_0^{\pi} |\varphi'(t)| dt \right)$$

thanks to $0 \leq |\cos\left(\left(m+\frac{1}{2}\right)t\right)| \leq 1 \quad \forall t \in (0, \pi]$,
and the triangle inequality for integrals.

$$\text{But } \frac{1}{m+1/2} (|\varphi(\pi)| + |\varphi(0)|) \xrightarrow{m \rightarrow +\infty} 0$$

$$\text{and } \frac{1}{m+1/2} \left(\int_0^{\pi} |\varphi'(t)| dt \right) \xrightarrow{m \rightarrow +\infty} 0 \quad \text{also}$$

So the inequality (*) and the squeezing theorem gives that

$$\lim_{m \rightarrow +\infty} \int_0^{\pi} \varphi(t) \cdot \sin\left(\left(m+\frac{1}{2}\right)t\right) dt = 0$$

□

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5) (This question was the hardest)

Let $n \in \mathbb{N}^*$.

$$\sum_{k=1}^n \frac{1}{k^2} = \sum_{k=1}^n \left(\int_0^\pi h(t) \cos(kt) dt \right)$$

Thanks to the question (4.B.2)

$$= \int_0^\pi h(t) \left(\sum_{k=1}^n \cos(kt) \right) dt$$

By linearity of the integral (it is just a finite sum here).

New thanks to (5) we can write $\sum_{k=1}^n \cos(kt)$ as

$$= \int_0^\pi h(t) \left(\frac{\sin((n+1/2)t)}{2 \sin(t/2)} - \frac{1}{2} \right) dt$$

So by linearity of the integral (again)

$$= \left(\int_0^\pi \frac{h(t)}{2 \sin(t/2)} \sin((n+1/2)t) dt \right) - \left(\frac{1}{2} \int_0^\pi h(t) dt \right)$$

But $\varphi_0(t) = \frac{h(t)}{2 \sin(t/2)}$ by definition

And thanks to (4.B.1), φ_0 is of class \mathcal{C}^1 on $[0, \pi]$

So thanks to (6) applied to $\varphi = \varphi_0$, we have

$$\int_0^\pi \frac{h(t)}{2 \sin(t/2)} \sin((n+1/2)t) dt \xrightarrow{n \rightarrow +\infty} 0$$

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$$5.) \text{ So } \sum_{k=1}^m \frac{1}{k^2} = \left(\int_0^{\pi} \frac{h(t)}{2 \sin(t/2)} \sin\left(\left(m + \frac{1}{2}\right)t\right) dt \right) - \frac{1}{2} \int_0^{\pi} h(t) dt$$

$$\xrightarrow{m \rightarrow \infty} \sum_{k=1}^{+\infty} \frac{1}{k^2} = z(2) = -\frac{1}{2} \int_0^{\pi} h(t) dt$$

Hence, that series converges, and we can compute

$$z(2) = -\frac{1}{2} \int_0^{\pi} h(t) dt$$

$$= -\frac{1}{2} \int_0^{\pi} \frac{t^2}{2\pi} dt + \frac{1}{2} \int_0^{\pi} t dt$$

$$= -\frac{1}{2} \left[\frac{t^3}{6\pi} \right]_0^{\pi} + \frac{1}{2} \left[\frac{t^2}{2} \right]_0^{\pi}$$

$$= -\frac{\pi^3}{12\pi} + \frac{\pi^2}{4} = \pi^2 \times \left(\frac{1}{4} - \frac{1}{12} \right)$$

$$= \pi^2 \times \frac{1}{4} \left(1 - \frac{1}{3} \right) = \pi^2 \times \frac{1}{4} \times \frac{2}{3} = \pi^2 \times \frac{1}{2} \times \frac{1}{3}$$

$$= \pi^2 / 6$$

Conclusion. We have, as tried, $z(2) = \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

$$6.) \pi \approx 3.1 \quad \text{so} \quad \pi^2 \approx (3.1)^2 = 9 + 0.01 + 2 \times 3 \times 0.1 = 9.61$$

$$\text{so } \frac{\pi^2}{6} \approx \frac{9.61}{6} \approx \frac{3.2}{2} = 1.6$$

$$\text{So } z(2) = \frac{\pi^2}{6} \approx 1.6$$

is > 1
is < 2
is not > 2
is < 5 .

7) Irrational

(but the proof is harder).

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(11)