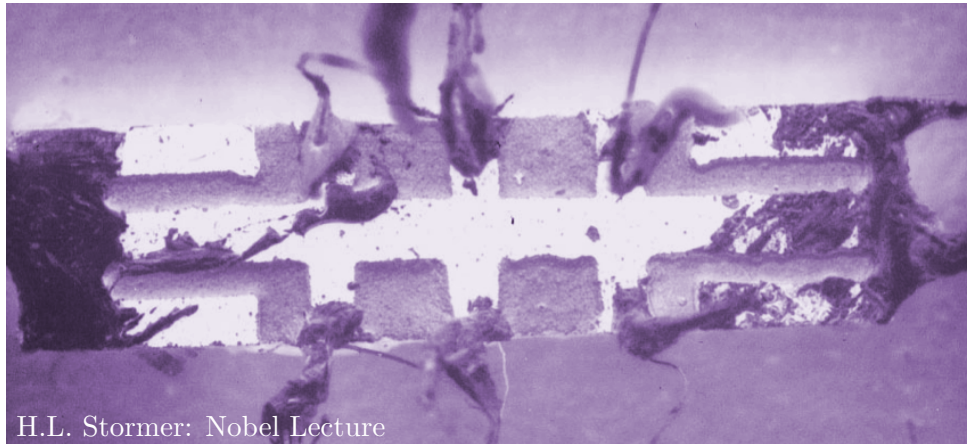


# Chapter 1

## The integer quantum Hall effect I



### Learning goals

- We know the basic phenomenology of the quantum Hall effect (QHE)
  - We know the structure of the lowest Landau level (LLL)
  - We understand the role of disorder for the QHE.
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- K. von Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**, 494 (1980)

In large parts of this chapter we follow reference [2].

### 1.1 Preliminaries

The Lorentz force acting on charged particles moving in a two-dimensional plane leads to a build-up of charges perpendicular to the direction of motion. This is the classical Hall effect first discussed by Edwin Hall in 1879 [3]. To understand this, let us consider a two-dimensional system which is translationally invariant. We move to a frame moving with  $-\mathbf{v}$  where we therefore see a current

$$\mathbf{J} = nev, \quad (1.1)$$

where  $n$  is the particle density and  $e$  the electron charge. In the laboratory frame we have  $\mathbf{E} = \mathbf{0}$  and  $\mathbf{B} = B\hat{\mathbf{z}}$ . Hence, in the moving frame we obtain

$$\mathbf{E} = \mathbf{v} \wedge \mathbf{B} \quad \text{and} \quad \mathbf{B} = B\hat{\mathbf{z}}. \quad (1.2)$$

We can express the electric field as

$$\mathbf{E} = \frac{B}{ne} \mathbf{J} \wedge \hat{\mathbf{z}}. \quad (1.3)$$

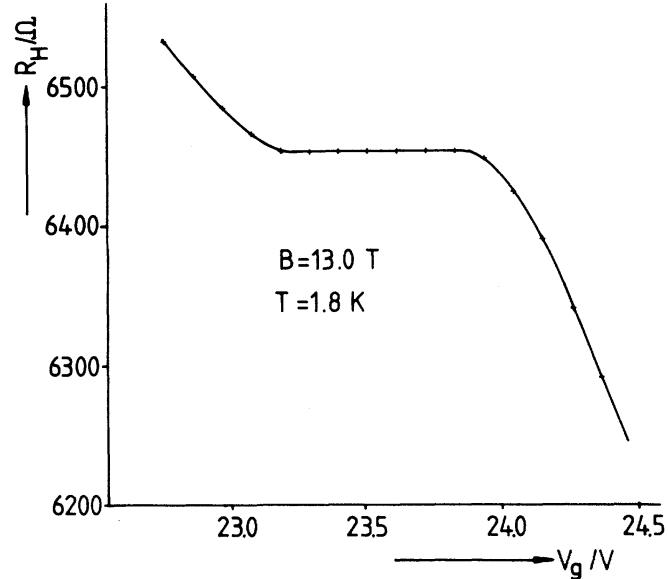


Figure 1.1: Measured Hall resistivity as a function of an applied back-gate which leads to a change in the particle density  $n$ . The pronounced plateau is the hallmark of the quantum Hall effect. Figure taken from Ref [1].

The resistivity  $\rho$  is defined as the relation between the current and the electric field  $E^\mu = \rho_{\mu\nu} J^\nu$ . We thus find

$$\rho = \frac{B}{ne} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad \sigma = \frac{ne}{B} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.4)$$

We see that owing to the non-zero  $\sigma_{xy}$  the longitudinal resistivity  $\rho_{xx} = \sigma_{xx} = 0$  is equal to the longitudinal conductivity. Moreover, the Hall resistivity is proportional to the magnetic field

$$\rho_{xy} = \frac{B}{ne}. \quad (1.5)$$

This is in striking contrast to the seminal discovery of von Klitzing and his co-workers in 1980 [1], see Fig. 1.1. The only ingredient in our theoretical model so far, however, was translational symmetry. In the following, we first take steps towards a quantum mechanical understanding of electrons in a magnetic field before we come back to the issue of translational symmetry breaking via disorder.

## 1.2 Classical Lagrangian

To motivate how the magnetic field enters our quantum mechanical description, we recall that the classical equations of motions are reproduced by the following Lagrangian  $\mathcal{L} = \frac{m}{2} \dot{x}^\mu \dot{x}^\mu - e \dot{x}^\mu A^\mu$ .

$$-\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \quad \Rightarrow \quad m\ddot{x} = -eB\dot{y} \quad \text{and} \quad m\ddot{y} = eB\dot{x}. \quad (1.6)$$

The canonical momentum is given by  $p^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m\dot{x}^\mu - eA^\mu$  and therefore the Hamiltonian reads

$$H(x^\mu, p^\mu) = \dot{x}^\mu p^\mu - \mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2m} (p^\mu + eA^\mu) (p^\mu + eA^\mu). \quad (1.7)$$

With this small detour into classical mechanics we are now in the position to tackle the quantum mechanical problem of a particle in a magnetic field.

### 1.3 Landau levels

We have to solve for the eigenstates of the following Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2. \quad (1.8)$$

As only the vector potential  $\mathbf{A}$  enters the Hamiltonian we have to choose an appropriate gauge. For now we choose the Landau gauge where  $A = xB\hat{y}$ . We check that  $\nabla \wedge \mathbf{A} \equiv \mathbf{B} = (\partial_x A_y - \partial_y A_x)\hat{z} = B\hat{z}$ . Inserted into the above Hamiltonian we obtain

$$H = \frac{1}{2m} \left[ p_x^2 + (p_y + exB)^2 \right]. \quad (1.9)$$

We immediately observe that this Hamiltonian has a translational symmetry in  $y$  direction. We therefore choose the following ansatz for the wave function  $\psi(x, y) = e^{iky} f_k(x)$ . With this ansatz we obtain a family of one-dimensional problems (one per momentum  $k$  in  $y$ -direction)

$$h_k = -\frac{\hbar^2 \partial_x^2}{2m} + \frac{1}{2} m \omega_c^2 (x + kl^2)^2 \quad \text{with} \quad \omega_c = \frac{eB}{m} \quad \text{and} \quad l = \sqrt{\frac{\hbar}{eB}}. \quad (1.10)$$

We see that we are dealing with a (displaced) one-dimensional harmonic oscillator. The characteristic frequency is known as the **the cyclotron frequency  $\omega_c$** . The displacement is proportional to the  $y$ -momentum and measured in the natural length scale, the **magnetic length  $l$** . Solving the harmonic oscillator we find that

1.  $\epsilon_k = \hbar\omega_c \left( s + \frac{1}{2} \right)$  with  $s \in \mathbb{N}$ .
2. For  $s = 0$ , i.e., the LLL the wave function is a Gaussian centered at  $X_k = -kl^2$

$$\psi(x, y) = \frac{1}{\sqrt{\pi^{1/2} L_y l}} e^{iky} e^{-\frac{1}{2l^2}(x+kl^2)^2} = \frac{1}{\sqrt{\pi^{1/2} L_y l}} e^{iky} e^{-\frac{1}{2l^2}(x-X_k)^2}, \quad (1.11)$$

where  $L_y$  is the extent in  $y$ -direction as shown in Fig. 1.2.

3. We have a vastly degenerate system. The number of degenerate states in each LL is given by

$$N = \frac{L_y}{2\pi} \int_0^{L_x/l^2} dk = \frac{L_x L_y}{2\pi l^2} = \frac{L_x L_y B}{\Phi_0}, \quad (1.12)$$

where  $\Phi_0 = h/e$  is the magnetic flux quantum. In other words, per magnetic flux quantum that penetrates the sample we have one state per Landau level.

Before we continue we should remind ourselves that in the case of huge degeneracies any perturbation might have dramatic effect. Moreover, the choice of basis can facilitate the description of these effects. For the case of a magnetic field, the choice of gauge determined the shape of the basis wave-functions. We will come back to this point later.

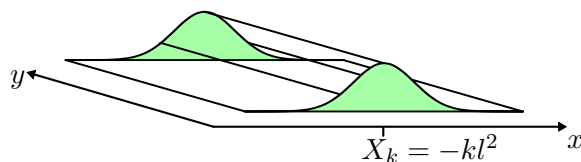


Figure 1.2: Eigenfunctions of the LLL in the Landau gauge.

## 1.4 Currents

We set out to understand the Hall conductivity. To make further progress, we need to calculate currents. We evaluate the current operator in  $y$  direction,  $J_y = -\frac{e}{m}(\mathbf{p}_y + eA_y)$ , in the LLL eigenfunctions

$$\langle \psi | J_y | \psi \rangle = -\frac{e}{m\pi^{1/2}l} \int dx e^{-\frac{(x-X_k)^2}{2l^2}} (\hbar k + eBx) e^{-\frac{(x-X_k)^2}{2l^2}} \quad (1.13)$$

$$= -\frac{e\omega_c}{\pi^{1/2}l} \int dx e^{-\frac{(x-X_k)^2}{l^2}} (x + kl^2) = -\frac{e\omega_c}{\pi^{1/2}l} \int d\alpha e^{-\frac{\alpha^2}{l^2}} \alpha = 0. \quad (1.14)$$

The last equality holds as the integrand is odd under  $\alpha \rightarrow -\alpha$ . In other words, no net current is flowing as shown in Fig. 1.3.

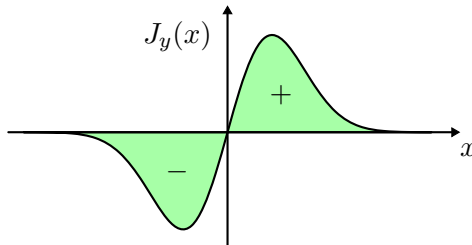


Figure 1.3: Current distribution in the lowest Landau level.

For a current to flow, we need to add an electric field in  $x$ -direction  $V(x) = eEx$ . We still are translationally invariant in  $y$ -direction and the one-dimensional problem is changed to

$$h_k = -\frac{\hbar^2 \partial_x^2}{2m} + \frac{1}{2}m\omega_c^2 (x + kl^2)^2 + eEx \quad (1.15)$$

$$= -\frac{\hbar^2 \partial_x^2}{2m} + \frac{1}{2}m\omega_c^2 \left( x + kl^2 + \frac{eE}{m\omega_c^2} \right)^2 - eEX'_k + \frac{1}{2}m\bar{v}^2, \quad (1.16)$$

where the center of the Gaussians is shifted

$$X'_k = -kl^2 - eE/m\omega_c^2 \quad (1.17)$$

and an additional energy  $\frac{1}{2}m\bar{v}^2$  with  $\bar{v} = -E/B$  arises from the drift of the electrons. The immediate conclusion is that the new energy depends on  $k$ , i.e., the huge degeneracy is lifted

$$\epsilon_k = \frac{1}{2}\hbar\omega_c + eEX'_k + \frac{1}{2}m\bar{v}^2. \quad (1.18)$$

With an energy that depends on  $k$  we can also calculate a non-zero group velocity

$$v_{\text{group}} = \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial k} = \frac{eE}{\hbar} \frac{\partial X'_k}{\partial k} = -\frac{eE}{\hbar} l^2 = -\frac{E}{B} = \bar{v}. \quad (1.19)$$

We this we reach the classical result

$$\langle J_y \rangle = -e\bar{v} \quad \Rightarrow \quad \sigma_{xy} = -\frac{ne}{B}. \quad (1.20)$$

We this result we reach the same conclusion as with the classical manipulations based entirely on translational symmetry in the beginning of this chapter. In order to make further progress we should take a closer look at the finite extent of a realistic sample as well as on disorder effects to understand the quantization of  $\sigma_{xy}$ .

## 1.5 Edge states

We try to build an understanding of the influence of the edges of a two dimensional sample by considering a strip which is finite in  $x$ -direction and infinite (or periodic) in  $y$ -direction. The basis wave functions of the LLL, or equivalently the gauge choice for  $\mathbf{A}$ , which we used above is optimally tailored to this geometry. Remember that the wave functions are localized in  $x$ -direction with a typical extent  $l$ . If we now consider a potential  $V(x)$  that confines the electrons to a finite region which is smooth over the length-scale  $l$ , we can expect the wave function to remain approximately Gaussian. However, the wave functions centered in the vicinity of the edges will be lifted in energy. As the position of the wave function is linked to the momentum  $k$  in  $y$ -direction we obtain dispersive edge channels.

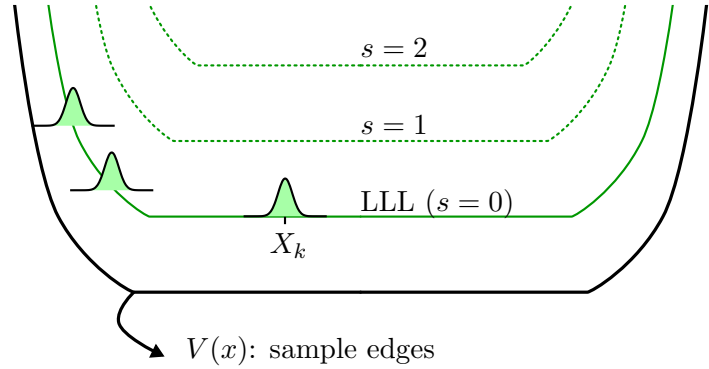


Figure 1.4: Edge states from curved Landau levels.

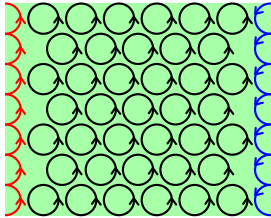


Figure 1.5: Classical skipping orbits.

In order to determine how the current is distributed we again calculate the group velocity.

$$v_{\text{group}} = \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial k} = \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial X_k} \frac{\partial X_k}{\partial k} = -\frac{l^2}{\hbar} \frac{\partial \epsilon_k}{\partial X_k} = \begin{cases} < 0 & \text{right edge} \\ > 0 & \text{left edge} \end{cases} \quad (1.21)$$

These simple manipulations reveal that the two opposite edges carry opposite current. This can also be understood from the classical “skipping orbits” picture as shown on the left.

In order to calculate  $\sigma_{xy}$  we now apply a voltage  $V_H$  between the two edges (in  $x$ -direction) and calculate the resulting current along the sample (in  $y$ -direction). Moreover, we assume that the Fermi energy  $E_F$  lies in between two Landau levels.

To obtain the total current  $I_y$  we sum over the contribution  $ev_k$  of all occupied states

$$I_y = -e \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\hbar} \frac{\partial \epsilon_k}{\partial k} n_k, \quad (1.22)$$

where  $n_k$  is the occupation probability of the  $k$ 'th mode. Under the assumption that we only fill the LLL and that we are at zero temperature the occupation numbers only take the values  $n_k = \{1, 0\}$ . Under these assumptions we arrive at

$$I_y = -\frac{e}{h} \int_{\mu_L}^{\mu_R} d\epsilon = -\frac{e}{h} (\mu_R - \mu_L), \quad (1.23)$$

where  $\mu_{R/L}$  are the respective chemical potentials on the two sides. As we can write  $eV_H = \mu_R - \mu_L$  we arrive at

$$I_y = -\frac{e^2}{h} V_H \quad \Rightarrow \quad \sigma_{xy} = -\frac{e^2}{h}. \quad (1.24)$$

Let us move now the Fermi energy in between any two LL and we immediately conclude that

$$\sigma_{xy} = -\nu \frac{e^2}{h}, \quad (1.25)$$

where the integer  $\nu$  counts the number of filled LL's.

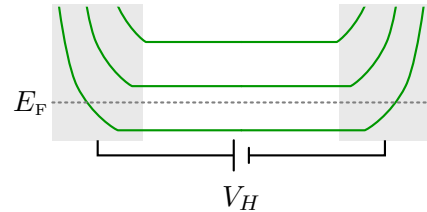


Figure 1.6: Voltage bias.

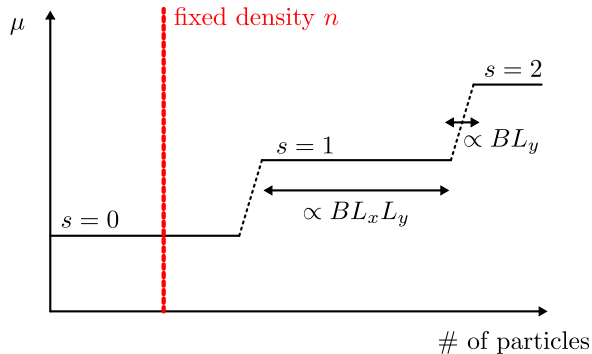


Figure 1.7: Chemical potential stuck to Landau levels.

### 1.5.1 The effect of disorder

The above result is strongly suggestive that one dimensional edge channels are responsible for the transport in the quantum Hall effect. Generically the current carried by a one-dimensional channel is given by

$$I = \frac{e^2}{h} |T|^2, \quad (1.26)$$

where  $|T|^2$  denotes the probability for an electron to be transmitted through a disordered region. However, our edge channels are *chiral* where the electrons have no way to be back-scattered and therefore  $|T|^2 = 1$ . These arguments explain why even in the case of a disordered sample  $\sigma_{xy}$  can be quantized. However, we did not yet reconcile a quantized  $\sigma_{xy}$  with the general result  $\sigma_{xy} = ne/B$  for a clean system.

We assumed the Fermi energy to lie between two Landau levels. Let us see under which conditions this can be the case. We assume the sample to be  $L_x$  wide and the edge region which is curved up to extend over the length  $W \ll L_x$ . From the finite size (or periodic) quantization in  $y$ -direction we know that the momentum can take the values  $k_i = \frac{2\pi}{L_y} i$  with  $i \in \mathbb{Z}$ . Hence, we find for the centers of the Localized wave functions  $X_i = \frac{2\pi}{L_y} l^2 i$ . We now count how many wave functions fit into the bulk and how many into the edge:

$$\text{edge} : \frac{W}{X_i - X_{i-1}} \propto L_y, \quad \text{bulk} : \frac{L_x}{X_i - X_{i-1}} \propto L_x L_y. \quad (1.27)$$

We see that there are extensively many bulk states but only a sub-extensive number of edge states as shown in Fig. 1.7.

Translated to a fixed density but varying magnetic field  $B$  we find that for almost all values of  $B$  the Fermi energy will lie in the bulk, not the edge! Meaning, our assumption that the we have a completely filled LL and the relevant physics is happening only on the edge was not justified. Hence we need to get a better understanding of disorder effects.

## References

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2. *Topological aspects of low dimensional systems* (eds Comtet, A., Jolicœur, T., Ouvry, S. & David, F.) (Springer-Verlag, Berlin, 1999).

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