

IVP $y' + \frac{2}{x}y = \frac{\cos x}{x^2}$, $y(\pi) = 0$

Standard form of First order linear ODE is

$$\frac{dy}{dx} + P(x)y = Q(x), y(x_0) = y_0$$

① clearly $P(x) = \frac{2}{x}$; $Q(x) = \frac{\cos x}{x^2}$ are continuous for all $x \in (-\infty, 0) \cup (0, \infty)$.

Given that $x_0 = \pi$. ($\pi > 0$)



\therefore Consider IVP in $(0, \infty)$.

So, the IVP has unique solution in $(0, \infty)$.

② consider $\frac{dy}{dx} + \frac{2}{x}y = 0$, $x \in (0, \infty)$

$$y' + \frac{2}{x}y = 0, \underline{y > 0}$$

$$\frac{y'}{y} = -\frac{2}{x}$$

on integrating from x_0 to x

$$\int_{x_0}^x \left(\frac{y'}{y}\right) dt = -\int_{x_0}^x \frac{2}{t} dt$$

$$\log y(t) \Big|_{x_0}^x = -2 \log t \Big|_{x_0}^x$$

$$\log y(x) - \log y(x_0) = -2(\log(x) - \log(x_0))$$

$$\Rightarrow \log y(x) = -2 \log x + \log c$$

where

$$\log c = 2 \log x_0 + \log y(x_0)$$

$$\log y(x) = \log(c/x^2)$$

$$y(x) = \frac{c}{x^2}, x \in (0, \infty), c \in \mathbb{R}(0, \infty)$$

Verification:

clearly $y(x) = \frac{c}{x^2}$ is differentiable on $(0, \infty)$.

$$y'(x) = \frac{-2c}{x^3}, c \in \mathbb{R}(0, \infty)$$

$$\text{So, } y' + \frac{2}{x}y = \frac{-2c}{x^3} + \frac{2}{x} \times \frac{c}{x^2} = 0$$

$\therefore y(x) = \frac{c}{x^2}$ is a solution of $y' + \frac{2}{x}y = 0$.

③ Consider $\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}$, $x \in (0, \infty)$

Let $y_p = \phi = \frac{c(x)}{x^2}$, where $c(x)$ is differentiable on $(0, \infty)$

$$\phi' = c'(x) \times x^{-2} + c(x) \times -2x^{-3} \quad ; \underline{y_p \text{ is a particular solution}}$$

$$\text{So, } \phi' + \frac{2}{x}\phi = \frac{\cos x}{x^2}$$

$$\Rightarrow c'(x) \times x^{-2} - \frac{2}{x^3} c(x) + \frac{2}{x} \times \frac{c(x)}{x^2} = \frac{\cos x}{x^2}$$

$$\rightarrow \frac{c'(x)}{x^2} = \frac{\cos x}{x^2}, \text{ since } x \neq 0.$$

$$\rightarrow c'(x) = \cos x \Rightarrow c(x) = \sin x + D, \quad \underline{D \in \mathbb{R}}.$$

\therefore Particular Solution is

$$y_p(x) = \phi(x) = \frac{1}{x^2} c(x) = \frac{1}{x^2} \left[\sin x + \frac{D}{1} \right]$$

$$= \frac{D}{x^2} + \frac{\sin x}{x^2}.$$

\therefore General solution of the non-homogeneous equation

is $y = y_c + y_p$

$$= \frac{C}{x^2} + \frac{D}{x^2} + \frac{\sin x}{x^2}$$

$$= \frac{C_1}{x^2} + \frac{\sin x}{x^2}, \text{ where } C_1 = (C + D) \in \mathbb{R}.$$

Given $y(\pi) = 0$.

$$\text{So, } 0 = \frac{C_1}{\pi^2} + \frac{\sin \pi}{\pi^2} \Rightarrow 0 = \frac{C_1}{\pi^2} + 0 \Rightarrow \boxed{C_1 = 0}$$

$$y = \frac{\sin x}{x^2}$$

④ Verification: we have $y = \frac{C_1}{x^2} + \frac{\sin x}{x^2}, x \in (0, \infty)$

$$y' = -\frac{2C_1}{x^3} + \frac{2C_1}{x^3} + \frac{\cos x}{x^2} - \frac{2 \sin x}{x^3}$$

Consider L.H.S

$$y' + \frac{2}{x} y = -\frac{2C_1}{x^3} + \frac{2C_1}{x^3} + \frac{\cos x}{x^2} - \frac{2 \sin x}{x^3} + \frac{2}{x} \frac{\sin x}{x^2}$$

$$= \frac{\cos x}{x^2} = \text{R.H.S.}$$

$\therefore y = \frac{C_1}{x^2} + \frac{\sin x}{x^2}$ is a general solution of

$$y' + \frac{2}{x} y = \frac{\cos x}{x^2}, x \in (0, \infty).$$