Are there infinite irrigation trees ?

M. Bernot [*], **V. Caselles** [†], **J.M. Morel** [‡]

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Abstract

In many natural or artificial flow systems, a fluid flow network succeeds in irrigating every point of a volume from a source. Examples are the blood vessels, the bronchial tree and many irrigation and draining systems. Such systems have raised recently a lot of interest and some attempts have been made to formalize their description, as a finite tree of tubes, and their scaling laws [25], [26]. In contrast, several mathematical models [5], [22], [10], propose an idealization of these irrigation trees, where a countable set of tubes irrigates any point of a volume with positive Lebesgue measure. There is no geometric obstruction to this infinitesimal model and general existence and structure theorems have been proved. As we show, there may instead be an energetic obstruction. Under Poiseuille law $R(s) = s^{-2}$ for the resistance of tubes with section s, the dissipated power of a volume irrigating tree cannot be finite. In other terms, infinite irrigation trees seem to be impossible from the fluid mechanics viewpoint. This also implies that the usual principle analysis performed for the biological models needs not to impose a minimal size for the tubes of an irrigating tree ; the existence of the minimal size can be proven from the only two obvious conditions for such irrigation trees, namely the Kirchhoff and Poiseuille laws.

1 Introduction

1.1 Irrigation networks made of tubes

The function of many natural or artificial irrigation or drainage systems is to connect by a fluid flow a finite size volume to a source. This happens, e.g., with the lungs [16] or with the blood circulation. A space filling hierarchical branching pattern is obviously required and observed. The resulting irrigation circuitry is a tree of tubes branching from a source and going as close as possible to any point of the irrigated volume. The following principles have been proposed to characterize such irrigation patterns :

(SF) Space filling requirement : The network supplies uniformly an entire volume of the organism.

(K) Kirchhoff law at branching (conservation of fluid mass).

(W) Energy minimization : the biological networks have evolved to minimize energy dissipation.

(MSU) Minimal size unit : the size of the final branches of the network is a size-invariant unit.

These principles are considered basic principles in all presentations of irrigation circuits [25], [26], [27], [4]. In the case of trees and plants, the energy criterion must be related to the mechanical stability of the trunk and branches in response to wind and gravity. In the case of irrigation or drainage networks, the energy criterion aims at a reduction of the overall resistance of the system, or, equivalently, to a minimization of the dissipated power.

[*] CMLA, ENS Cachan, 61, Av. du Président Wilson 94235 Cachan Cedex France, bernot@cmla.ens-cachan.fr
[†] Departament de Tecnologia, Universitat Pompeu-Fabra, vicent.caselles@upf.edu
[‡] CMLA, ENS Cachan, 61, Av. du Président Wilson 94235 Cachan Cedex France, morel@cmla.ens-cachan.fr

In the mentioned papers, several additional assumptions are usually made to derive conclusions from this set of principles, namely that

(H) Homogeneous tree : The irrigation system is assumed to be a tree made of tubes, fully homogeneous in lengths and sections.

Let us describe in some detail this homogeneous framework and its consequences. We denote by $k \in I\!N$, $k \le k_{max} (\le \infty)$ the branching level in the tree. The tubes at the final level k_{max} will be called the capillaries. There is a single tube at level 0, and N_k tubes at level k. By (H), at each level k all tubes (which we shall refer to as k-tubes) are equal and are described by the same parameters: l_k, r_k, f_k, namely the common value of their length, radius and flow. We shall also use the variable $s_k = r_k^2$ which is proportional to the area of the constant section of the tube. With these variables, the power dissipated by the irrigation network is expressed as

$$W = \sum_{k=1}^{k_{max}} N_k l_k s_k^{-\beta} f_k^2. \tag{1.1}$$

Although we treat β as a free parameter, Poiseuille law states that for all newtonian fluids in laminar mode, $\beta = 2$. The homogeneity of the irrigation tree can be rendered still more specific by imposing the realistic

(CB) Constant branching : $\frac{N_{k+1}}{N_k} = \nu = constant.$

The space filling requirement can be formalized in a rough way by stating that the k-th tube irrigates a volume proportional to l_k^3. This is a possible interpretation of (SF) which we shall call $(SF1)$. So we can summarize as a set of equations the constraints usually proposed for homogeneous trees

(H) Homogeneous tree with unknown $k,\ l_k,\ r_k,\ f_k,\ k \le k_{max}$.

(K) Kirchhoff $N_k f_k = constant.$

(SF1) Space filling $N_k l_k^3 = constant.$

(MSU) Minimal size capillaries : $k_{max} < \infty$.

(CB) Constant branching (optional), $\frac{N_{k+1}}{N_k} = \nu = constant.$

The aim of this set of assumptions was in [25], [26], [27] to prove that the network has a fractal-like structure with self-similar properties. In the mentioned papers, it is claimed that the minimization of the energy (1.1) with prescribed volume $\sum_k N_k l_k r_k^2 = V$ leads to self-similar properties, namely constant ratios $\frac{l_{k+1}}{l_k} = constant$, $\frac{r_{k+1}}{r_k} = constant$, so that also $\frac{l_k}{r_k} = constant$, namely the tubes have a scale invariant shape and all quantities scale as powers of n. Actually, such results were not proven, the main focus of the mentioned papers being rather to discuss scaling laws in animal metabolism. A mathematically more comprehensive study of the consequences of the above mentioned axioms is given in [7] where the correct consequences are drawn. We shall recall these results and extend their techniques in section 2.

The above axiomatic of irrigation systems is simple and efficient enough, but has some weak points. There is no mention of the tube non-intersection constraint. From that point of view, the homogeneity assumption is probably not quite realistic. Also, the space filling assumption does not take into account the volume occupied by the network itself. In short, the realistic embedding of the circuit in a volume is not directly considered and the Lagrangian calculus involved in the mentioned papers is done as though all lengths, radii and even branching numbers could move freely. This is certainly not the case for a realistic embedded circuit. Thus, it would be good to get rid of the homogeneity assumption (H)

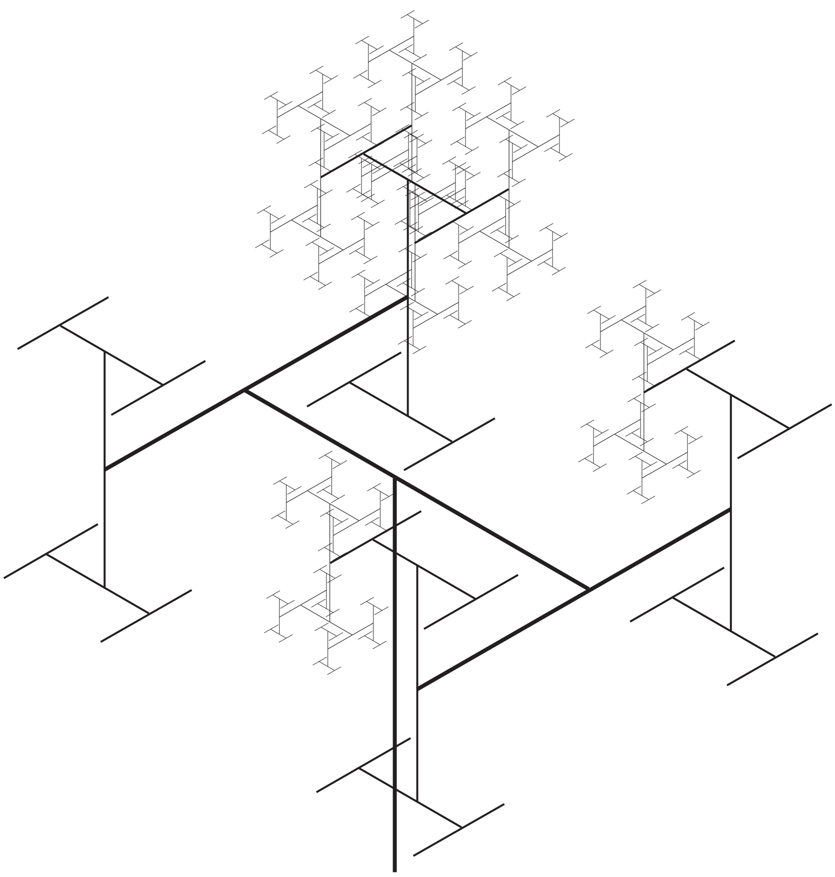

Figure 1: An irrigating tree

which clearly should be derived as a property from the first four principles. Also, the question arises of whether the four basic principles (SF), (K), (W) and (MSU) are redundant or not. One of the main outcomes of our discussion here will be to eliminate (MSU), that is, the minimal size constraint for the capillaries. The (MSU) assumption, essential in the above mentioned physical models, was simply written as $k_{max} < \infty$ and forbids infinite branching. It also actually excludes a volume direct irrigation and only permits any point of a volume to be "close enough" to a capillary.

There is, however, no geometric obstruction for the existence of infinite trees irrigating a positive volume K in a strong sense, namely with a branch of the tree (a sequence of tubes) arriving at every point of K. Such tube trees can be constructed by rather explicit rules ; they can satisfy the Kirchhoff law and can even have the fluid speed decrease and be null at the tips of capillaries. Such constructions can be found (e.g.) in [4], [14] and [5]. See Figure 1 for an example.

We shall prove that the only obstruction to infinite trees is the infinite resistance of such circuits. We assume without loss of generality that Poiseuille law holds throughout the circuit : it is generally acknowledged that this law is valid in all biological circuits, at least for the smaller tubes [12]. We shall prove :

Theorem 1 *Let $\beta \geq 2$. Then $W = +\infty$ for any set of tubes obeying (K) and (W) and irrigating a positive volume.*

(See Theorem 3 for a more precise statement.)

This result may invalidate the infinitesimal models, admitting infinite branching, proposed in several recent mathematical works [5], [22], [10]. Now, as we shall see, the tools developed in the mentioned paper turn out to be quite handy to perform the present axiomatic discussion. And, of course, nothing hinders the consideration of other resistance laws than Poiseuille law for other human built transportation circuits. Poiseuille law states that for fluids, the resistance $R(s) = Cs^{-2}$ of a tube with section s scales as the inverse second power of s. If we instead consider $R(s) = Cs^{-\beta}$, then infinitesimal circuits are possible. The power $\beta = 2$ is the limit exponent.

Two of the mentioned mathematical models, [22] and [10], do not involve the radius of the tubes. They instead express a "cost" of the flow directly as

$$\tilde{W} = \sum_{i \in I} l_i f_i^\alpha$$

where $0 < \alpha < 1$ and I denotes the countable set of all tubes. There is, however, a way to relate this expression of the cost to the energy W, at least for optimal and homogeneous circuits.

Proposition 1 *Let us consider an irrigation network which optimizes the dissipated power W given by (1.1) under the constraints of fixed volume V and prescribed lengths of tubes l_i and flows f_i. Then $s_i = C_1 f_i^{2/(\beta+1)}$, and $W = C_2 \sum l_i f_i^{2/(\beta+1)}$ for some constants $C_1, C_2 > 0$. (Poiseuille law corresponds to the case $\beta = 2$ in dimension 3, and in this case $s_i = C f_i^{2/3}$ for some constant $C > 0$).*

The proof of this proposition is easy and can be done along the lines of the proof of Proposition 3 in Section 2. The model equivalence thus obtained is not quite satisfactory : we are not a priori allowed to move freely the radii in an optimal embedded circuit, since we do not take into account the fact that the tubes should not intersect. Let us concede anyway some validity to the model equivalence thus indicated. Then we see that there is no contradiction with the existence results in [22] and [10]. Indeed, these authors assume (in dimension 3) $\alpha > \frac{2}{3}$ which corresponds to $\beta < 2$, and we prove that $\beta \geq 2$ is not compatible with Poiseuille law.

1.2 Mathematical, infinitesimal approaches

Let us give some details on the existing mathematical formalizations of the problem, since we shall use some of them. The model proposed in [5] is directly compatible, but more general than the above tube model. It directly considers the problem of finding a maximal irrigated volume with minimal cost. Let D be an open domain of $\mathbb{R}^d$ (of course, d=2 or 3). A point source $S \in D$ is fixed. Say that a compact set $E \subset D$ is *irrigable* if the complementary open set $U = D \setminus E$ is connected and contains S. U is called the irrigation network and is nothing but an open set at this point. The authors then fix an "accessibility profile", namely a function $f(s) : \mathbb{R}_+ \to \mathbb{R}_+$, increasing and such that $f(0) = 0$. A point $x \in E$ is said f-irrigable if there is a path $x(s)$ such that $x(0) = x$, $x(L) = S$, and for every $s \in [0, L]$, $B(x(s), f(s)) \subset U$, where $B(x, r)$ denotes the ball with center x and radius r. Such paths exist in the physical tube model as a branch of the irrigation tree. In other terms, there is a thick path inside U leading to x. This path becomes thinner when approaching the irrigated point x, but with a thinning rate uniformly bounded from below. The authors show first that if f slightly super-linear at 0 (e.g. $f(s) = s^\alpha$, $0 < \alpha < 1$) then the problem of irrigating a maximal positive volume is well posed. Namely : there exists K with maximal volume among all f-irrigable sets.

From this paper, we shall retain the following result which will be a main ingredient here.

Proposition 2 *Let $x \in E$ be irrigable from S with profile f. Assume that x is a Lebesgue point of E. Then $\limsup_{s \to 0+} f(s)/s = 0$. As a consequence $\int_0^R \frac{1}{f(s)}\, dr = \infty$ for all $R > 0$.*

Almost every point of a measurable set is a Lebesgue point, and this yields a generic constraint on accessibility profiles, not taken into account in finite models, but handy in infinitesimal ones. In the terminology of homogeneous trees of tubes, this constraint yields under (H)

$$\sum_k \frac{l_k}{r_k} = +\infty. \tag{1.2}$$

We shall not use Qinglan Xia's formalism [22], but it is particularly elegant and efficient. This author starts with finite atomic measures a and b and defines a "path" from a to b as a flow on a finite embedded graph whose end vertices end up on a or b. He denotes by e the (straight) edges of the graph, by $w(e)$ the flow inside, and by $\vec{e}$ a unit vector oriented by e. He calls $[e] = \mathcal{H}^1_{|e}\vec{e}$ the vectorial measure obtained as the product of the Hausdorff measure restricted to e and of the vector $\vec{e}$. Then the embedded path from a to b can be written as vectorial measure

$$G = \sum_e w(e)[[e]].$$

The Kirchhoff law (K) is simply expressed as

$$div(G) = a - b.$$

Calling $\mathcal{G}_\Lambda$ the set of all such paths between atomic measures a and b in a compact and convex subset of $\mathbb{R}^n$, the author defines on $\mathcal{G}_\Lambda$ the cost functional

$$M^\alpha(G) = \sum_e w(e)^\alpha \text{length}(e).$$

Then, the author proceeds to define transport paths connecting two Radon measures μ^+ and μ^- with equal total mass. He says that a vector measure T is a transport path between μ^+ and μ^- if there are sequences of atomic measures a_i and b_i and paths G_i connecting a_i to b_i such that a_i and b_i converge weakly to μ^+ and μ^- and $G_i \to T$ weakly in the sense of vector measures. This implies $div(T) = \mu^+ - \mu^-$ in the distribution sense. The energy of any such path is defined by relaxation as

$$M^\alpha(T) := \inf \liminf_{i\to\infty} M^\alpha(G_i),$$

where the infimum is taken over the set of all possible approximating graph sequences $a_i,\ b_i,\ G_i$ of T. Qinglan Xia's existence theorem states the existence of an optimal transport path between any two Radon measures μ^+ and μ^- with same mass, provided $1 - \frac{1}{d} < \alpha \leq 1$. Further, it is to be mentioned that Qinglan Xia proved regularity results about the structure of an optimal transport path (see [23] and [24]).

Equivalence, or rather complementarity, can be established between the results established in [22] and [10]. The proof of this equivalence is sketched in [10] and actually follows from the tools developed in [22] and [10]. Let us now describe the formalism of [10], which we shall use for our aims here. This paper describes an irrigation system as a (usually uncountable) set of paths or "fibers" starting from a point source S and arriving at every point of the support of the irrigated measure. The fibers represent either the trajectory in $\mathbb{R}^d$ of a fluid particle, or a fiber of a tree. Each fiber is parameterized as $\chi(\omega, l) \in \mathbb{R}^d$, where l is time (or length along the fiber) and ω denotes a particle, belonging to an abstract probability space Ω. A stopping time (or length) $\sigma_\chi(\omega)$ is associated with each fiber. This permits to define the irrigation measure as a density measure of the fibers stopping in any given volume : set $T(\omega) = \chi(\omega, \sigma_\chi(\omega))$. Then the amount of fluid irrigating a Borel set A is the measure of $T^{-1}(A)$ in Ω.

The authors define χ-vessels, or branches, as equivalence classes by the equivalence relation $\omega \simeq \omega'$ if $\chi(\omega, s)$ and $\chi(\omega', s)$ coincide up to time l. The branches at time l simply correspond to the tubes at geodesic distance l from the origin. We shall use this formalism to derive expressions of energy and volume as integrals over the set of fibers Ω.

Our plan is as follows : Section 2 is devoted to the classical physical tube models, the derivation of scaling laws and the proof of our result in the homogeneous case (hypothesis (H)). Section 3 gives all elements we need from [10] to perform integration on the set of fibers. Section 4 constructs from any embedded set of tubes a set of fibers $\chi(\omega, l)$. Section 5 proves the main result. Three small appendices are devoted, for a sake of completeness, to a proof of Proposition 2, the optimality of circular section for tubes and the derivation of Poiseuille law in tubes.

2 Dissipated power in a homogeneous network of tubes

In this section, we take the standard notation given in the introduction. We consider an homogeneous irrigation network as a set of tubes which are organized as a hierarchical branching system from level 0 up to a final level $k_{max}(\leq \infty)$. There is a single tube at level 0, and N_k tubes at level k. At each level k, all tubes (which we shall refer to as k-tubes) are equal and are described by their length l_k, radius r_k, and flow f_k We set $s_k = r_k^2$ which is proportional to the area of the constant section of the tube. With these variables, the power dissipated by the irrigation network is expressed as

$$W = \sum_{k=1}^{k_{max}} N_k l_k s_k^{-\beta} f_k^2 \tag{2.1}$$

for some $\beta > 0$ (Poiseuille law corresponds to $\beta = 2$). As proposed in [25], [26], [27], if we prescribe the volume occupied by the irrigation network, physical networks are designed to minimize the dissipated power W, and satisfy the following assumptions:

(K) Kirchhoff's law: the fluid is conserved as it flows through the system, that is, $N_k f_k = N_{k+1} f_{k+1}$ for each k. In other words, Kirchhoff's law holds in the network.

$(SF1)$ Space filling requirement: at each level k the volume supplied by the set of k-tubes is independent of k and is approximately given by the sum of the volumes of N_k spheres of diameter $l_k/2$. This total volume is $N_k l_k^3$ and we assume that this quantity is a constant.

For an homogeneous irrigation network satisfying (K) and $(SF1)$, there are constants $C, C' > 0$ such that

$$f_k = \frac{C'}{N_k}, \quad l_k = C N_k^{-1/3}$$

and the dissipated power may be written as

$$W(s_k) = C'^2 C \sum_{k=1}^{k_{max}} N_k^{-4/3} s_k^{-\beta}.$$

In the same way the volume $V = \sum_{k=1}^{k_{max}} N_k l_k s_k$ can be written as

$$V(s_k) = C \sum_{k=1}^{k_{max}} N_k^{2/3} s_k.$$

We shall consider the geometry of the network as given, i.e. the values N_k are prescribed, hence the dissipated power is only a function of the variables s_k. Under the constraint of given volume, we consider an optimal irrigation network as a minimizer of the dissipated power W.

Proposition 3 *Assume that $\beta \geq 2$. Under the assumptions (K) and $(SF1)$, an optimal homogeneous irrigation network with prescribed volume satisfies $k_{max} < \infty$ and $N_k r_k^{\beta+1} = constant$.*

We observe that if $k_{max} < \infty$, then the relation $N_k r_k^{\beta+1} = constant$ does not require to assume that $\beta \geq 2$.

In particular, accepting that assumption (K) is a sound one, this proposition proves that the assumption $(SF1)$ cannot be fulfilled if we want to consider infinite trees. If we accept it, we have to assume that capillaries cannot be infinitely thin.

Proof. Assume first that $k_{max} < \infty$. For simplicity, let us assume that $C'^2C = 1$. Then, by Lagrange multiplier's Theorem, there is a constant $\lambda \in I\!R$ such that $\frac{\partial W}{\partial s_k} = \lambda \frac{\partial V}{\partial s_k}$, that is,

$$-\beta N_k^{-4/3} s_k^{-(\beta+1)} = \lambda C N_k^{2/3}.$$

Hence $N_k^2 s_k^{\beta+1} = -\frac{\beta}{\lambda C}$, and therefore $N_k r_k^{\beta+1} = constant$.

Assume that $k_{max} = \infty$, and there exists an homogeneous irrigation network with specified volume $V = V_0 < \infty$ and finite dissipated power. Then the dissipated power has a minimum in the set $\mathcal{S} = \{(s_k)_{k=1}^\infty : s_k > 0,\ C\sum_{k=1}^\infty N_k^{2/3} s_k = V_0\}$. Indeed, since the infimum of W in $\mathcal{S}$ is finite, let $\{(s_k(n))_k\}_n$ be a minimizing sequence of elements in $\mathcal{S}$. By extracting a subsequence, if necessary, we may assume that $s_k(n) \to s_k$ as $n \to \infty$ for all k. If $s_k = 0$ for some k, then $W(s_k(n)) \geq N_k^{-4/3}(s_k(n))^{-\beta} \to \infty$ as $n \to \infty$, a contradiction with the fact that $(s_k(n))_k$ is a minimizing sequence. Hence $s_k > 0$ for all k. Now, for each $p \geq 1$, we have

$$C\sum_{k=1}^{p} N_k^{2/3} s_k \leq \lim_n C\sum_{k=1}^{p} N_k^{2/3} s_k(n) \leq \lim_n C\sum_{k=1}^{\infty} N_k^{2/3} s_k(n) = V_0.$$

Thus $M := C\sum_{k=1}^\infty N_k^{2/3} s_k \leq V_0$. If $M < V_0$, we define $S_k = \frac{V_0}{M} s_k$ and we have $V(S_k) = V_0$. Now,

$$\begin{aligned}\sum_{k=1}^{p} N_k^{-4/3} S_k^{-\beta} &= (\frac{M}{V_0})^\beta \sum_{k=1}^{p} N_k^{-4/3} s_k^{-\beta} = (\frac{M}{V_0})^\beta \lim_n \sum_{k=1}^{p} N_k^{-4/3}(s_k(n))^{-\beta} \\ &\leq (\frac{M}{V_0})^\beta \lim_n \sum_{k=1}^{\infty} N_k^{-4/3}(s_k(n))^{-\beta} = (\frac{M}{V_0})^\beta \inf_{\mathcal{S}} W.\end{aligned}$$

In particular, we deduce that $\inf_{\mathcal{S}} W > 0$, and

$$W(S_k) \leq (\frac{M}{V_0})^\beta \inf_{\mathcal{S}} W < \inf_{\mathcal{S}} W.$$

This contradiction proves that $M = V_0$, hence $(s_k) \in \mathcal{S}$, and $(s_k)_k$ is a minimum of W in $\mathcal{S}$. Let us denote $\vec{s} = (s_k)_k$, and for each $p \geq 1$, $-s_p N_p^{2/3} < \epsilon < s_{p+1} N_{p+1}^{2/3}$, $\vec{s}_\epsilon^{\,p} = (s_1, \ldots, s_p + \frac{\epsilon}{N_p^{2/3}}, s_{p+1} - \frac{\epsilon}{N_{p+1}^{2/3}}, s_{p+2}, \ldots)$. Then computing

$$\lim_{\epsilon \to 0+} \frac{W(\vec{s}_\epsilon) - W(\vec{s})}{\epsilon} = 0,$$

we obtain that

$$N_p^2 s_p^{\beta+1} = N_{p+1}^2 s_{p+1}^{\beta+1}.$$

Since this holds for all p, we obtain that $N_k^2 s_k^{\beta+1} = constant$, hence also $N_k^{2/3} s_k^{(\beta+1)/3} = constant$. Using that $(\beta+1)/3 \geq 1$, we have

$$N_k^{2/3} s_k^{(\beta+1)/3} \leq N_k^{2/3} s_k,$$

when $s_k < 1$. We conclude that $V(s_k) = \infty$. Notice that we also have $W(s_k) = C\sum_{k=1}^\infty s_k^{(2-\beta)/3} = \infty$ since $\beta \geq 2$ and $s_k \to 0$ as $k \to \infty$. $\square$

Remark 1 Observe that no relation of the type $N_{k+1}/N_k = constant$ follows for optimal irrigation trees under the assumption (K) and $(SF1)$ as suggested in [25], [26], [27]. This fact has also been observed in [7]. Now, if we add the assumption of constant branching

(CB) $\frac{N_{k+1}}{N_k} = \nu$,

we obtain the relations (written modulo multiplicative constants) $N_k = \nu^k$, $s_k = \nu^{-2k/(\beta+1)}$, $W = \sum_k \nu^{(\frac{2\beta}{\beta+1} - \frac{4}{3})k}$, $V = \sum_k \nu^{(\frac{2}{3} - \frac{2}{\beta+1})k}$ and both quantities are infinite if $k_{max} = \infty$, and $\beta \geq 2$.

We shall replace the space filling assumption $(SF1)$ by a different assumption which is related to the existence of a positive volume irrigated by the network. Indeed, we assume

$(SF2)$ $k_{max} = \infty$ and $\sum_{k=1}^{\infty} \frac{l_k}{r_k} = \infty$.

This implies that the length of the tubes cannot be too small compared to its radius. Our analysis in Subsection B will prove that this assumption holds for networks irrigating a positive volume.

Proposition 4 *Assume that $\beta \geq 2$. Under the assumptions (K) and $(SF2)$, both V and W cannot be finite.*

Proof. Recall that $V = \sum_{k=1}^{\infty} N_k l_k s_k$. Since $N_k f_k = C$ for some constant $C > 0$, we may write $W = C^2 \sum_{k=1}^{\infty} \frac{l_k s_k^{-\beta}}{N_k}$, using Cauchy-Schwarz inequality, we have

$$\sqrt{W}\sqrt{V} \geq C \sum_{k=1}^{\infty} \sqrt{l_k^2 s_k^{1-\beta}} = C \sum_{k=1}^{\infty} \frac{l_k}{r_k^{\beta-1}} = \infty,$$

and the conclusion follows. □

Remark 2 The proof of Proposition 4 can also be done using Lagrange multiplier's theorem as we did in the proof of Proposition 3.

3 A model of abstract tree

The purpose of this section is to recall the formalization defining "set of fibers" in the sense of [10]. In Section 4, we shall make the link with irrigation trees. To do this, we shall describe the tree made of tubes as a tree of segments, each segment being the medial axis of a tube. We shall also keep the flow information inside each tube. These informations are enough, as we shall see, to associate with the concrete tree an abstract "set of fibers". The reason for making this association will become clear in Section 5 : we wish to compute the volume or the dissipated power by integrating along the sections of the irrigation tree. These computations are facilitated by the "set of fibers" formalism.

Let us recall the main concepts introduced in [10]. Let $(\Omega, |.|)$ be a probability space which we interpret as the reference configuration of a fluid incompressible material body. We can also interpret it as the trunk section of a tree, this trunk being thought of as a set of fibers (which can bifurcate into branches). A *set of fibers of Ω with source point $S \in I\!R^d$* is a mapping

$$\chi : \Omega \times I\!R_+ \to I\!R^d$$

such that :

C1) For a.e. *material point* $p \in \Omega$, $\chi_p(l) : t \mapsto \chi(p, l)$ is Lipschitz continuous with Lipschitz constant less than or equal to one.

C2) For a.e. $p \in \Omega$: $\chi_p(0) = S$.

We shall consider the source point $S \in I\!R^d$ as given and we will denote by $\mathbf{C}_S(\Omega)$ the set of all the set of fibers of Ω.

Definition 1 *[10] Given $l \in I\!R_+$, we shall say that two points $p,\ q \in \Omega$ belong to the same χ-vessel of value l and we will write $p \simeq_l q$ if*

$$\chi_p(s) = \chi_q(s) \text{ for all } s \in [0,l].$$

For every $l \in I\!R_+$, the equivalence relation $\simeq_l$ induces a decomposition of Ω into equivalence classes X. We will call *χ-vessels* such classes.

Definition 2 *[10] Given $p \in \Omega$ and $l \in I\!R_+$, the equivalence class of $\simeq_l$ which contains p and which will be denoted by $[p]_t$ will be named χ-vessel of the point p at l.*

Given $\chi \in \mathbf{C}_S(\Omega)$ and $l > 0$, we shall denote by $\Omega_l(\chi)$ the set of all the χ-vessels at the value l, that is

$$\Omega_t(\chi) := \Omega/\simeq_l.$$

The decomposition of Ω induced by $\simeq_l$ can be viewed as dividing the body in parts which are mapped, through χ, into tube-like regions of $I\!R^d$ which we identify with rectifiable curves. Since we control only the total amount of fluid carried by these regions, we describe them by giving their axial curves. Thus, at each l a set of fibers χ can be regarded as a set of curves, obtained by varying $[p]_l$. Indeed, by Definition 1, on the interval $[0,l]$, χ_p coincides with any other function χ_q for q varying in the set $[p]_l$. A set of fibers can also be interpreted as modelling a tree, in which case the χ-vessels represent the branches.

Definition 3 *[10] Let $\chi \in \mathbf{C}_S(\Omega)$ be given. The function $\sigma_\chi : \Omega \to I\!R_+$ defined as follows*

$$\sigma_\chi(p) := \inf\{l \in I\!R_+ \mid \chi_p(s) \text{ is constant on } [l, +\infty[\}$$

will be called absorption time. We shall say that a point $p \in \Omega$ is absorbed when $\sigma_\chi(p) < +\infty$. A point $p \in \Omega$ is absorbed at the time l if $\sigma_\chi(p) \le l$. We denote $A_l(\chi)$ the set of absorbed points at time l, and A_χ the set of absorbed points at some time.

Lemma 1 *[10] Let $f : \Omega \times I\!R_+ \to I\!R$ be such that $f(\cdot, l)$ is measurable for l in a dense subset $D \subset I\!R_+$ and $f(p, \cdot)$ is continuous for a.e. $p \in \Omega$. Then f is a measurable mapping.*

Theorem 2 *[10] For every set of fiber $\chi \in \mathbf{C}_S(\Omega)$ the following statements are equivalent.*

1. *χ is measurable.*
2. *$\chi(\cdot, l)$ is measurable for every l in a dense subset $D \subset I\!R_+$.*
3. *$\chi(\cdot, l)$ is measurable for every $l \in I\!R_+$.*

In the following we only consider measurable sets of fibers.

Proposition 5 *[10] For every $\chi \in \mathbf{P}_S(\Omega)$, the absorption function σ_χ is a measurable mapping.*

Let $\chi \in \mathbf{P}_S(\Omega)$. We introduce the *irrigation function* defined on the set A_χ of the absorbed points :

$$i_\chi(p) = \chi(p, \sigma_\chi(p))$$

We have $i_\chi(p) = \lim_{t\to\infty} \chi(p,l)$ and so $i_\chi : A_\chi \to I\!R^d$ is a measurable function, as a pointwise limit of a sequence of measurable functions. The function i_χ induces the image (push-forward) measure μ_χ defined by the formula

$$\mu_\chi(A) := |i_\chi^{-1}(A)|$$

for any Borel set $A \subset I\!R^d$. We shall refer to μ_χ as to the *irrigation measure* induced by χ.

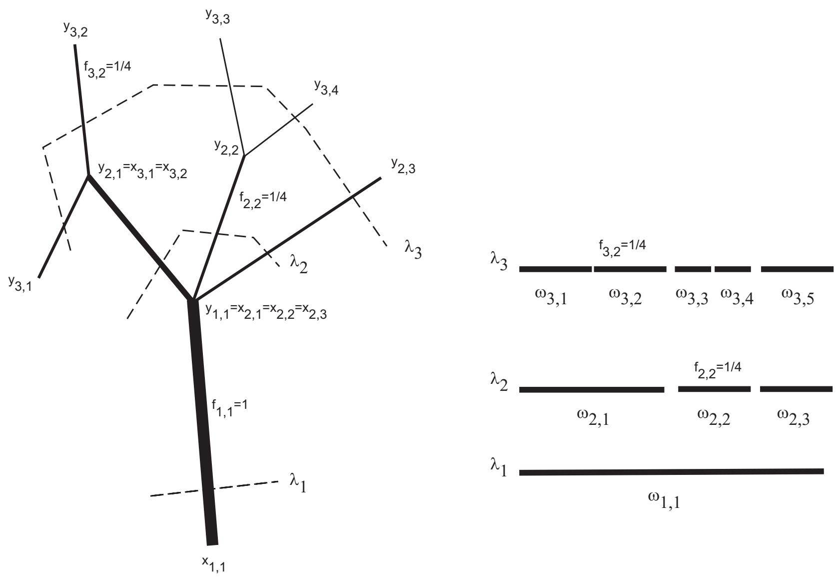

Figure 2: Skeleton of a tree of tubes

4 The set of fibers associated to the skeleton of a tree of tubes

Our purpose in this section is to obtain an abstract description of a physically realized tree. We first introduce the skeleton of a tree of tubes which is a depurate description of an embedded tree (the tree being viewed as a set of tubes). The skeleton description of a tree permits to associate a set of fibers to it, in the sense of [10] as described in the section 3. Integration of functions which are constants on any tube is then allowed and made easier with this formalism.

Since notation here is necessarily a bit cumbersome, we refer to Figure 2.

4.1 Embedded irrigation tree through its skeleton

Definition 4 *Let $\{[x_n^k, y_n^k] \mid n \geq 1, k \in [1, N(n)]\}$ be a family of segments in $\mathbb{R}^d$ such that $]x_n^k, y_n^k]$ are disjoint. We shall say that the set $S = \cup_{n=1}^{\infty} \cup_{k=1}^{N(n)} [x_n^k, y_n^k]$ is a skeleton if there are increasing surjective functions $\phi_n : [1, N(n)] \to [1, N(n-1)]$ such that $x_n^k = y_{n-1}^{\phi_n(k)}$.*
The number $N(n)$ will be called the number of branches at generation n. The segment $[x_n^k, y_n^k]$ will be called the (n,k) tube. We will consider skeletons such that $N(1) = 1$.
The set $K_N = \cup_{n=1}^{N} \cup_{k=1}^{N(n)} [x_n^k, y_n^k]$ will be called the partial tree at generation N of the skeleton.

We shall consider skeletons with a flow attached to each tube so that Kirchhoff's law is satisfied at each bifurcation.

Definition 5 *Let S be a skeleton. We say that S is a skeleton with a flow F if the family $F = \{f_n^k \mid n \in \mathbb{N},\ k \in [1, N(n)]\}$ is such that $\max_k f_n^k \to 0$ as $n \to \infty$ and satisfies Kirchhoff's law, i.e.,*

$$\sum_{l \in \phi_{n+1}^{-1}(k)} f_{n+1}^l = f_n^k.$$

We shall say that f_1^1 is the total flow on S. In the sequel we normalize the total flow $f_1^1 = 1$.
We associate to a skeleton with a flow the family $R = \{r_n^k \mid n \in \mathbb{N},\ k \in [1, N(n)]\}$, where r_n^k represents the radius of the (n,k) tube. We shall assume that $\sup_{n,k} r_n^k < \infty$.

4.2 Correspondence between a skeleton and a filtration of [0,1)

The idea is to associate to each tube of generation n of the tree some interval $\omega_n^k \subset [0,1]$, so that σ-algebras $\mathcal{A}_n$ generated by the finite number of sets ω_n^k form a filtration. A point of $[0,1)$ will then correspond to a path in the tree. This construction follows [10].

Proposition 6 *Let S be a skeleton with a flow F. We assume that the total flow is 1. Then, there is a family ω_n^k such that $|\omega_n^k| = f_n^k$ for all $k \in [1, N(n)]$ and the family of intervals $\{\omega_n^k : k \in [1, N(n)]\}$ forms a partition of $\Omega = [0,1)$. Moreover, the σ-algebras $\mathcal{A}_n$ generated by $\{\omega_n^k \mid k \in [1, N(n)]\}$ form a filtration and the σ-algebra $\mathcal{A}$ generated by $\bigcup_n \mathcal{A}_n$ coincides with the σ-algebra of Borel sets of $[0,1)$.*

Proof. Let $\omega_1^1 = [0,1)$. Suppose that ω_n^k are defined for all $k \in [1, N(n)]$. We have to define ω_{n+1}^l for all $l \in [1, N(n+1)]$. Let $k \in [1, N(n)]$. Then, if $\omega_n^k = [a,b)$, for all $r \in \phi_{n+1}^{-1}(k) = [l_k + 1, l_{k+1}]$, we define

$$\omega_{n+1}^r = [\, a + \sum_{i=l_k+1}^{r-1} f_n^i \,,\; a + \sum_{i=l_k+1}^{r} f_n^i \,)$$

From the definition, $|\omega_{n+1}^r| = f_{n+1}^r$, and ω_{n+1}^r are intervals of the form $[c,d)$ forming partition of ω_n^k because of Kirchhoff's law. Repeating the same construction for all $k \in [1, N(n)]$ we obtain the family $\{\omega_{n+1}^l\}$.

By construction, the σ-algebras $\mathcal{A}_n$ generated by $\{\omega_n^k \mid k \in [1, N(n)]\}$ form a filtration. Since $\max_k |\omega_n^k| = \max_k f_n^k$ converges towards 0 when n goes to infinity, the σ-algebra $\mathcal{A}$ generated by $\bigcup_n \mathcal{A}_n$ coincides with the σ-algebra of Borel subsets of $[0,1)$. Moreover, if $\omega \in \Omega = [0,1)$ there is a unique decreasing family of intervals $\{\omega_n^{k(n)}, n \geq 1\}$ such that $\omega = \bigcap_n \omega_n^{k(n)}$, or, in other words, paths of the tree are in a one-to-one correspondence with points of $[0,1)$. Note that $\mathcal{A}_1 = \{\omega_1^1\}$ with $\omega_1^1 = [0,1)$. □

4.3 Construction of the set of fibers associated to the skeleton. The equality of supply.

Let S be a skeleton with a flow, and let $l_n^k = |x_n^k - y_n^k|$ be the length of the (n,k) tube. By definition of skeletons, there is a unique path from x_n^k to the source x_1^1, that is to say, given ω_n^k, there is a unique family of intervals $\omega_i^{k(i)}$ such that $\omega_n^k \subset \omega_i^{k(i)}$ for all $i \leq n$. Notice that $\omega_n^{k(n)} = \omega_n^k$. We shall denote by L_n^k the sum of lengths corresponding to the tubes $\{\omega_i^{k(i)} : i \leq n\}$, i.e., $L_n^k = \sum_{i=1}^n l_i^{k(i)}$. We also set $L_n^{k*} = \sum_{i=1}^{n-1} l_i^{k(i)}$. More generally, for all $\omega \in \Omega$, there exists a unique sequence $k(n)$ such that $\omega = \cap_n \omega_n^{k(n)}$. We define $L(\omega) = \sum_n l_n^{k(n)} \in I\!R \cup \{\infty\}$ to be the length of the path ω.

Proposition 7 *Let S be a skeleton with a flow. Let us define by recursively*

$$\chi_1(\omega, l) = \begin{cases} x_1^1 + l \frac{y_1^1 - x_1^1}{|y_1^1 - x_1^1|} & \textit{if } l \leq l_1^1 \\ y_1^1 & \textit{if } l > l_1^1 \end{cases}$$

and, for $n \geq 2$, $\omega \in \omega_n^k$, let

$$\chi_n(\omega, l) = \begin{cases} \chi_{n-1}(\omega, l) & \textit{if } l \leq L_n^{k*} \\ x_n^k + (l - L_n^{k*}) \frac{y_n^k - x_n^k}{|y_n^k - x_n^k|} & \textit{if } l \in [L_n^{k*}, L_n^k] \\ y_n^k & \textit{if } l > L_n^k \end{cases}$$

Then the pointwise limit $\chi(\omega, l) := \lim_n \chi_n(\omega, l)$ exists for any $(\omega, l) \in [0,1] \times I\!R^+$, and it is measurable. Hence χ is a measurable set of fibers in the sense [10].

Proof. Let us prove that χ_n is $\mathcal{A}_n \times \mathcal{B}(I\!R^+)$ measurable. First, since for any given l, $\chi_n(\cdot, l)$ is constant on every interval ω_n^k, the inverse image of any subset of $I\!R^d$ is a finite union of intervals ω_n^k, hence it is in $\mathcal{A}_n$. Thus $\chi_n(\cdot, l)$ is measurable for any l. Moreover, since for any $\omega \in \Omega$, $\chi_n(\omega, \cdot)$ is 1-Lipschitz, by Lemma 1, we obtain that χ_n is measurable, hence it is a set of fibers.

Let us prove that the pointwise limit $\chi(\omega, l) = \lim_n \chi_n(\omega, l)$ exists for any $(\omega, l) \in [0,1] \times I\!R^+$. If $l < L(\omega)$, and $\omega = \cap_m \omega_m^{k(m)}$, then there is an integer n such that $l \in (L_n^{k(n)*}, L_n^{k(n)}]$. The sequence $\{\chi_i(\omega, l)\}_{i \geq n}$ is constant, hence it is convergent. If $l \geq L(\omega)$, then $\chi_n(\omega, l)$ is a Cauchy sequence. Indeed,

$$|\chi_n(\omega, l) - \chi_m(\omega, l)| = |\chi_n(\omega, L_n^{k(n)}) - \chi_m(\omega, L_m^{k(m)})| = |y_m^{k(m)} - y_n^{k(n)}|$$

and the conclusion follows from the fact that $y_n^{k(n)}$ is a Cauchy sequence ; this being so because $L(\omega) \leq l < \infty$. We conclude that χ is Borel measurable, being a pointwise limit of Borel measurable functions. $\square$

Let χ_n and χ be the set of fibers associated to the skeleton S constructed in Proposition 7. Let us define the functions

$$r_{\chi_n}(\omega, l) = \sum_{\{(m,k): m \leq n,\, k \leq N(m)\}} 1_{\omega_m^k}(\omega) 1_{(L_m^{k*}, L_m^k]}(l) r_m^k,$$

$$f_{\chi_n}(\omega, l) = \sum_{\{(m,k): m \leq n,\, k \leq N(m)\}} 1_{\omega_m^k}(\omega) 1_{(L_m^{k*}, L_m^k]}(l) f_m^k,$$

for $(\omega, l) \in [0,1] \times I\!R^+$. Observe that

$$r_{\chi_n}(\omega, l) = r_{\chi_{n-1}}(\omega, l) \quad \text{if } \omega \in \omega_n^k,\ l \leq L_n^{k*},$$

and similarly for $f_{\chi_n}(\omega, l)$. Thus if $\omega = \cap_m \omega_m^{k(m)}$, and $l < L(\omega)$, there is an integer n such that $l \in (L_n^{k(n)*}, L_n^{k(n)}]$, and $r_{\chi_i}(\omega, l) = r_n^{k(n)}$, $f_{\chi_i}(\omega, l) = f_n^{k(n)}$ for all $i \geq n$. Thus the pointwise limits

$$r_\chi(\omega, l) = \lim_n r_{\chi_n}(\omega, l),$$

$$f_\chi(\omega, l) = \lim_n f_{\chi_n}(\omega, l)$$

exist for any $(\omega, l) \in [0,1] \times I\!R^+$ such that $l < L(\omega)$. If $l \geq L(\omega)$, then $r_{\chi_n}(\omega, l) = 0$, $f_{\chi_n}(\omega, l) = 0$, and we may define

$$r_\chi(\omega, l) = 0,$$

$$f_\chi(\omega, l) = 0.$$

Observe that the functions r_{χ_n}, f_{χ_n} are measurable, and, hence, r_χ, f_χ are also measurable.

The function $L(\omega)$ can be seen as an absorption length since it may be written as

$$L(\omega) = \inf\{l \in I\!R_+ \mid \chi(\omega, l) \text{ is constant on } [l, +\infty)\}.$$

Then, by Proposition 5, it is also Lebesgue measurable. As in [10] and Section 3, we define the irrigation measure μ by $\mu(A) = |T^{-1}(A)|$, where $T : \omega \to \chi(\omega, L(\omega))$.

Definition 6 *Let S be a skeleton with a flow. We shall say that S satisfies weak equality of supply if its associated set of fibers defines an image measure μ such that $\mu = f(x)\lambda$ where λ is the Lebesgue measure in $I\!R^d$ and $f \in L^1(I\!R^d)$, $f \geq 0$, $f \neq 0$.*

We say that S satisfies the equality of supply $f = c1_K$ where K is some set of positive measure. In the general case, we shall denote by $K := \{x \in I\!R^d : f(x) > 0\}$.

Remark 3 The set K can be taken as being a subset of $T(\Omega)$, indeed

$$\int_{K\setminus T(\Omega)} f(x)d\lambda = \mu(K\setminus T(\Omega)) = |T^{-1}(K\setminus T(\Omega))| = |\emptyset| = 0.$$

Since $f>0$ on K, we have that $K\subset T(\Omega)$ almost everywhere and we may write $\mu = f(x)1_{K\cap T(\Omega)}\lambda$. Thus, replacing K by $K\cap T(\Omega)$ if necessary, we may assume that $K\subset T(\Omega)$.

The aim of the above construction is to be able to reformulate the energy and the volume of the tube network as Lebesgue integrals of adequate functions defined on the set Ω of paths, as we shall see in the next section.

5 Source to volume transfer energy

There are technical difficulties if one wants to make calculations on a tree. For instance, if one wants to write the volume of a tree as an integral, either one writes it as an integral of the total sections over all branches, from the source to the tips; or we write it as an integral over $[0,1)\times I\!R^+$, i.e., an integral along the paths of the tree. The construction of the last section will enable us to follow the latter approach. In what follows, we introduce the volume and the dissipated power of a skeleton with a flow. It is to be mentioned that these definitions only intend to be of the same order as the exact volume and dissipated power of an associated embedded tree.

Definition 7 *Let S be a skeleton with a flow. Let l_n^k, r_n^k, f_n^k be the length, radius, and flow, respectively, of the (n,k) tubes. We define the volume of the tree associated to S by $V=\sum_{n=1}^{\infty}\sum_{k=1}^{N(n)} l_n^k s_n^k$, and its dissipated power associated with a resistance law $R(s)$ by $W=\sum_{n=1}^{\infty}\sum_{k=1}^{N(n)} l_n^k R(s_n^k)(f_n^k)^2$, where $s_n^k=(r_n^k)^{d-1}$ (the quantities are taken modulo constants).*

To prepare the proof of Theorem 3, it will be convenient to write them as double integrals over l and ω as follows.

Proposition 8 *We may express the volume and the dissipated power of the tree by the formulas $V=\int_0^\infty\int_0^1 Q_1(\omega,l)dld\omega$ where $Q_1(\omega,l)=\frac{r_\chi(\omega,l)^{d-1}}{f_\chi(\omega,l)}$ for $l\le L(\omega)$ and $Q_1(\omega,l)=0$ for $l>L(\omega)$, and $W=\int_0^\infty\int_0^1 Q_2(\omega,l)dld\omega$ where $Q_2(\omega,l)=f_\chi(\omega,l)R(s_\chi(\omega,l))$, where $s_\chi(\omega,l)=r_\chi(\omega,l)^{d-1}$, for $l\le L(\omega)$ and $Q_2(\omega,l)=0$ for $l>L(\omega)$.*

Proof. Let us define

$$\frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)}=0$$

when both terms are 0. Then it is easy to check that

$$\frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)} = \sum_{\{(m,k):m\le n,\, k\le N(m)\}} 1_{\omega_m^k}(\omega)1_{(L_m^{k*},L_m^k]}(l)\frac{(r_n^k)^{d-1}}{f_n^k} \qquad (\omega,l)\in[0,1]\times I\!R^+,$$

and

$$\int_0^\infty\int_0^1 \frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)}\,dl\,d\omega = \sum_{m=1}^{n}\sum_{k=1}^{N(m)} l_m^k (r_m^k)^{d-1} \tag{5.1}$$

for each $n\ge 1$. Since $\frac{r_{\chi_n}(\omega,l)^{d-1}}{f_{\chi_n}(\omega,l)}\uparrow Q_1(\omega,l)$ pointwise as $n\to\infty$, letting $n\to\infty$ in (5.1) we deduce that

$$V=\int_0^\infty\int_0^1 Q_1(\omega,l)dld\omega.$$

In a similar way, we prove that $W=\int_0^\infty\int_0^1 Q_2(\omega,l)dld\omega$. $\square$

Definition 8 *We shall say that S is a skeleton with almost surely finite paths if $L(\omega) < \infty$ for almost every $\omega \in \Omega$.*

Theorem 3 *Let $0 < \alpha \leq 1 - \frac{1}{d}$. Let us assume that the resistivity function is $R(s) = s^{(\alpha-2)/\alpha}$. Let S be a skeleton with a flow which has almost surely finite paths and satisfies weak equality of supply. Then, V and W cannot be simultaneously finite.*

Proof. Observe that we have $Q_1 Q_2 = r_\chi(\omega, l)^{2(d-1)(1-\alpha^{-1})}$. Hence $Q_1 Q_2 \geq \frac{c^2}{r_\chi(\omega,l)^2}$ when $r_\chi(\omega, l) > 1$ where $c = (\sup_{\omega,l} r_\chi(\omega, l))^{(d-1)(1-\alpha^{-1})}$. By Cauchy-Schwarz inequality we have

$$\sqrt{V}\sqrt{W} \geq \int_0^\infty \int_0^1 \sqrt{Q}_1 \sqrt{Q}_2 = \int_0^\infty \int_0^1 r_\chi(\omega, l)^{(d-1)(1-\alpha^{-1})} d\omega dl \geq c \int_0^\infty \int_0^1 \frac{1}{r_\chi(\omega, l)} d\omega dl.$$

Let K be the set where $f > 0$, f being the function such that $\mu = f\lambda$, where μ is the irrigation measure defined by the set of fibers associated to the skeleton (see Definition 6). Let us decompose K as $K = A \cup R$, where A are the points of Lebesgue density 1 of K, R has zero Lebesgue measure and, because of weak equality of supply, it is also of μ measure zero. Then $|T^{-1}(R)| = 0$, so that $|T^{-1}(A)|$ is of non zero measure. By Proposition 2 in the Appendix B, the profile of an irrigating branch must decrease faster than linearly and $\int_0^\infty \frac{1}{r_\chi(\omega,l)} dl$ is infinite for all ω such that $T(\omega) = x \in A$. Then it turns out that

$$\int_0^1 \int_0^\infty \frac{1}{r_\chi(\omega, l)} dl d\omega \geq \int_{T^{-1}(A)} \int_0^\infty \frac{1}{r_\chi(\omega, l)} dl d\omega \geq \infty.$$

We conclude that

$$\sqrt{V}\sqrt{W} \geq \int_0^1 \int_0^\infty \frac{1}{r_\chi(\omega, l)} dl d\omega = \infty.$$

□

Thus, the exponent $\alpha = 1 - \frac{1}{d}$ is critical relatively to the fact that a tree cannot irrigate a volume at finite cost. This results is coherent with the forthcoming work [17].

A Flows in tubes

In this appendix, we shall consider a fluid with laminar flow in a tube. We recall how Poiseuille law can be derived from Navier-Stokes equation. Next, we discuss the optimality of the circular section.

A.1 Poiseuille law

Let us consider a tube of constant circular section with a straight axis. We take (x, y, l) as coordinates in the tube, where $l \in [0; L]$ is the distance along the axis and $(x, y) \in D(0, r)$ are orthogonal cartesian coordinates.

We assume a stationary regime and that the flow is laminar, that is to say the velocity is oriented by the axis and is constant on all trajectories, so that $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$. The velocity v at a point of a tube along the z-axis is given by Navier-Stokes equation

$$-\triangle v(l)(x, y) = \frac{1}{\eta}\frac{\partial p}{\partial z}, \text{ where } \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Hence, $\frac{\partial p}{\partial z} = constant$ (where η denotes the viscosity coefficient). Thus, the gradient of pressure has the form $\frac{[p]}{L}$ where $[p]$ denotes the pressure difference at the ends of the tube, and we shall denote it by $\bigtriangledown p$. In other words, p is a linear interpolation of the initial and final pressures in the tube. We assume that the pressure is constant on the initial and ending sections of the tube, so that the pressure is constant on each section of the tube. For simplicity, let us take $\eta = 1$.

Under these hypotheses, we can calculate the velocity and the corresponding flow through the whole tube

$$v(x,y,l) = \frac{(r^2 - (x^2 + y^2))}{4} \bigtriangledown p$$

$$f = \int_{D(0,r)} v(x,y,l) = \frac{1}{4} r^4 \bigtriangledown p = r^2 v_{max}$$

The power dissipated by the steady flow is $W = fL \bigtriangledown p$. This is to be identified with $W = Lf^2 R$ where by definition R stands for the resistivity of the tube. Thus we obtain $R = 4/r^4$: Poiseuille law says that the resistivity of a tube scales as the inverse fourth power of the radius.

A.2 Optimality of the circular section

What is the optimal form of the section of a tube? If we prescribe the pressure at both ends of a tube of constant section, the circular form ensures the maximal flow. We briefly present the result obtained in [19] and [1].

Let us recall the definition of the rearrangement of a set (see [9]). If $A \subset I\!R^d$, we denote by A^* the ball $B(0,r) = \{x \mid |x| < r\}$ such that $|B(0,r)| = |A|$. If $f : I\!R^d \to I\!R$ is a Borel measurable function vanishing at infinity, we define the symmetric decreasing rearrangement of f by $f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt$. It results from the definition that $|\{x \mid |f(x)| > t\}| = |\{x \mid f^*(x) > t\}|$ and $||f||_p = ||f^*||_p$.

Let u be such that $-\triangle u(x,y) = \bigtriangledown p$ in the domain Ω. Let v be such that $-\triangle v(x,y) = (\bigtriangledown p)^* = \bigtriangledown p$ in Ω^*. Then, it can be shown that $u^* \leq v$ [19]. As a consequence, the flow in a tube of section Ω is such that $\int_\Omega u = \int_{\Omega^*} u^* \leq \int_{\Omega^*} v$. Then a circular section is always more advantageous from the point of view of the flow.

In [1], the authors prove the uniqueness of the optimal form : if $\max u = \max v$, then there is x_0 such that $\Omega = x_0 + \Omega^*$ and $u = v(\cdot + x_0)$. Then, if Ω is an optimal form, we have $\int_\Omega u = \int_{\Omega^*} v$ and $u^* \leq v$, hence $\max u = \max u^* = \max v$ necessarily. Then there is x_0 such that $\Omega = x_0 + \Omega^*$, and, therefore, the circular form is the unique optimum.

B Irrigable points and local density

In this appendix, a model of irrigating tree based on irrigation at the tips is considered. A point is said to be accessible if it is at the end of a tip, the thickness profile of which is prescribed. Conditions are given on the profile so that it prevents a set from irrigating a set of positive measure.

We denote by $B(x,r)$ the open ball of center $x \in I\!R^d$ and radius $r > 0$. If $E \subseteq I\!R^d$ is Lebesgue-measurable and $x \in I\!R^d$, the upper and lower densities of x in E are defined by

$$\overline{d}(E,x) := \limsup_{\rho\to 0^+} \frac{|E \cap B(x,\rho)|}{|B(x,\rho)|}$$

$$\underline{d}(E,x) := \liminf_{\rho\to 0^+} \frac{|E \cap B(x,\rho)|}{|B(x,\rho)|}.$$

When the upper and lower limits are equal, we denote their common value by $d(E,x)$ and we call it the density of E at x. By Lebesgue density theorem [11], both densities are equal to 1 at almost every point of E.

Let $f : [0,\infty) \to [0,\infty)$ be an increasing continuous function such that $f(0) = 0$.

Definition 9 *Let E be a measurable set in $\mathbb{R}^d$, $S \in \mathbb{R}^d \setminus E$. We say that $x \in E$ is irrigable in E from S with profile given by f if there is a curve $\gamma : [0, L(\gamma)] \to \mathbb{R}^d$ parameterized by its arclength such that $\gamma(0) = x$, $\gamma(L(\gamma)) = S$,*

$$B(\gamma(s), f(s)) = (\gamma(s) + B(0, f(s))) \subseteq \mathbb{R}^d \setminus E \tag{B.1}$$

for all $s \in (0, L(\gamma)]$.

Proposition 9 *Let $x \in E$ be irrigable from S with profile f. Assume that $d(E, x) = 1$. Then* $\limsup_{r\to 0+} f(r)/r = 0$. *As a consequence $\int_0^R \frac{1}{f(r)}\, dr = \infty$, $R > 0$.*

Proof. Let γ be a curve of accessibility to x, and $r < L(\gamma)$. Then, since γ is parametrized by its arclength, we have $\gamma(\frac{r}{2}) \in \overline{B(x, \frac{r}{2})}$. As a consequence, $B(\gamma(\frac{r}{2}), \frac{r}{2}) \subset B(x, r)$.

If $f(\frac{r}{2}) < \frac{r}{2}$, then $B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \subset B(x, r)$, so that $B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \cap B(x, r) = B(\gamma(\frac{r}{2}), f(\frac{r}{2}))$. If $f(\frac{r}{2}) \geq \frac{r}{2}$, then $B(\gamma(\frac{r}{2}), \frac{r}{2}) \subset B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \cap B(x, r)$. In both cases, we have $B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \subset U$, hence

$$\frac{|(\mathbb{R}^d \setminus E) \cap B(x, r)|}{|B(x, r)|} \geq \frac{|B(\gamma(\frac{r}{2}), f(\frac{r}{2})) \cap B(x, r)|}{|B(x, r)|} \geq \frac{min(r/2, f(\frac{r}{2}))^d}{r^d}$$

Taking the limsup, the inequality yields $\overline{d}(\mathbb{R}^d \setminus E, x) \geq \frac{1}{2^d} min(\limsup_{r\to 0+} f(r)/r, 1)^d$. Then, $d(E, x) = 1$ implies that $\limsup_{r\to 0+} f(r)/r = 0$.

Finally, observe that, for some $R > 0$, $\frac{f(r)}{r} < 1$ for all $r < R$; otherwise we would have $\limsup_{r\to 0+} f(r)/r \geq 1$. It follows that $\frac{1}{r} < \frac{1}{f(r)}$ for all $r < R$, and thus $\int_0^R \frac{1}{f(r)}\, dr = \infty$. $\square$

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