

Memento on cell-complexes

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October 1, 2009

X is a Hausdorff space. D^n is a n -dimensional closed ball. Δ^n is the n -dimensional standard simplex. For a subspace A of X , \bar{A} is the closure of A and $\text{int } A$ its interior. A *characteristic map* of X is a continuous function $f_\alpha : D^n \rightarrow X$ whose restriction to $\text{int } D^n$ is an embedding. A *cell* e_α^n of X is the image of a characteristic map f_α of X on the open ball $\text{int } D^n$. Notice that $\bar{e}_\alpha^n = f_\alpha(D_\alpha^n)$.

Cell-complex. A *cell-complex* K is a Hausdorff space which is the disjoint union of *cells* e_α^n which satisfy the following condition

$$(\star) \quad \bar{e}_\alpha^n \setminus e_\alpha^n \text{ is contained in } K^{n-1}$$

where K^n is the n^{th} skeleton of K which consists of all the cells of dimension $\leq n$ ($K^{-1} = \emptyset$). A cell-complex is called *regular* if its characteristic maps can be chosen to be embeddings. A cell-complex is *finite* if its disjoint union of cells is finite. The *dimension* of K is the minimum m such that $K^m = K$. A *subcomplex* $L \subset K$ is the union of a subcollection of the cells of K such that, if $e \subset L$ then $\bar{e} \subset L$. K^n is a subcomplex of K for any n .

The examples in Fig 3 illustrate the hypothesis in the definition of a cell-complex. In C_1 , the two edges are not disjoint. In C_2 or C_3 , it is impossible to map continuously the closed segment D^1 onto the open segment or the circle. In C_4 , e^2 is a cell but e^1 is not because it is not homeomorphic to the open segment. In C_5 , the condition (\star) is falsified because ∂e_2^1 is contained in K^1 but not in K^0 . Notice that (\star) is satisfied for e^1 in C_2 and in C_3 and for e^2 in C_5 . In C_6 , the cell e^2 is homeomorphic to $\text{int } D^2$ but such an homeomorphism cannot be extended to an embedding of D^2 so C_6 is not regular. C_7 and C_8 are two cell complexes where ∂e_2 is not a union of cells.

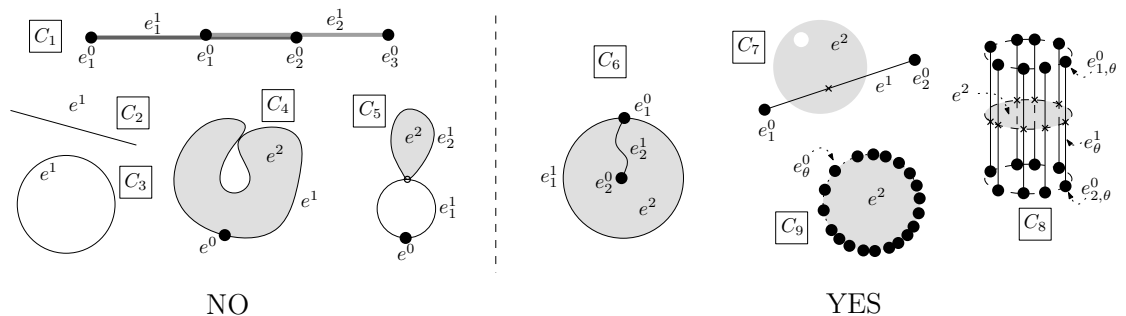


Figure 1: Examples and counter-examples of cell-complexes

Lavalle's definition. Lavalle gives an alternative definition of cell-complex. A *L-complex* is a Hausdorff space K which is the disjoint union of cells e_α^n satisfying the two following condition:

- (a) K^n is a closed subset of K ,
- (b) e_α^n is an open subset of K^n .

Lemma 1. *A L-complex is a cell-complex.*

Proof. Let consider a L-complex. As e_α^n is an subset of K^n which is a closed subset of K (hypothesis (a)), its closure \bar{e}_α^n (in K) is contained in K^n so ∂e_α^n is contained in K^n . Suppose now by contradiction that a point x of ∂e_α^n is contained in some cells e_β^n . As e_β^n is an open subset of K^n (hypothesis (b)), its complementary $K^n \setminus e_\beta^n$ has to be a closed subset of K^n . But x is an accumulation point of e_α^n which is disjoint of e_β^n , this leads to a contradiction. Finally ∂e_α^n is contained in K^{n-1} and the property (\star) is satisfied. \square

CW properties. A *CW-complex* is a cell-complex which satisfy the following properties:

- *Closure-finiteness.* For each cell e_α^n , its frontier ∂e_α^n is contained in the union of a finite number of cells of K^{n-1} .
- *Weak topology.* A subset of K is closed if and only if it meets the closure of each cell of K in a closed set.

Any finite complex K is a CW-complex: it is obviously closure finite and it has the weak topology since $A \subset K$ is the union of the finite number of sets $X \cap \bar{e}$.

Lemma 2. *A CW-complex is a L-complex.*

Proof. In a CW-complex, all skeleton K^n are closed CW-complexes (see [2]) so condition (a) of L-complex is satisfied. Let show that $(e_\alpha^n)^c = K^n \setminus e_\alpha^n$ is closed in K^n by using the weak topology property of K_n . The condition (b) will be also satisfied.

- *Cells $e_\beta^j \neq e_\alpha^n$ in K^n .* Such a cell is contained in $(e_\alpha^n)^c$ by the disjointness of cells. By condition (*star*), its boundary ∂e_β^j is contained in $K^{j-1} \subset (e_\alpha^n)^c$, so the closed set \bar{e}_β^j is contained in $(e_\alpha^n)^c$.
- *Cell e_α^n .* We have $\bar{e}_\alpha^n \cap (e_\alpha^n)^c = \partial e_\alpha^n$. By closure-finiteness property, ∂e_α^n is contained in a finite union of cells e_γ^k with $k < n$. Since \bar{e}_α^n is a closed set of K^n , $\bar{e}_\alpha^n \cap \bar{e}_\beta^j = \partial e_\alpha^n \cap \bar{e}_\beta^j$ is a closed set for all cells $e_\beta^j \neq e_\alpha^n$. We obtain

$$\bar{e}_\alpha^n \cap (e_\alpha^n)^c = \partial e_\alpha^n = \bigcup_{\beta \neq \alpha} \partial e_\alpha^n \cap \bar{e}_\beta^j = \bigcup_{\gamma} \partial e_\alpha^n \cap \bar{e}_\gamma^k$$

and the last union is finite so it is a closed set of K_n .

\square

About non-compact cells. In affine space, we wish to consider unbounded cells (as in Voronoi diagrams) but in the formalism of a cell-complex, a cell has compact closure. We can embed the affine space in the projective space or we can compactify it to get back the formalism of complexes. In each case, we have to add ideal cells.

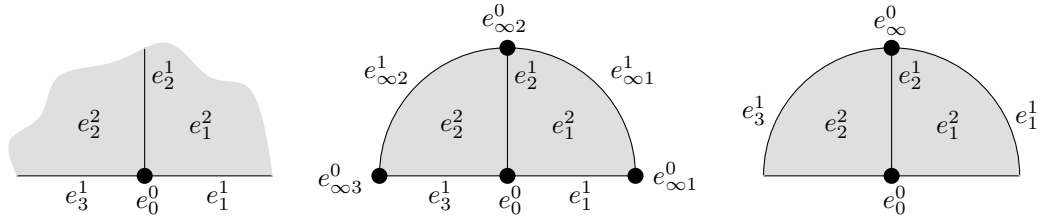


Figure 2: From left to right: decomposition in affine space, associated cell-complex in projective space, associated cell-complex in the compactification

Triangulations. A Δ -complex is a (CW-)complex whose characteristic maps satisfy the additional property: each cell e_{α}^n is provided with a characteristic map $f_{\alpha} : \Delta^n \rightarrow \bar{e}_{\alpha}^n$ such that the restriction of f_{α} to each face Δ^{n-1} of Δ^n is f_{β} for some cell e_{β}^{n-1} . If the ordering of vertices of simplices does not matter, the Δ -complex is *unordered*. In the literature unordered Δ -complexes are called *generalized triangulations*.

Regular complexes are homeomorphic to Δ -complexes but the converse is false (à vérifier, homéo cellulaire?). A *simplicial complex* (or simplicial multicomplex, or multicomplex) is a regular unordered Δ -complex in which each simplex is uniquely determined by its vertices. (ie on a des simplexes bien droits?)

A *singular Δ -complex* is a (CW-)complex with characteristic maps $f_{\alpha} : \Delta^n \rightarrow \bar{e}_{\alpha}^n$ whose restrictions to faces Δ^{n-1} are compositions $f_{\beta} \circ q : \Delta^{n-1} \rightarrow \Delta^k \rightarrow K$ for q a linear surjection taking vertices to vertices (and preserving orders).

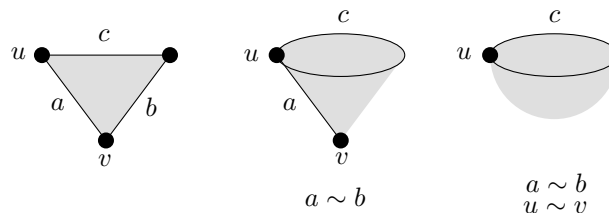


Figure 3: From left to right: simplicial complex, Δ -complex, singular complex

References

- [1] A. Hatcher, *Algebraic Topology*.
- [2] J. H. C. Whitehead, Combinatorial Homotopy, *Bull. Amer. Math. Soc.*, 55(3), 1949, 213–245.
- [3] S. M. LaValle, *Planning Algorithms*.