# MANY-BODY THEORY OF NON-EQUILIBRIUM SYSTEMS 

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## 1. Introduction

### 1.1. Motivation and outline

These lectures are devoted to the Keldysh formalism for the treatment of out-of-equilibrium many-body systems. The name of the technique takes its origin from the 1964 paper of L. V. Keldysh [1]. Among the earlier approaches that are closely related to the Keldysh technique, one should mention J. Schwinger [2] and R. P. Feynman and F. L. Vernon [3]. The classical counterparts of the Keldysh technique are extremely useful and interesting on their own. Among them the Wyld diagrammatic technique [4] and the Martin-Siggia-Rose method [5] for stochastic systems.

There are a number of pedagogical presentations of the method [6-8]. The emphasis of these notes is on the functional integration approach. It makes the structure of the theory clearer and more transparent. The notes also cover modern applications such as the Usadel equation and the nonlinear $\sigma$-model. Great attention is devoted to exposing connections to other techniques such as the equilibrium Matsubara method and the classical Langevin and Fokker-Planck equations, as well as the Martin-Siggia-Rose technique.

The Keldysh formulation of the many-body theory is useful for the following tasks:

- Treatment of systems that are not in thermal equilibrium (either due to the presence of external fields, or in a transient regime) $[1,6,8]$.
- Calculation of the full counting statistics of a quantum mechanical observable (as opposed to an average value or correlators) $[9,10]$.
- As an alternative to the replica and the supersymmetry methods in the theory of disordered and glassy systems [11-15].
- Treatment of equilibrium problems, in which the Matsubara analytical continuation may prove to be cumbersome.

The outline of these lectures is as follows. The technique is introduced and explained for the simplest possible system, that of a single bosonic state (harmonic oscillator), which is later generalized to real (phonons), or complex (atoms) bosonic fields. Their action and Green functions are introduced in Chapter 2 Boson interactions, the diagrammatic technique and the quantum kinetic equation
are treated in Chapter 3 Chapter 4 is devoted to a bosonic particle in contact with a dissipative environment (bath). This example is used to establish connections with the classical methods (Langevin and Fokker-Planck) as well with the quantum equilibrium technique (Matsubara). Fermions and fermion-boson systems are treated in Chapter 5] Covered topics include the random phase approximation and the quantum kinetic equation. Non-interacting Fermions in the presence of quenched disorder are treated in Chapter 6 with the help of the Keldysh nonlinear $\sigma$-model.

### 1.2. Closed time contour

The standard construction of the zero temperature (or equilibrium) many-body theory (see e.g. [7,16]) involves the adiabatic switching "on" of interactions at a distant past, and "off" at a distant future. A typical correlation function has the form of a time ordered product of operators in the Heisenberg representation: $C\left(t, t^{\prime}\right) \equiv\langle 0| T \hat{A}(t) \hat{B}\left(t^{\prime}\right)|0\rangle$, where $|0\rangle$ is the ground-state (or thermal equilibrium state) of the interacting Hamiltonian, $\hat{H}$. This state is supposed to be given by $|0\rangle=\hat{S}(0,-\infty)| \rangle_{0}$, where $\left\rangle_{0}\right.$ is the (known) ground-state of the non-interacting Hamiltonian, $\hat{H}_{0}$, at $t=-\infty$. The $\hat{S}$-matrix operator $\hat{S}\left(t, t^{\prime}\right)=e^{i \hat{H}_{0} t} e^{-i \hat{H}\left(t-t^{\prime}\right)} e^{-i \hat{H}_{0} t^{\prime}}$ describes the evolution due to the interaction Hamiltonian, $\hat{H}-\hat{H}_{0}$, and is thus responsible for the adiabatic switching "on" of the interactions. An operator in the Heisenberg representation is given by $\hat{A}(t)=[\hat{S}(t, 0)]^{\dagger} \hat{\mathcal{A}}(t) \hat{S}(t, 0)=\hat{S}(0, t) \hat{\mathcal{A}}(t) \hat{S}(t, 0)$, where $\hat{\mathcal{A}}(t)$ is the operator in the interaction representation. As a result, the correlation function takes the form:

$$
\begin{align*}
C\left(t, t^{\prime}\right) & ={ }_{0}\langle | T \hat{S}(-\infty, 0) \hat{S}(0, t) \hat{\mathcal{A}}(t) \hat{S}\left(t, t^{\prime}\right) \hat{\mathcal{B}}\left(t^{\prime}\right) \hat{S}\left(t^{\prime}, 0\right) \hat{S}(0,-\infty)| \rangle_{0} \\
& =\frac{{ }_{0}\langle | T \hat{\mathcal{A}}(t) \hat{\mathcal{B}}\left(t^{\prime}\right) \hat{S}(\infty,-\infty)| \rangle_{0}}{{ }_{0}\langle | \hat{S}(\infty,-\infty)| \rangle_{0}} \tag{1.1}
\end{align*}
$$

where one employed: ${ }_{0}\langle | \hat{S}(-\infty, 0)=e^{-i L}{ }_{0}\langle | \hat{S}(\infty,-\infty) \hat{S}(-\infty, 0)$, and interchanged the order of operators, which is always allowed under the $T$-operation (time ordering). The idea is that, starting at $t=-\infty$ at the ground (or equilibrium) state, $\left\rangle_{0}\right.$, of the non-interacting system and then adiabatically switching interactions "on" and "off", one arrives at $t=+\infty$ at the state $|\infty\rangle$. The crucial assumption is that this state is unique, independent of the details of the switching procedure and is again the ground-state, up to a possible phase factor: $e^{i L}={ }_{0}\langle ||\infty\rangle={ }_{0}\langle | \hat{S}(\infty,-\infty)| \rangle_{0}$.

Clearly this is not the case out of equilibrium. Starting from some arbitrary non-equilibrium state and then switching interactions "on" and "off", the system


Fig. 1. The closed time contour $\mathcal{C}$. Dots on the forward and the backward branches of the contour denote discrete time points.
evolves to some unpredictable state. The latter depends, in general, on the peculiarities of the switching procedure. The entire construction sketched above fails since we have no knowledge of the final state.

One would like, thus, to build a theory that avoids references to the state at $t=+\infty$. Since traces are calculated, one still needs to know the final state. Schwinger's suggestion is to take the final state to be exactly the same as the initial one. The central idea is to let the quantum system evolve first in the forward direction in time and then to "unwind" its evolution backwards, playing the "movie" in the backward direction. One ends up, thus, with the need to construct a theory with the time evolution along the two-branch contour, $\mathcal{C}$, depicted on Fig. 1.2 Then, no matter what the state at $t=+\infty$ is, after the backward evolution the system returns back to the known initial state. As a result, the unitary evolution operator, $\hat{U}_{t, t^{\prime}} \equiv e^{-i H\left(t-t^{\prime}\right)}$, along such a closed time contour is always a unit operator:

$$
\begin{equation*}
\hat{U}_{\mathcal{C}} \equiv 1 \tag{1.2}
\end{equation*}
$$

In this construction there is no switching of interactions in the future. Both switchings "on" and "off" take place in the past: "on" - at the forward branch of the contour and "off" - at the backward one. This way the absence of information about the $t=+\infty$ state is bypassed. There is a price to pay for such luxury: a doubling of degrees of freedom. Indeed at every moment of time one needs to specify a field residing on the forward branch as well as on the backward branch of the contour. As a result, the algebraic structure of the theory is more complicated. The difficulties may be minimized, however, by a proper choice of variables based on the internal symmetries of the theory.

## 2. Free boson systems

### 2.1. Partition function

Let us consider the simplest possible many-body system: bosonic particles occupying a single quantum state with an energy $\omega_{0}$. It is completely equivalent, of course, to a harmonic oscillator. The secondary quantized Hamiltonian has the form:

$$
\begin{equation*}
\hat{H}=\omega_{0} a^{\dagger} a \tag{2.1}
\end{equation*}
$$

where $a^{\dagger}$ and $a$ are bosonic creation and annihilation operators with the commutation relation $\left[a, a^{\dagger}\right]=1$. Let us define "partition function" as:

$$
\begin{equation*}
Z=\frac{\operatorname{Tr}\left\{\hat{U}_{\mathcal{C}} \hat{\rho}\right\}}{\operatorname{Tr}\{\hat{\rho}\}} \tag{2.2}
\end{equation*}
$$

If one assumes that all external fields are exactly the same on the forward and backward branches of the contour, then $\hat{U}_{\mathcal{C}}=1$ and therefore $Z=1$.

The initial density matrix $\hat{\rho}=\hat{\rho}(\hat{H})$ is some operator-valued function of the Hamiltonian. To simplify the derivations one may choose it to be the equilibrium density matrix, $\hat{\rho}_{0}=\exp \{-\beta(\hat{H}-\mu \hat{N})\}=\exp \left\{-\beta\left(\omega_{0}-\mu\right) a^{\dagger} a\right\}$. Since arbitrary external perturbations may be switched on (and off) at a later time, the choice of the equilibrium initial density matrix does not prevent one from treating non-equilibrium dynamics. For the equilibrium initial density matrix:

$$
\begin{equation*}
\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}=\sum_{n=0}^{\infty} e^{-\beta\left(\omega_{0}-\mu\right) n}=\left[1-\rho\left(\omega_{0}\right)\right]^{-1} \tag{2.3}
\end{equation*}
$$

where $\rho\left(\omega_{0}\right)=e^{-\beta\left(\omega_{0}-\mu\right)}$. An important point is that, in general, $\operatorname{Tr}\{\hat{\rho}\}$ is an interaction- and disorder-independent constant. Indeed, both interactions and disorder are supposed to be switched on (and off) on the forward (backward)

Reminder: the bosonic coherent state $|\phi\rangle(\langle\phi|)$, parameterized by a complex number $\phi$, is defined as a right (left) eigenstate of the annihilation (creation) operator: $a|\phi\rangle=\phi|\phi\rangle\left(\langle\phi| a^{\dagger}=\langle\phi| \bar{\phi}\right)$. The matrix elements of a normally ordered operator, such as the Hamiltonian, take the form
$\langle\phi| \hat{H}\left(a^{\dagger}, a\right)\left|\phi^{\prime}\right\rangle=H\left(\bar{\phi}, \phi^{\prime}\right)\left\langle\phi \mid \phi^{\prime}\right\rangle$. The overlap between two coherent states is [17] $\left\langle\phi \mid \phi^{\prime}\right\rangle=\exp \left\{\bar{\phi} \phi^{\prime}\right\}$. Since the coherent state basis is overcomplete, the trace of an operator, $\hat{A}$, is calculated with the weight:
$\operatorname{Tr}\{\hat{A}\}=\pi^{-1} \iint d(\Re \phi) d(\Im \phi) e^{-|\phi|^{2}}\langle\phi| \hat{A}|\phi\rangle$.
parts of the contour sometime after (before) $t=-\infty$. This constant is, therefore, frequently omitted - it never causes a confusion.

The next step is to divide the $\mathcal{C}$ contour into $(2 N-2)$ time steps of length $\delta_{t}$, such that $t_{1}=t_{2 N}=-\infty$ and $t_{N}=t_{N+1}=+\infty$ as shown in Fig. 1.2 One then inserts the resolution of unity in the coherent state overcomplete basis [17]

$$
\begin{equation*}
1=\iint \frac{d\left(\Re \phi_{j}\right) d\left(\Im \phi_{j}\right)}{\pi} e^{-\left|\phi_{j}\right|^{2}}\left|\phi_{j}\right\rangle\left\langle\phi_{j}\right| \tag{2.4}
\end{equation*}
$$

at each point $j=1,2, \ldots, 2 N$ along the contour. For example, for $N=3$ one obtains the following sequence in the expression for $\operatorname{Tr}\left\{\hat{U}_{\mathcal{C}} \hat{\rho}_{0}\right\}$ (read from right to left):
$\left\langle\phi_{6}\right| \hat{U}_{-\delta_{t}}\left|\phi_{5}\right\rangle\left\langle\phi_{5}\right| \hat{U}_{-\delta_{t}}\left|\phi_{4}\right\rangle\left\langle\phi_{4}\right| \hat{1}\left|\phi_{3}\right\rangle\left\langle\phi_{3}\right| \hat{U}_{+\delta_{t}}\left|\phi_{2}\right\rangle\left\langle\phi_{2}\right| \hat{U}_{+\delta_{t}}\left|\phi_{1}\right\rangle\left\langle\phi_{1}\right| \hat{\rho}_{0}\left|\phi_{6}\right\rangle$,
where $\hat{U}_{ \pm \delta_{t}}$ is the evolution operator during the time interval $\delta_{t}$ in the positive (negative) time direction. Its matrix elements are given by:
$\left\langle\phi_{j+1}\right| \hat{U}_{ \pm \delta_{t}}\left|\phi_{j}\right\rangle \equiv\left\langle\phi_{j+1}\right| e^{\mp i \hat{H}\left(a^{\dagger}, a\right) \delta_{t}}\left|\phi_{j}\right\rangle \approx\left\langle\phi_{j+1} \mid \phi_{j}\right\rangle e^{\mp i H\left(\bar{\phi}_{j+1}, \phi_{j}\right) \delta_{t}}$,
where the last approximate equality is valid up to the linear order in $\delta_{t}$. Obviously this result is not restricted to the toy example, Eq 2.1, but holds for any normally-ordered Hamiltonian. Notice that there is no evolution operator inserted between $t_{N}$ and $t_{N+1}$. Indeed, these two points are physically indistinguishable and thus the system does not evolve during this time interval.

> | Exercise: show that $\langle\phi\| e^{\kappa a^{\dagger} a}\left\|\phi^{\prime}\right\rangle=\exp \left\{\bar{\phi} \phi^{\prime} e^{\kappa}\right\}$. Putting $\kappa=-\beta\left(\omega_{0}-\mu\right)$, one finds |
| :---: |
| $\left\langle\phi_{1}\right\| \hat{\rho}_{0}\left\|\phi_{2 N}\right\rangle=\exp \left\{\bar{\phi}_{1} \phi_{2 N} \rho\left(\omega_{0}\right)\right\}$. |

Combining all such matrix elements along the contour together with the exponential factors from the resolutions of unity, Eq. [2.4), one finds for the partition function (2.2):

$$
\begin{equation*}
Z=\frac{1}{\operatorname{Tr}\left\{\rho_{0}\right\}} \iint \prod_{j=1}^{2 N}\left[\frac{d\left(\Re \phi_{j}\right) d\left(\Im \phi_{j}\right)}{\pi}\right] e^{i \sum_{j, j^{\prime}=1}^{2 N} \bar{\phi}_{j} G_{j j^{\prime}}^{-1} \phi_{j^{\prime}}}, \tag{2.7}
\end{equation*}
$$

where the $2 N \times 2 N$ matrix $i G_{j j^{\prime}}^{-1}$ stands for:

$$
i G_{j j^{\prime}}^{-1} \equiv\left[\begin{array}{rrr|rrr}
-1 & & & & & \rho\left(\omega_{0}\right)  \tag{2.8}\\
1-h & -1 & & & & \\
& 1-h & -1 & & & \\
\hline & & 1 & -1 & & \\
& & & 1+h & -1 & \\
& & & & 1+h & -1
\end{array}\right]
$$

and $h \equiv i \omega_{0} \delta_{t}$. It is straightforward to evaluate the determinant of such a matrix

$$
\begin{equation*}
\operatorname{det}\left[i G^{-1}\right]=1-\rho\left(\omega_{0}\right)\left(1-h^{2}\right)^{N-1} \approx 1-\rho\left(\omega_{0}\right) e^{\left(\omega_{0} \delta_{t}\right)^{2}(N-1)} \rightarrow 1-\rho\left(\omega_{0}\right), \tag{2.9}
\end{equation*}
$$

where one has used that $\delta_{t}^{2} N \rightarrow 0$ if $N \rightarrow \infty$ (indeed, the assumption was $\delta_{t} N \rightarrow$ const). Employing Eqs. A. 1) and (2.3), one finds:

$$
\begin{equation*}
Z=\frac{\operatorname{det}^{-1}\left[i G^{-1}\right]}{\operatorname{Tr}\left\{\rho_{0}\right\}}=1 \tag{2.10}
\end{equation*}
$$

as it should be, of course. Notice, that keeping the upper-right element of the discrete matrix, Eq. (2.8), is crucial to maintain this normalization identity.

One may now take the limit $N \rightarrow \infty$ and formally write the partition function in the continuous notations, $\phi_{j} \rightarrow \phi(t)$ :

$$
\begin{equation*}
Z=\int \mathcal{D} \bar{\phi} \phi e^{i S[\bar{\phi}, \phi]}=\int \mathcal{D} \bar{\phi} \phi \exp \left\{i \int_{\mathcal{C}}\left[\bar{\phi}(t) G^{-1} \phi(t)\right] d t\right\} \tag{2.11}
\end{equation*}
$$

where according to Eqs. 2.7) and 2.8 the action is given by
$S[\bar{\phi}, \phi]=\sum_{j=2}^{2 N}\left[i \bar{\phi}_{j} \frac{\phi_{j}-\phi_{j-1}}{\delta t_{j}}-\omega_{0} \bar{\phi}_{j} \phi_{j-1}\right] \delta t_{j}+i \bar{\phi}_{1}\left(\phi_{1}-\rho\left(\omega_{0}\right) \phi_{2 N}\right)$
where $\delta t_{j} \equiv t_{j}-t_{j-1}= \pm \delta_{t}$. Thus a continuous form of the operator $G^{-1}$ is:

$$
\begin{equation*}
G^{-1}=i \partial_{t}-\omega_{0} \tag{2.13}
\end{equation*}
$$

It is important to remember that this continuous notation is only an abbreviation that represents the large discrete matrix, Eq. (2.8). In particular, the upper-right element of the matrix (the last term in Eq. (2.12), that contains the information about the distribution function, is seemingly absent in the continuous notations.

To avoid integration along the closed time contour, it is convenient to split the bosonic field $\phi(t)$ into the two components $\phi_{+}(t)$ and $\phi_{-}(t)$ that reside on
the forward and the backward parts of the time contour correspondingly. The continuous action may then be rewritten as

$$
\begin{equation*}
S=\int_{-\infty}^{\infty} d t \bar{\phi}_{+}(t)\left[i \partial_{t}-\omega_{0}\right] \phi_{+}(t)-\int_{-\infty}^{\infty} d t \bar{\phi}_{-}(t)\left[i \partial_{t}-\omega_{0}\right] \phi_{-}(t) \tag{2.14}
\end{equation*}
$$

where the relative minus sign comes from the reverse direction of the time integration on the backward part of the contour. Once again, the continuous notations are somewhat misleading. Indeed, they create an undue impression that the $\phi_{+}(t)$ and $\phi_{-}(t)$ fields are completely independent from each other. In fact, they are connected due to the presence of the non-zero off-diagonal blocks in the discrete matrix, Eq. (2.8).

### 2.2. Green functions

One would like to define the Green functions as:

$$
\begin{equation*}
G\left(t, t^{\prime}\right)=-i \int \mathcal{D} \bar{\phi} \phi e^{i S[\bar{\phi}, \phi]} \phi(t) \bar{\phi}\left(t^{\prime}\right) \equiv-i\left\langle\phi(t) \bar{\phi}\left(t^{\prime}\right)\right\rangle \tag{2.15}
\end{equation*}
$$

where both time arguments reside somewhere on the Keldysh contour. Notice, the absence of the factor $Z^{-1}$ in comparison with the analogous definition in the equilibrium theory [17]. Indeed, in the present construction $Z=1$. This seemingly minor difference turns out to be the major issue in the theory of disordered systems, Chapter 6

According to the general property of Gaussian integrals (see Appendix A, the Green function is the inverse of the correlator matrix $G^{-1}$, Eq. (2.8), standing in the quadratic action. Thus, one faces the unpleasant task of inverting the large $2 N \times 2 N$ matrix, Eq. 2.8. It may seem more attractive to invert the differential operator, Eq. (2.13). Such an inversion, however, is undefined due to the presence of the zero mode ( $\sim e^{-i \omega_{0} t}$ ). The necessary regularization is provided by the off-diagonal blocks of the discrete matrix. The goal is to develop a formalism that avoids dealing with the large discrete matrices and refers to the continuous notations only.

The easiest way to proceed is to recall [17] that the Green functions are traces of time-ordered products of the field operators (in the Heisenberg representation), where the ordering is done along the contour $\mathcal{C}$. Recalling also that the time arguments on the backward branch are always after those on the forward, one finds:
$\left\langle\phi_{+}(t) \bar{\phi}_{-}\left(t^{\prime}\right)\right\rangle \equiv i G^{<}\left(t, t^{\prime}\right)=\frac{\operatorname{Tr}\left\{a^{\dagger}\left(t^{\prime}\right) a(t) \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}}=\frac{\operatorname{Tr}\left\{e^{i \hat{H} t^{\prime}} a^{\dagger} e^{i \hat{H}\left(t-t^{\prime}\right)} a e^{-i \hat{H} t} \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}}$

$$
\begin{gather*}
=\frac{e^{-i \omega_{0}\left(t-t^{\prime}\right)}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}} \sum_{m=0}^{\infty} m\left[\rho\left(\omega_{0}\right)\right]^{m}=n e^{-i \omega_{0}\left(t-t^{\prime}\right)} ; \\
\left\langle\phi_{-}(t) \bar{\phi}_{+}\left(t^{\prime}\right)\right\rangle \equiv i G^{>}\left(t, t^{\prime}\right)=\frac{\operatorname{Tr}\left\{a(t) a^{\dagger}\left(t^{\prime}\right) \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}}=\frac{\operatorname{Tr}\left\{e^{i \hat{H} t} a e^{i \hat{H}\left(t^{\prime}-t\right)} a^{\dagger} e^{-i \hat{H} t^{\prime}} \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}} \\
=\frac{e^{i \omega_{0}\left(t^{\prime}-t\right)}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}} \sum_{m=0}^{\infty}(m+1)\left[\rho\left(\omega_{0}\right)\right]^{m}=(n+1) e^{-i \omega_{0}\left(t-t^{\prime}\right)} ; \\
\left\langle\phi_{+}(t) \bar{\phi}_{+}\left(t^{\prime}\right)\right\rangle \equiv i G^{T}\left(t, t^{\prime}\right)=\frac{\operatorname{Tr}\left\{T\left[a(t) a^{\dagger}\left(t^{\prime}\right)\right] \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}}  \tag{2.16}\\
\quad=\theta\left(t-t^{\prime}\right) i G^{>}\left(t, t^{\prime}\right)+\theta\left(t^{\prime}-t\right) i G^{<}\left(t, t^{\prime}\right) ; \\
\left\langle\phi_{-}(t) \bar{\phi}_{-}\left(t^{\prime}\right)\right\rangle \equiv i G^{\tilde{T}}\left(t, t^{\prime}\right)=\frac{\operatorname{Tr}\left\{\tilde{T}\left[a(t) a^{\dagger}\left(t^{\prime}\right)\right] \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}} \\
\quad=\theta\left(t^{\prime}-t\right) i G^{>}\left(t, t^{\prime}\right)+\theta\left(t-t^{\prime}\right) i G^{<}\left(t, t^{\prime}\right) ;
\end{gather*}
$$

where the symbols $T$ and $\tilde{T}$ denote time-ordering and anti-time-ordering correspondingly. Hereafter the time arguments reside on the open time axis $t \in]-$ $\infty, \infty\left[\right.$. The Planck occupation number $n$ stands for $n\left(\omega_{0}\right) \equiv \rho\left(\omega_{0}\right) /\left(1-\rho\left(\omega_{0}\right)\right)$.

Notice the presence of non-zero off-diagonal Green functions $\left\langle\phi_{-} \bar{\phi}_{+}\right\rangle$and $\left\langle\phi_{+} \bar{\phi}_{-}\right\rangle$. This is seemingly inconsistent with the continuous action, Eq. 2.14. This is due to the presence of the off-diagonal blocks in the discrete matrix, that are lost in the continuous notations. The existence of the off-diagonal Green functions does not contradict to continuous notations. Indeed, $\left[i \partial_{t}-\omega_{0}\right] G^{>,<}=$ 0 , while $\left[i \partial_{t}-\omega_{0}\right] G^{T, \tilde{T}}= \pm \delta\left(t-t^{\prime}\right)$. Therefore in the obvious $2 \times 2$ matrix notations $G^{-1} \circ G=1$, as it should be. The point is that the inverse of the operator $\left[i \partial_{t}-\omega_{0}\right]$ is not well-defined (due to the presence of the eigenmode ( $\sim \exp \left\{-i \omega_{0} t\right\}$ ) with zero eigenvalue). A regularization must be specified and the off-diagonal blocks of the discrete matrix do exactly this.

The $\theta$-function in Eq. 2.16 is the usual Heaviside step function. There is an uncertainty, however, regarding its value at coinciding time arguments. To resolve it, one needs to refer to the discrete representation one last time. Since the fields $\bar{\phi}$ always appear one time step $\delta_{t}$ after the fields $\phi$ on the Keldysh contour, cf. Eq. 2.6, the proper convention is:

$$
\begin{equation*}
G^{T}(t, t)=G^{\tilde{T}}(t, t)=G^{<}(t, t)=n \tag{2.17}
\end{equation*}
$$

Obviously not all four Green functions defined above are independent. Indeed, a direct inspection shows that for $t \neq t^{\prime}$ :

$$
\begin{equation*}
G^{T}+G^{\tilde{T}}=G^{>}+G^{<} . \tag{2.18}
\end{equation*}
$$

One would like therefore to perform a linear transformation of the fields to benefit explicitly from this relation. This is achieved by the Keldysh rotation.

### 2.3. Keldysh rotation

Define new fields as

$$
\begin{equation*}
\phi_{c l}(t)=\frac{1}{\sqrt{2}}\left(\phi_{+}(t)+\phi_{-}(t)\right) ; \quad \phi_{q}(t)=\frac{1}{\sqrt{2}}\left(\phi_{+}(t)-\phi_{-}(t)\right) \tag{2.19}
\end{equation*}
$$

with the analogous transformation for the conjugated fields. The subscripts " $c l$ " and " $q$ " stand for the classical and the quantum components of the fields correspondingly. The rationale for these notations will become clear shortly. First, a simple algebraic manipulation of Eq. 2.16) shows that

$$
-i\left\langle\phi_{\alpha}(t) \bar{\phi}_{\beta}\left(t^{\prime}\right)\right\rangle \equiv \hat{G}^{\alpha \beta}=\left(\begin{array}{cc}
G^{K}\left(t, t^{\prime}\right) & G^{R}\left(t, t^{\prime}\right)  \tag{2.20}\\
G^{A}\left(t, t^{\prime}\right) & 0
\end{array}\right)
$$

where hereafter $\alpha, \beta=(c l, q)$. The cancellation of the $(q, q)$ element of this matrix is a manifestation of identity 2.18. Superscripts $R, A$ and $K$ stand for the retarded, advanced and Keldysh components of the Green function correspondingly. These three Green functions are the fundamental objects of the Keldysh technique. They are defined as

$$
\begin{align*}
G^{R}\left(t, t^{\prime}\right) & =\frac{1}{2}\left(G^{T}-G^{\tilde{T}}-G^{<}+G^{>}\right)=\theta\left(t-t^{\prime}\right)\left(G^{>}-G^{<}\right) \\
G^{A}\left(t, t^{\prime}\right) & =\frac{1}{2}\left(G^{T}-G^{\tilde{T}}+G^{<}-G^{>}\right)=\theta\left(t^{\prime}-t\right)\left(G^{<}-G^{>}\right)  \tag{2.21}\\
G^{K}\left(t, t^{\prime}\right) & =\frac{1}{2}\left(G^{T}+G^{\tilde{T}}+G^{>}+G^{<}\right)=G^{<}+G^{>}
\end{align*}
$$

In the discrete representation each of these three Green functions is represented by an $N \times N$ matrix. Since both $G^{<}$and $G^{>}$are, by definition, antiHermitian (cf. Eq. 2.16), Eq. 2.21) implies:

$$
\begin{equation*}
G^{A}=\left[G^{R}\right]^{\dagger} \quad G^{K}=-\left[G^{K}\right]^{\dagger} \tag{2.22}
\end{equation*}
$$

where the Hermitian conjugation includes complex conjugation, as well as transposition of the time arguments. The retarded (advanced) Green function a is lower (upper) triangular matrix. Due to the algebra of triangular matrices, a product of any number of upper (lower) triangular matrices is again an upper (lower) triangular matrix. This leads to the simple rule:

$$
\begin{align*}
G_{1}^{R} \circ G_{2}^{R} \circ \ldots \circ G_{l}^{R} & =G^{R} \\
G_{1}^{A} \circ G_{2}^{A} \circ \ldots \circ G_{l}^{A} & =G^{A}, \tag{2.23}
\end{align*}
$$



Fig. 2. Graphic representation of $G^{R}, G^{A}$, and $G^{K}$ correspondingly. The full line represents the classical field component, $\phi_{c l}$, while the dashed line - the quantum component, $\phi_{q}$.
where the circular multiplication signs are understood as integrations over intermediate times (discrete matrix multiplication). At coinciding time arguments, one finds (cf. Eqs. 2.17) and 2.21):

$$
\begin{equation*}
G^{R}(t, t)+G^{A}(t, t)=0 \tag{2.24}
\end{equation*}
$$

Although in the discrete representation both $G^{R}$ and $G^{A} d o$ contain non-zero (pure imaginary, due to Eqs. (2.22), (2.24) main diagonals (otherwise the matrix $\hat{G}$ is not invertible), the proper continuous convention is: $\theta(0)=0$. The point is that in any diagrammatic calculation, $G^{R}(t, t)$ and $G^{A}(t, t)$ always come in symmetric combinations and cancel each other due to Eq. 2.24. It is thus a convenient and noncontradictory agreement to take $\theta(0)=0$.

It is useful to introduce graphic representations for the three Green functions. To this end, let us denote the classical component of the field by a full line and the quantum component by a dashed line. Then the retarded Green function is represented by a full-arrow-dashed line, the advanced by a dashed-arrow-full line and the Keldysh by full-arrow-full line, see Fig. 2.3 Notice, that the dashed-arrow-dashed line, that would represent the $\left\langle\phi_{q} \bar{\phi}_{q}\right\rangle$ Green function, is identically zero due to identity (2.18). The arrow shows the direction from $\phi_{\alpha}$ towards $\bar{\phi}_{\beta}$.

For the single bosonic state (cf. Eq. (2.16)): $G^{>}=-i(n+1) e^{-i \omega_{0}\left(t-t^{\prime}\right)}$ and $G^{<}=-i n e^{-i \omega_{0}\left(t-t^{\prime}\right)}$, where $n=n\left(\omega_{0}\right)=\rho\left(\omega_{0}\right) /\left(1-\rho\left(\omega_{0}\right)\right)$ is the Planck occupation number (since the system is non-interacting the initial distribution function does not evolve). Therefore:

$$
\begin{align*}
G^{R}\left(t, t^{\prime}\right) & =-i \theta\left(t-t^{\prime}\right) e^{-i \omega_{0}\left(t-t^{\prime}\right)} \\
G^{A}\left(t, t^{\prime}\right) & =i \theta\left(t^{\prime}-t\right) e^{-i \omega_{0}\left(t-t^{\prime}\right)}  \tag{2.25}\\
G^{K}\left(t, t^{\prime}\right) & =-i\left(1+2 n\left(\omega_{0}\right)\right) e^{-i \omega_{0}\left(t-t^{\prime}\right)}
\end{align*}
$$

Notice that the retarded and advanced components contain information only about the spectrum and are independent of the occupation number, whereas the Keldysh
component does depend on it. Such a separation is common for systems that are not too far from thermal equilibrium. Fourier transforming with respect to $\left(t-t^{\prime}\right)$ to the energy representation, one finds:

$$
\begin{align*}
& G^{R(A)}(\epsilon)=\left(\epsilon-\omega_{0} \pm i 0\right)^{-1} ;  \tag{2.26}\\
& G^{K}(\epsilon)=\left(1+2 n\left(\omega_{0}\right)\right)(-2 \pi i) \delta\left(\epsilon-\omega_{0}\right)=(1+2 n(\epsilon))(-2 \pi i) \delta\left(\epsilon-\omega_{0}\right)
\end{align*}
$$

Therefore for the case of thermal equilibrium, one notices therefore that

$$
\begin{equation*}
G^{K}(\epsilon)=\operatorname{coth} \frac{\epsilon}{2 T}\left(G^{R}(\epsilon)-G^{A}(\epsilon)\right) \tag{2.27}
\end{equation*}
$$

The last equation constitutes the statement of the fluctuation-dissipation theorem (FDT). As it is shown below, the FDT is a general property of thermal equilibrium that is not restricted to the toy example, considered here. It implies the rigid relation between the response and correlation functions.

In general, it is convenient to parameterize the anti-Hermitian (see Eq. 2.22) Keldysh Green function by a Hermitian matrix $F=F^{\dagger}$, as:

$$
\begin{equation*}
G^{K}=G^{R} \circ F-F \circ G^{A} \tag{2.28}
\end{equation*}
$$

where $F=F\left(t^{\prime}, t^{\prime \prime}\right)$ and the circular multiplication sign implies integration over the intermediate time (matrix multiplication). The Wigner transform (see below), $f(\tau, \epsilon)$, of the matrix $F$ is referred to as the distribution function. In thermal equilibrium: $f(\epsilon)=\operatorname{coth}(\epsilon / 2 T)$.

### 2.4. Keldysh action and causality

The Keldysh rotation from the ( $\phi_{+}, \phi_{-}$) field components to ( $\phi_{c l}, \phi_{q}$ ) considerably simplifies the structure of the Green functions (cf. Eqs. (2.16) and 2.20). It is convenient, therefore, to write the action in terms of $\phi_{c l}, \phi_{q}$ as well. A simple way of doing it is to apply the Keldysh rotation, Eq. 2.19, to the continuous action, Eq. 2.14, written in terms of $\phi_{+}, \phi_{-}$. However, as was discussed above, the continuous action, Eq. 2.14, loses the crucial information about the offdiagonal blocks of the discrete matrix, Eq. (2.8). To keep this information, one may invert the matrix of Green functions, Eq. (2.20), and use the result as the correlator in the quadratic action. The inversion is straightforward:

$$
\hat{G}^{-1}=\left(\begin{array}{cc}
G^{K} & G^{R}  \tag{2.29}\\
G^{A} & 0
\end{array}\right)^{-1}=\left(\begin{array}{cc}
0 & {\left[G^{-1}\right]^{A}} \\
{\left[G^{-1}\right]^{R}} & {\left[G^{-1}\right]^{K}}
\end{array}\right)
$$

where the three components of the inverted Green function, labelled in advance as $A, R$ and $K$, satisfy:

$$
\begin{equation*}
\left[G^{-1}\right]^{R(A)}=\left[G^{R(A)}\right]^{-1}=i \partial_{t}-\omega_{0} \pm i 0 \tag{2.30}
\end{equation*}
$$

$$
\left[G^{-1}\right]^{K}=-\left[G^{R}\right]^{-1} \circ G^{K} \circ\left[G^{A}\right]^{-1}=\left[G^{R}\right]^{-1} \circ F-F \circ\left[G^{A}\right]^{-1}
$$

where parameterization 2.28 was employed in the last line. It is easy to see that $\left[G^{R}\right]^{-1}$ and $\left[G^{A}\right]^{-1}$ are lower and upper triangular matrices correspondingly, thus justifying their superscripts. The continuous notations may create an impression that $\left[G^{-1}\right]^{K}=(2 i 0) F$ and thus may be omitted. One should remember, however, that this component is non-zero in the discrete form and therefore it is important to acknowledge its existence (even if it is not written explicitly).

Once the correlator Eqs. 2.29, 2.30 is established, one may immediately write down the corresponding action:

$$
S\left[\phi_{c l}, \phi_{q}\right]=\iint_{-\infty}^{\infty} d t d t^{\prime}\left(\bar{\phi}_{c l}, \bar{\phi}_{q}\right)_{t}\left(\begin{array}{cc}
0 & {\left[G^{A}\right]^{-1}} \\
{\left[G^{R}\right]^{-1}} & {\left[G^{-1}\right]^{K}}
\end{array}\right)_{t, t^{\prime}}\binom{\phi_{c l}}{\phi_{q}}_{t^{\prime}},(2.31)
$$

where it is acknowledged that the correlator is, in general, a non-local function of time. The Green functions, Eq. 2.20, follow from the Gaussian integral with this action, by construction. Notice that the presence of $\left[G^{-1}\right]^{K}=(2 i 0) F$ (with a positive imaginary part) is absolutely necessary for the convergence of the corresponding functional integral.

The structure of the Gaussian action given by Eq. (2.31) is very general and encodes regularization of the functional integral. Since the Keldysh component carries the information about the density matrix, there is no further need to recall the discrete representation. The main features of this structure are:

- The $c l-c l$ component is zero.

This zero may be traced back to identity 2.18. It has, however, a much simpler interpretation. It reflects the fact that for a pure classical field configuration ( $\phi_{q}=$ $0)$ the action is zero. Indeed, in this case $\phi_{+}=\phi_{-}$and the action on the forward part of the contour is cancelled exactly by that on the backward part. The very general statement is, therefore, that

$$
\begin{equation*}
S\left[\phi_{c l}, \phi_{q}=0\right]=0 \tag{2.32}
\end{equation*}
$$

Obviously Eq. 2.32) is not restricted to a Gaussian action.

- The $c l-q$ and $q-c l$ components are mutually Hermitian conjugated upper and lower (advanced and retarded) triangular matrices in the time representation. This property is responsible for the causality of the response functions as well as for protecting the $\mathrm{cl}-\mathrm{cl}$ component from a perturbative renormalization (see below).
- The $q-q$ component is an anti-Hermitian matrix (cf. Eq. 2.22) with a positive-definite imaginary spectrum. It is responsible for the convergence of the functional integral. It also keeps the information about the distribution function.

As was already mentioned, these three items are generic and reproduce themselves in every order of perturbation theory. For the lack of a better terminology, we'll refer to them as the causality structure.

### 2.5. Free bosonic fields

It is a straightforward matter to generalize the entire construction to bosonic systems with more than one state. Suppose the states are labelled by an index $k$, that may be, e.g., a momentum vector. Their energies are given by a function $\omega_{k}$, for example $\omega_{k}=k^{2} /(2 m)$, where $m$ is the mass of the bosonic atoms. One introduces then a doublet of complex fields (classical and quantum) for every state $k:\left(\phi_{c l}(t ; k), \phi_{q}(t ; k)\right)$ and writes down the action in the form of Eq. (2.31) including a summation over the index $k$. Away from equilibrium, the Keldysh component may be non-diagonal in the index $k: F=F\left(t, t^{\prime} ; k, k^{\prime}\right)$. The retarded (advanced) component, on the other hand, has the simple form $\left[G^{R(A)}\right]^{-1}=i \partial_{t}-\omega_{k}$.

If $k$ is momentum, it is more instructive to Fourier transform to real space and to deal with $\left(\phi_{c l}(t ; r), \phi_{q}(t ; r)\right)$. Introducing a combined time-space index $x=(t ; r)$, one may write down for the action of the free complex bosonic field (atoms):

$$
S_{0}=\iint d x d x^{\prime}\left(\bar{\phi}_{c l}, \bar{\phi}_{q}\right)_{x}\left(\begin{array}{cc}
0 & {\left[G^{A}\right]^{-1}}  \tag{2.33}\\
{\left[G^{R}\right]^{-1}} & {\left[G^{-1}\right]^{K}}
\end{array}\right)_{x, x^{\prime}}\binom{\phi_{c l}}{\phi_{q}}_{x^{\prime}}
$$

where in the continuous notations

$$
\begin{equation*}
\left[G^{R(A)}\right]^{-1}\left(x, x^{\prime}\right)=\delta\left(x-x^{\prime}\right)\left(i \partial_{t^{\prime}}+\frac{1}{2 m} \nabla_{r^{\prime}}^{2}\right) \tag{2.34}
\end{equation*}
$$

while in the discrete form it is a lower (upper) triangular matrix in time (not in space). The $\left[G^{-1}\right]^{K}$ component for the free field is only the regularization factor, originating from the (time) boundary terms. It is, in general, non-local in $x$ and $x^{\prime}$, however, being a pure boundary term it is frequently omitted. It is kept here as a reminder that the inversion, $\hat{G}$, of the correlator matrix must posses the causality structure, Eq. (2.20).

In an analogous way, the action of free real bosons (phonons) is (cf. Eq. (B.9)):

$$
S_{0}=\iint d x d x^{\prime}\left(\varphi_{c l}, \varphi_{q}\right)_{x}\left(\begin{array}{cc}
0 & {\left[D^{A}\right]^{-1}}  \tag{2.35}\\
{\left[D^{R}\right]^{-1}} & {\left[D^{-1}\right]^{K}}
\end{array}\right)_{x, x^{\prime}}\binom{\varphi_{c l}}{\varphi_{q}}_{x^{\prime}}
$$

where

$$
\begin{equation*}
\left[D^{R(A)}\right]^{-1}\left(x, x^{\prime}\right)=\delta\left(x-x^{\prime}\right)\left(-\partial_{t^{\prime}}^{2}+v_{s}^{2} \nabla_{r^{\prime}}^{2}\right), \tag{2.36}
\end{equation*}
$$

in the continuous notations. In the discrete representations $\left[D^{R(A)}\right]^{-1}$ are again the lower (upper) triangular matrices. Here too the Keldysh component, $\left[D^{-1}\right]^{K}$, here too is just a regularization, originating from the (time) boundary terms. It is kept in Eq. 2.35) to emphasize the casuality structure of the real boson Green function $\hat{D}\left(x, x^{\prime}\right)$, analogous to Eq. 2.20):

$$
\hat{D}\left(x, x^{\prime}\right)=\left(\begin{array}{cc}
D^{K} & D^{R}  \tag{2.37}\\
D^{A} & 0
\end{array}\right) ; \quad \begin{aligned}
& D^{R(A)}(\epsilon, k)=\left((\epsilon \pm i 0)^{2}-v_{s}^{2} k^{2}\right)^{-1} \\
& D^{K}=D^{R} \circ F-F \circ D^{A}
\end{aligned}
$$

where $F=F\left(t, t^{\prime} ; r, r^{\prime}\right)$ is a symmetric distribution function matrix.

## 3. Collisions and kinetic equation

### 3.1. Interactions

The short range two-body collisions of bosonic atoms are described by the local "four-boson" Hamiltonian: $H_{\text {int }}=\lambda \sum_{r} a_{r}^{\dagger} a_{r}^{\dagger} a_{r} a_{r}$, where index $r$ "numerates" spatial locations. The interaction constant, $\lambda$, is related to a commonly used $s-$ wave scattering length, $a_{s}$, as $\lambda=4 \pi a_{s} / m$ [18]. The corresponding term in the continuous Keldysh action takes the form:

$$
\begin{equation*}
S_{\text {int }}=-\lambda \int d r \int_{\mathcal{C}} d t(\bar{\phi} \phi)^{2}=-\lambda \int d r \int_{-\infty}^{\infty} d t\left[\left(\bar{\phi}_{+} \phi_{+}\right)^{2}-\left(\bar{\phi}_{-} \phi_{-}\right)^{2}\right] . \tag{3.1}
\end{equation*}
$$

It is important to remember that there are no interactions in the distant past, $t=-\infty$ (while they are present in the future, $t=+\infty$ ). The interactions are supposed to be adiabatically switched on and off on the forward and backward branches correspondingly. That guarantees that the off-diagonal blocks of the matrix, Eq. 2.8, remain intact. Interactions modify only those matrix elements of the evolution operator, Eq. 2.6, that are away from $t=-\infty$. It is also worth remembering that in the discrete time form the $\bar{\phi}$ fields are taken one time step $\delta_{t}$ after the $\phi$ fields along the Keldysh contour $\mathcal{C}$. Therefore the two terms on the r.h.s. of the last equation should be understood as $\left(\bar{\phi}_{+}\left(t+\delta_{t}\right) \phi_{+}(t)\right)^{2}$ and $\left(\bar{\phi}_{-}(t) \phi_{-}\left(t+\delta_{t}\right)\right)^{2}$ correspondingly. Performing the Keldysh rotation, Eq. 2.19, one finds

$$
\begin{equation*}
S_{i n t}\left[\phi_{c l}, \phi_{q}\right]=-\lambda \int_{-\infty}^{\infty} d t\left[\bar{\phi}_{q} \bar{\phi}_{c l}\left(\phi_{c l}^{2}+\phi_{q}^{2}\right)+c . c .\right] \tag{3.2}
\end{equation*}
$$



Fig. 3. Graphic representation of the two interaction vertexes of the $|\phi|^{4}$ theory. There are also two conjugated vertexes with a reversed direction of all arrows.
where $c . c$. stands for the complex conjugate of the first term. The collision action, Eq. (3.2), obviously satisfies the causality condition, Eq. (2.32). Diagrammatically the action (3.2) generates two types of vertexes depicted in Fig. 3.1(as well as two complex conjugated vertexes, obtained by reversing the direction of the arrows): one with three classical fields (full lines) and one quantum field (dashed line) and the other with one classical field and three quantum fields.

Let us demonstrate that adding the collision term to the action does not violate the fundamental normalization, $Z=1$. To this end one may expand $e^{i S_{\text {int }}}$ in powers of $\lambda$ and then average term by term with the Gaussian action, Eq. (2.33). To show that the normalization, $Z=1$, is not altered by the collisions, one needs to show that $\left\langle S_{i n t}\right\rangle=\left\langle S_{i n t}^{2}\right\rangle=\ldots=0$. Applying the Wick theorem, one finds for the terms that are linear order in $\lambda:\left\langle\bar{\phi}_{q} \bar{\phi}_{c l} \phi_{c l}^{2}+c . c.\right\rangle \sim$ $\left[G^{R}(t, t)+G^{A}(t, t)\right] G^{K}(t, t)=0$, and $\left\langle\bar{\phi}_{q} \bar{\phi}_{c l} \phi_{q}^{2}+c . c\right\rangle=0$. The first term vanishes due to identity (2.24), while the second one vanishes because $\left\langle\phi_{q} \bar{\phi}_{q}\right\rangle=0$. There are two families of terms that are second order in $\lambda$. The first one is $\left\langle\bar{\phi}_{q} \bar{\phi}_{c l} \phi_{c l}^{2} \phi_{q}^{\prime} \phi_{c l}^{\prime}\left(\bar{\phi}_{c l}^{\prime}\right)^{2}\right\rangle \sim G^{R}\left(t^{\prime}, t\right) G^{A}\left(t^{\prime}, t\right)\left[G^{K}\left(t, t^{\prime}\right)\right]^{2}$, while the second is $\left\langle\bar{\phi}_{q} \bar{\phi}_{c l} \phi_{c l}^{2} \phi_{q}^{\prime} \phi_{c l}^{\prime}\left(\bar{\phi}_{q}^{\prime}\right)^{2}\right\rangle \sim\left[G^{R}\left(t, t^{\prime}\right)\right]^{2} G^{R}\left(t^{\prime}, t\right) G^{A}\left(t^{\prime}, t\right)$, where $\phi_{\alpha}^{\prime} \equiv \phi_{\alpha}\left(t^{\prime}\right)$. Both of these terms are zero, because $G^{R}\left(t^{\prime}, t\right) \sim \theta\left(t^{\prime}-t\right)$, while $G^{A}\left(t^{\prime}, t\right) \sim$ $G^{R}\left(t, t^{\prime}\right)^{*} \sim \theta\left(t-t^{\prime}\right)$ and thus their product has no support ${ }^{1}$. It is easy to see that, for exactly the same reasons, all higher order terms vanish and thus the normalization is unmodified (at least in a perturbative expansion).

[^0]

Fig. 4. Graphic representation of the two interaction vertexes of the $\varphi^{3}$ theory. Notice the relative factor of one third between them.

As another example, consider the real boson field, Eq. (2.35), with the cubic nonlinearity:
$S_{\text {int }}=\frac{\kappa}{6} \int d r \int_{\mathcal{C}} d t \varphi^{3}=\frac{\kappa}{6} \int d r \int_{-\infty}^{\infty} d t\left[\varphi_{+}^{3}-\varphi_{-}^{3}\right]=\kappa \int d r \int_{-\infty}^{\infty} d t\left[\varphi_{c l}^{2} \varphi_{q}+\frac{1}{3} \varphi_{q}^{3}\right]$
The causality condition, Eq. 2.32, is satisfied again. Diagrammatically the cubic nonlinearity generates two types of vertexes, Fig. 3.1 one with two classical fields (full lines) and one quantum field (dashed line), and the other with three quantum fields. The former vortex carries the factor $\kappa$, while the latter has a weight of $\kappa / 3$. Notice that for a real field the direction of the lines is not specified by arrows.

Exercise: Show that there are no corrections of second order in $\kappa$ to the partition function, $Z=1$. Check, that the same is true for the higher orders, as well.

### 3.2. Saddle point equations

Before developing the perturbation theory further, one has to discuss the saddle points of the action. According to Eq. 2.32, there are no terms in the action
that have zero power of both $\bar{\phi}_{q}$ and $\phi_{q}$. The same is obviously true regarding $\delta S / \delta \bar{\phi}_{c l}$ and therefore one of the saddle point equations:

$$
\begin{equation*}
\frac{\delta S}{\delta \bar{\phi}_{c l}}=0 \tag{3.4}
\end{equation*}
$$

may always be solved by

$$
\begin{equation*}
\phi_{q}=0 \tag{3.5}
\end{equation*}
$$

irrespectively of what the classical component, $\phi_{c l}$, is. One may check that this is indeed the case for the action given by Eqs. (2.33) plus (3.2). Under condition (3.5) the second saddle point equation takes the form:

$$
\begin{equation*}
\frac{\delta S}{\delta \bar{\phi}_{q}}=\left(\left[G^{R}\right]^{-1}-\lambda\left|\phi_{c l}\right|^{2}\right) \phi_{c l}=\left(i \partial_{t}+\frac{\nabla_{r}^{2}}{2 m}-\lambda\left|\phi_{c l}\right|^{2}\right) \phi_{c l}=0 \tag{3.6}
\end{equation*}
$$

This is the non-linear time-dependent (Gross-Pitaevskii) equation [18], that uniquely determines the classical field configuration, provided some initial and boundary conditions are specified.

The message is that among the possible solutions of the saddle-point equations for the Keldysh action, there is always one with a zero quantum component and with a classical component that obeys the classical (non-linear) equations of motion. We shall call such a saddle point - "classical". Thanks to Eqs. 2.32 and (3.5), the action at the classical saddle-point field configurations is identically zero. As was argued above, the perturbative expansion in small fluctuations around the classical saddle point leads to a properly normalized partition function, $Z=1$. This seemingly excludes the possibility of having any other saddle points. Yet, this conclusion is premature. The system may posses "non-classical" saddle points - such that $\phi_{q} \neq 0$. Such saddle points do not contribute to the partition function (and thus do not alter the fundamental normalization, $Z=1$ ), however, they may contribute to the correlation functions. In general, the action at a non-classical saddle point is non-zero. Its contribution is thus associated with exponentially small (or oscillatory) terms. Examples include: tunnelling, thermal activation (considered in the next chapter), Wigner-Dyson level statistics, etc.

Let us develop now a systematic perturbative expansion in deviations from the classical saddle point. As was discussed above, it does not bring any new information about the partition function. It does, however, provide information about the Green functions (and thus various observables). Most notably, it generates the kinetic equation for the distribution function. To simplify further consideration, let us assume that $\phi_{c l}=0$ is the proper solution of the classical saddle-point equation (3.6 (i.e. there is no Bose condensate).

### 3.3. Dyson equation

The goal is to calculate the dressed Green function, defined as:

$$
\begin{equation*}
\mathbf{G}^{\alpha \beta}\left(t, t^{\prime}\right)=-i \int \mathcal{D} \bar{\phi} \phi e^{i\left(S_{0}+S_{i n t}\right)} \phi_{\alpha}(t) \bar{\phi}_{\beta}\left(t^{\prime}\right) \tag{3.7}
\end{equation*}
$$

where $\alpha, \beta=(c l, q)$ and the action is given by Eqs. 2.33) and 3.2) (or for real bosons: Eqs. 2.35) and (3.3), with $\phi \rightarrow \varphi$ ). To this end one may expand the exponent in deviations from the classical saddle point: $\phi_{q} \equiv 0$ and (in the simplest case) $\phi_{c l}=0$. The functional integration with the remaining Gaussian action is then performed using the Wick theorem. This leads to the standard diagrammatic series. Combining all one-particle irreducible diagrams into the self-energy matrix $\hat{\Sigma}$, one obtains:

$$
\begin{equation*}
\hat{\mathbf{G}}=\hat{G}+\hat{G} \circ \hat{\Sigma} \circ \hat{G}+\hat{G} \circ \hat{\Sigma} \circ \hat{G} \circ \hat{\Sigma} \circ \hat{G}+\ldots=\hat{G} \circ(\hat{1}+\hat{\Sigma} \circ \hat{\mathbf{G}}) \tag{3.8}
\end{equation*}
$$

where $\hat{G}$ is given by Eq. 2.20 and the circular multiplication sign implies integrations over intermediate times and coordinates as well as a $2 \times 2$ matrix multiplication. The only difference compared with the text-book [17] diagrammatic expansion is the presence of the $2 \times 2$ Keldysh matrix structure. The fact that the series is arranged as a sequence of matrix products is of no surprise. Indeed, the Keldysh index, $\alpha=(c l, q)$, is just one more index in addition to time, space, spin, etc. Therefore, as with any other index, there is a summation (integration) over all of its intermediate values, hence the matrix multiplication. The concrete form of the self-energy matrix, $\hat{\Sigma}$, is of course specific to the Keldysh technique and is discussed below in some details.

Multiplying both sides of Eq. 3.8 by $\hat{G}^{-1}$ from the left, one obtains the Dyson equation for the exact dressed Green function, $\hat{\mathbf{G}}$ :

$$
\begin{equation*}
\left(\hat{G}^{-1}-\hat{\Sigma}\right) \circ \hat{\mathbf{G}}=\hat{1} \tag{3.9}
\end{equation*}
$$

where $\hat{1}$ is the unit matrix. The very non-trivial feature of the Keldysh technique is that the self energy matrix, $\hat{\Sigma}$, possesses the same causality structure as $\hat{G}^{-1}$, Eq. (2.29):

$$
\hat{\Sigma}=\left(\begin{array}{cc}
0 & \Sigma^{A}  \tag{3.10}\\
\Sigma^{R} & \Sigma^{K}
\end{array}\right)
$$

where $\Sigma^{R(A)}$ are lower (upper) triangular matrices in the time domain, while $\Sigma^{K}$ is an anti-Hermitian matrix. This fact will be demonstrated below. Since both $\hat{G}^{-1}$ and $\hat{\Sigma}$ have the same structure, one concludes that the dressed Green
function, $\hat{\mathbf{G}}$, also possesses the causality structure, like Eq. 2.20. As a result, the Dyson equation acquires the form:

$$
\left(\begin{array}{cc}
0 & {\left[G^{A}\right]^{-1}-\Sigma^{A}}  \tag{3.11}\\
{\left[G^{R}\right]^{-1}-\Sigma^{R}} & -\Sigma^{K}
\end{array}\right) \circ\left(\begin{array}{cc}
\mathbf{G}^{\mathbf{K}} & \mathbf{G}^{\mathbf{R}} \\
\mathbf{G}^{\mathbf{A}} & 0
\end{array}\right)=\hat{1}
$$

where one took into account that $\left[G^{-1}\right]^{K}$ is a pure regularization $(\sim i 0 F)$ and thus may be omitted in the presence of a non-zero $\Sigma^{K}$. Employing the specific form of $\left[G^{R(A)}\right]^{-1}$, Eq. 2.34, one obtains for the retarded (advanced) components:

$$
\begin{equation*}
\left(i \partial_{t}+\frac{1}{2 m} \nabla_{r}^{2}\right) \mathbf{G}^{\mathbf{R}(\mathbf{A})}=\Sigma^{R(A)} \circ \mathbf{G}^{\mathbf{R}(\mathbf{A})} . \tag{3.12}
\end{equation*}
$$

Provided the self-energy component $\Sigma^{R(A)}$ is known (in some approximation), Eq. 3.12 constitutes a closed equation for the retarded (advanced) component of the dressed Green function. The latter carries the information about the spectrum of the interacting system.

To write down the equation for the Keldysh component, it is convenient to parameterize it as $\mathbf{G}^{\mathbf{K}}=\mathbf{G}^{\mathbf{R}} \circ \mathbf{F}-\mathbf{F} \circ \mathbf{G}^{\mathbf{A}}$, where $\mathbf{F}$ is a Hermitian matrix in the time domain. The equation for the Keldysh component then takes the form: $\left(\left[G^{R}\right]^{-1}-\Sigma^{R}\right) \circ\left(\mathbf{G}^{\mathbf{R}} \circ \mathbf{F}-\mathbf{F} \circ \mathbf{G}^{\mathbf{A}}\right)=\Sigma^{K} \circ \mathbf{G}^{\mathbf{A}}$. Multiplying it from the right by $\left(\left[G^{A}\right]^{-1}-\Sigma^{A}\right)$ and employing Eq. 3.12, one finally finds:

$$
\begin{equation*}
\left[\mathbf{F},\left(i \partial_{t}+\frac{1}{2 m} \nabla_{r}^{2}\right)\right]_{-}=\Sigma^{K}-\left(\Sigma^{R} \circ \mathbf{F}-\mathbf{F} \circ \Sigma^{A}\right) \tag{3.13}
\end{equation*}
$$

where the symbol $[,]_{-}$stands for the commutator. This equation is the quantum kinetic equation for the distribution matrix $\mathbf{F}$. Its l.h.s. is called the kinetic term, while the r.h.s. is the collision integral (up to a factor). As is shown below, $\Sigma^{K}$ has the meaning of an "incoming" term, while $\Sigma^{R} \circ \mathbf{F}-\mathbf{F} \circ \Sigma^{A}$ is an "outgoing" term. In equilibrium these two channels cancel each other (the kinetic term vanishes) and the self-energy has the same structure as the Green function: $\Sigma^{K}=\Sigma^{R} \circ \mathbf{F}-\mathbf{F} \circ \Sigma^{A}$. This is not the case, however, away from the equilibrium.

### 3.4. Self-energy

Let us demonstrate in the case of one specific example, that the self-energy matrix, $\hat{\Sigma}$, indeed possesses the causality structure, Eq. 3.10. To this end, we consider the real boson action, Eq. (2.35), with the $\kappa \varphi^{3}$ nonlinearity, Eq. 3.3, and perform the calculations up to the second order in the parameter, $\kappa$. Employing the two vertexes of Fig. 3.1 one finds that:

d)
e)



Fig. 5. Self-energy diagrams for the $\varphi^{3}$ theory.
the $\mathrm{cl}-\mathrm{cl}$ component is given by the single diagram, depicted in Fig. 3.4.4. The corresponding analytic expression is $\Sigma^{c l-c l}\left(t, t^{\prime}\right)=4 i \kappa^{2} D^{R}\left(t, t^{\prime}\right) D^{A}\left(t, t^{\prime}\right)=0$. Indeed, the product $D^{R}\left(t, t^{\prime}\right) D^{A}\left(t, t^{\prime}\right)$ has no support (see, however, the footnote in section 3.1.
the cl-q (advanced) component is given by the single diagram, Fig. 3.4p. The corresponding expression is:

$$
\begin{equation*}
\Sigma^{A}\left(t, t^{\prime}\right)=4 i \kappa^{2} D^{A}\left(t, t^{\prime}\right) D^{K}\left(t, t^{\prime}\right) \tag{3.14}
\end{equation*}
$$

Since $\Sigma^{A}\left(t, t^{\prime}\right) \sim D^{A}\left(t, t^{\prime}\right) \sim \theta\left(t^{\prime}-t\right)$, it is, indeed, an advanced (upper triangular) matrix. There is a combinatoric factor of 4 , associated with the diagram ( 4 ways of choosing external legs $\times 2$ internal permutations $\times 1 /(2!$ ) for having two identical vertexes).
the $q-c l$ (retarded) component is given by the diagram of Fig. 3.4::

$$
\begin{equation*}
\Sigma^{R}\left(t, t^{\prime}\right)=4 i \kappa^{2} D^{R}\left(t, t^{\prime}\right) D^{K}\left(t, t^{\prime}\right) \tag{3.15}
\end{equation*}
$$

that could be obtained, of course, by the Hermitian conjugation of Eq. 3.14 with the help of Eq. 2.22): $\Sigma^{R}=\left[\Sigma^{A}\right]^{\dagger}$. Since $\Sigma^{R}\left(t, t^{\prime}\right) \sim D^{R}\left(t, t^{\prime}\right) \sim \theta\left(t-t^{\prime}\right)$, it is, indeed, a retarded (lower triangular) matrix.
the $q-q$ (Keldysh) component is given by the three diagrams, Fig. 3.4d-f. The corresponding expressions are:
$\Sigma^{K}\left(t, t^{\prime}\right)=2 i \kappa^{2}\left[D^{K}\left(t, t^{\prime}\right)\right]^{2}+6 i\left(\frac{\kappa}{3}\right) \kappa\left[D^{A}\left(t, t^{\prime}\right)\right]^{2}+6 i \kappa\left(\frac{\kappa}{3}\right)\left[D^{R}\left(t, t^{\prime}\right)\right]^{2}$

$$
\begin{equation*}
=2 i \kappa^{2}\left(\left[D^{K}\left(t, t^{\prime}\right)\right]^{2}+\left[D^{R}\left(t, t^{\prime}\right)-D^{A}\left(t, t^{\prime}\right)\right]^{2}\right) \tag{3.16}
\end{equation*}
$$

where the combinatoric factors are: 2 for diagram d and 6 for e and f . In the last equality, the fact that $G^{R}\left(t, t^{\prime}\right) G^{A}\left(t, t^{\prime}\right)=0$, due to the absence of support in the time domain, has been used again. Employing Eq. 2.22, one finds $\Sigma^{K}=$ $-\left[\Sigma^{K}\right]^{\dagger}$. This completes the proof of the statement that $\hat{\Sigma}$ possesses the same structure as $\hat{D}^{-1}$. One may check that the statement holds in higher orders as well. In Eqs. 3.14-3.16) one has omitted the spatial coordinates, that may be restored in an obvious way.

Exercise: Calculate the self-energy matrix for the $|\phi|^{4}$ theory to the second order in $\lambda$. Show that it possesses the causality structure.

### 3.5. Kinetic term

To make further progress in the discussion of the kinetic equation it is convenient to perform the Wigner transformation (WT). The WT of a distribution function matrix, $\mathbf{F}\left(t, t^{\prime} ; r, r^{\prime}\right)$, is a function: $\mathbf{f}(\tau, \epsilon ; \rho, k)$, where $\tau$ and $\rho$ are the "center of mass" time and coordinate correspondingly. According to definition (2.28), the $\mathbf{F}$ matrix appears in a product with $G^{R}-G^{A}$ (or $D^{R}-D^{A}$ ). Since the latter is a sharply peaked function at $\epsilon=\omega_{k}$ (cf. Eq. 2.26 for free particles, while for interacting systems this is the condition for having well-defined quasi-particles), one frequently writes $\mathbf{f}(\tau, \rho, k)$, understanding that $\epsilon=\omega_{k}$.

To rewrite the kinetic term (the l.h.s. of Eq. 3.13) in the Wigner representation, one notices that the WT of $i \partial_{t}$ is $\epsilon$, while the WT of $\nabla_{r}^{2}$ is $-k^{2}$. Then

Reminder: The Wigner transform of a matrix $A\left(r, r^{\prime}\right)$ is defined as

$$
a(\rho, k) \equiv \int d r_{1} A\left(\rho+\frac{r_{1}}{2}, \rho-\frac{r_{1}}{2}\right) e^{i k r_{1}}
$$

One may show that the Wigner transform of the matrix $C=A \circ B$ is equal to:

$$
c(\rho, k)=\iint d r_{1} d r_{2} \iint \frac{d k_{1} d k_{2}}{(2 \pi)^{2 d}} a\left(\rho+\frac{r_{1}}{2}, k+k_{1}\right) b\left(\rho+\frac{r_{2}}{2}, k+k_{2}\right) e^{i\left(k_{1} r_{2}-k_{2} r_{1}\right)}
$$

Expanding the functions under the integrals in $k_{i}$ and $r_{i}$, one finds:

$$
c(\rho, k)=a(\rho, k) b(\rho, k)+(2 i)^{-1}\left(\nabla_{\rho} a \nabla_{k} b-\nabla_{k} a \nabla_{\rho} b\right)+\ldots
$$

e.g. $\left[\mathbf{F}, \nabla_{r}^{2}\right]_{-} \rightarrow\left[k^{2}, \mathbf{f}\right]_{-}+i \nabla_{k} k^{2} \nabla_{\rho} \mathbf{f}=2 i k \nabla_{\rho} \mathbf{f}$, where the commutator vanishes, since WT's commute. In a similar way: $\left[\mathbf{F}, i \partial_{t}\right]_{-} \rightarrow-i \partial_{\tau} \mathbf{f}$. If there is a scalar potential $V(r) a_{r}^{\dagger} a_{r}$ in the Hamiltonian, it translates into the term $-V\left(\bar{\phi}_{c l} \phi_{q}+\bar{\phi}_{q} \phi_{c l}\right)$ in the action and thus $-V(r)$ is added to $\left[G^{R(A)}\right]^{-1}$. This, in turn, brings the term $-[\mathbf{F}, V]_{-}$to the l.h.s. of the Dyson equation 3.13], or after the WT: $i E \nabla_{k} \mathbf{f}$, where $E \equiv-\nabla_{\rho} V$ is the electric field. As a result, the WT of the Dyson equation (3.9) takes the form:

$$
\begin{equation*}
\left(\partial_{\tau}-v_{k} \nabla_{\rho}-E \nabla_{k}\right) \mathbf{f}(\tau, \rho, k)=I_{c o l}[\mathbf{f}] \tag{3.17}
\end{equation*}
$$

where $v_{k} \equiv k / m$ and $I_{\text {col }}[\mathbf{f}]$ is the WT of the r.h.s. of Eq. 3.13 (times $i$ ). This is the kinetic equation for the distribution function.

For real bosons with the dispersion relation $\epsilon=\omega_{k}$, the kinetic term is (cf. Eq. (2.36): $\left[\epsilon^{2}-\omega_{k}^{2}, \mathbf{F}\right]_{-} \rightarrow 2 i\left(\epsilon \partial_{\tau}-\omega_{k}\left(\nabla_{k} \omega_{k}\right) \nabla_{\rho}\right) \mathbf{f}=2 i \epsilon\left(\partial_{\tau}-v_{k} \nabla_{\rho}\right) \mathbf{f}$, where $v_{k} \equiv \nabla_{k} \omega_{k}$ is the group velocity. As a result, the kinetic equation takes the form: $\left(\partial_{\tau}-v_{k} \nabla_{\rho}\right) \mathbf{f}(\tau, \rho, k)=I_{\text {col }}[\mathbf{f}]$, where the collision integral $I_{\text {col }}[\mathbf{f}]$ is the WT of the r.h.s. of Eq. (3.13), divided by $-2 i \epsilon$.

### 3.6. Collision integral

Let us discuss the collision integral, using the $\varphi^{3}$ theory calculations of section 3.4 as an example. To shorten the algebra, let us consider a system that is spatially uniform and isotropic in momentum space. One, thus, focuses on the energy relaxation only. In this case the distribution function is $\mathbf{f}(\tau, \rho, k)=\mathbf{f}\left(\tau, \omega_{k}\right)=$ $\mathbf{f}(\tau, \epsilon)$, where the dependence on the modulus of the momentum is substituted by the $\omega_{k}=\epsilon$ argument. Employing Eqs. 3.14-3.16, one finds for the WT of the r.h.s. of Eq. $3.13{ }^{2}$ :

$$
\begin{align*}
& \Sigma^{R} \circ \mathbf{F}-\mathbf{F} \circ \Sigma^{A} \rightarrow-2 i \mathbf{f}(\tau, \epsilon) \int d \omega M(\tau, \epsilon, \omega)(\mathbf{f}(\tau, \epsilon-\omega)+\mathbf{f}(\tau, \omega)) \\
& \Sigma^{K} \rightarrow-2 i \int d \omega M(\tau, \epsilon, \omega)(\mathbf{f}(\tau, \epsilon-\omega) \mathbf{f}(\tau, \omega)+1) \tag{3.18}
\end{align*}
$$

where the square of the transition matrix element is given by:

$$
\begin{equation*}
M(\tau, \epsilon, \omega)=2 \pi \kappa^{2} \sum_{q} \boldsymbol{\Delta}_{\mathbf{d}}(\tau, \epsilon-\omega ; k-q) \boldsymbol{\Delta}_{\mathbf{d}}(\tau, \omega ; q) . \tag{3.19}
\end{equation*}
$$

[^1]Here $\boldsymbol{\Delta}_{\mathbf{d}} \equiv i\left(\mathbf{d}^{\mathbf{R}}-\mathbf{d}^{\mathbf{A}}\right) /(2 \pi)$ and $\mathbf{d}^{\mathbf{R}(\mathbf{A})}(\tau, \epsilon ; k)$ is the WT of the retarded (advanced) Green function. One has substituted the dressed Green functions into Eqs. 3.14-3.16 instead of the bare ones to perform a partial resummation of the diagrammatic series. (This trick is sometimes called the self-consistent Born approximation. It still neglects the vertex corrections.) Assuming the existence of well defined quasi-particles at all times, one may regard $\boldsymbol{\Delta}_{\mathbf{d}}(\tau, \epsilon, k)$ as a sharply peaked function at $\epsilon=\omega_{k}$. In this case Eq. (3.19) simply reflects the fact that an initial particle with $\epsilon=\omega_{k}$ decays into two real (on mass-shell) particles with energies $\omega=\omega_{q}$ and $\epsilon-\omega=\omega_{k-q}$. As a result, one finally obtains for the kinetic equation:

$$
\begin{equation*}
\frac{\partial \mathbf{f}(\epsilon)}{\partial \tau}=\int d \omega \frac{M(\epsilon, \omega)}{\epsilon}[\mathbf{f}(\epsilon-\omega) \mathbf{f}(\omega)+1-\mathbf{f}(\epsilon)(\mathbf{f}(\epsilon-\omega)+\mathbf{f}(\omega))] \tag{3.20}
\end{equation*}
$$

where the time arguments are suppressed for brevity.
Due to the identity: $\operatorname{coth}(a-b) \operatorname{coth}(b)+1=\operatorname{coth}(a)(\operatorname{coth}(a-b)+\operatorname{coth}(b))$, the collision integral is identically nullified by

$$
\begin{equation*}
\mathbf{f}(\epsilon)=\operatorname{coth} \frac{\epsilon}{2 T} . \tag{3.21}
\end{equation*}
$$

where $T$ is a temperature. This is the thermal equilibrium distribution function. According to the kinetic equation 3.20, it is stable for any temperature (the latter is determined either by an external reservoir, or, for a closed system, from the total energy conservation). Since the equilibrium distribution obviously nullifies the kinetic term, according to Eq. (3.13) the exact self-energy satisfies $\Sigma^{K}=$ $\operatorname{coth}(\epsilon /(2 T))\left(\Sigma^{R}-\Sigma^{A}\right)$. Since also the bare Green functions obey the same relation, Eq. 2.27, one concludes that in thermal equilibrium the exact dressed Green function satisfies:

$$
\begin{equation*}
\mathbf{D}^{\mathbf{K}}=\operatorname{coth} \frac{\epsilon}{2 T}\left(\mathbf{D}^{\mathbf{R}}-\mathbf{D}^{\mathbf{A}}\right) \tag{3.22}
\end{equation*}
$$

This is the statement of the fluctuation-dissipation theorem (FDT). Its consequence is that in equilibrium the Keldysh component does not contain any additional information with respect to the retarded one. Therefore, the Keldysh technique may be, in principle, substituted by a more compact construction - the Matsubara method. The latter does not work, of course, away from equilibrium.

Returning to the kinetic equation (3.20, one may identify "in" and "out" terms in the collision integral. Most clearly it is done by writing the collision integral in terms of the occupation numbers $\mathbf{n}_{k}$, defined as $\mathbf{f}=1+2 \mathbf{n}$. The expression in the square brackets on the r.h.s. of Eq. 3.20 takes the form: $4\left[\mathbf{n}_{\epsilon-\omega} \mathbf{n}_{\omega}-\mathbf{n}_{\epsilon}\left(\mathbf{n}_{\epsilon-\omega}+\mathbf{n}_{\omega}+1\right)\right]$. The first term: $\mathbf{n}_{\epsilon-\omega} \mathbf{n}_{\omega}$, gives a probability that a particle with energy $\epsilon-\omega$ absorbs a particle with energy $\omega$ to populate a
state with energy $\epsilon$ - this is the "in" term of the collision integral. It may be traced back to the $\Sigma^{K}$ part of the self-energy. The second term: $-\mathbf{n}_{\epsilon}\left(\mathbf{n}_{\epsilon-\omega}+\mathbf{n}_{\omega}+1\right)$, says that a state with energy $\epsilon$ may be depopulated either by stimulated emission of particles with energies $\epsilon-\omega$ and $\omega$, or by spontaneous emission (unity). This is the "out" term, that may be traced back to the $\Sigma^{R(A)}$ contributions.

Finally, let us discuss the approximations involved in the Wigner transformations. Although Eq. 3.13) is formally exact, it is very difficult to extract any useful information from it. Therefore, passing to an approximate, but much more tractable, form like Eqs. 3.17) or 3.20 is highly desirable. In doing it, one has to employ the approximate form of the WT. Indeed, a formally infinite series in $\nabla_{k} \nabla_{\rho}$ operators is truncated, usually by the first non-vanishing term. This is a justified procedure as long as $\delta k \delta \rho \gg 1$, where $\delta k$ is a characteristic microscopic scale of the momentum dependence of $\mathbf{f}$, while $\delta \rho$ is a characteristic scale of its spatial variations. One may ask if there is a similar requirement in the time domain: $\delta \epsilon \delta \tau \gg 1$, with $\delta \epsilon$ and $\delta \tau$ being the characteristic energy and the time scale of $\mathbf{f}$, correspondingly? Such a requirement is very demanding, since typically $\delta \epsilon \approx T$ and at low temperature it would allow to treat only very slow processes: with $\delta \tau \gg 1 / T$. Fortunately, this is not the case. Because of the peaked structure of $\boldsymbol{\Delta}_{\mathbf{d}}(\epsilon, k)$, the energy argument $\epsilon$ is locked to $\omega_{k}$ and does not have its own dynamics as long as the peak is sharp. The actual criterion is therefore that $\delta \epsilon$ is much larger than the width of the peak in $\boldsymbol{\Delta}_{\mathbf{d}}(\epsilon, k)$. The latter is, by definition, the quasi-particle life-time, $\tau_{q p}$, and therefore the condition is $\tau_{q p} \gg 1 / T$. This condition is indeed satisfied by many systems with the interactions that are not too strong.

## 4. Particle in contact with an environment

### 4.1. Quantum dissipative action

Consider a particle with the coordinate $\Phi(t)$, living in a potential $U(\Phi)$ and attached to a harmonic string $\varphi(t ; x)$. The particle may represent a collective degree of freedom, such as the phase of a Josephson junction or the charge on a quantum dot. On the other hand, the string serves to model a dissipative environment. The advantage of the one-dimensional string is that it is the simplest continuum system, having a constant density of states. Due to this property it mimics, for example, interactions with a Fermi sea. A continuous reservoir with a constant density of states at small energies is sometimes called an "Ohmic" environment (or bath). The environment is supposed to be in thermal equilibrium.

The Keldysh action of such a system is given by the three terms (cf. Eqs. B. 5)
and (2.35):

$$
\begin{align*}
& S_{p}[\hat{\Phi}]=\int_{-\infty}^{\infty} d t\left[-2 \Phi_{q} \frac{d^{2} \Phi_{c l}}{d t^{2}}-U\left(\Phi_{c l}+\Phi_{q}\right)+U\left(\Phi_{c l}-\Phi_{q}\right)\right] \\
& S_{s t r}[\hat{\varphi}]=\int_{-\infty}^{\infty} d t \int d x \hat{\varphi}^{T} \hat{D}^{-1} \hat{\varphi}  \tag{4.1}\\
& S_{i n t}[\hat{\Phi}, \hat{\varphi}]=\left.2 \sqrt{\gamma} \int_{-\infty}^{\infty} d t \hat{\Phi}^{T}(t) \hat{\sigma}_{1} \nabla_{x} \hat{\varphi}(t, x)\right|_{x=0}
\end{align*}
$$

where we have introduced vectors of classical and quantum components, e.g. $\hat{\Phi}^{T} \equiv\left(\Phi_{c l}, \Phi_{q}\right)$ and the string correlator, $\hat{D}^{-1}$, is the same as in Eqs. 2.35, 2.36. The interaction term between the particle and the string is taken to be the local product of the particle coordinate and the string stress at $x=0$ (so the force on the particle is proportional to the local stress of the string). In the time domain the interaction is instantaneous, $\left.\Phi(t) \nabla_{x} \varphi(t, x)\right|_{x=0} \rightarrow \Phi_{+} \nabla \varphi_{+}-\Phi_{-} \nabla \varphi_{-}$on the Keldysh contour. Transforming to the classical-quantum notations leads to: $2\left(\Phi_{c l} \nabla \varphi_{q}+\Phi_{q} \nabla \varphi_{c l}\right)$, that satisfies the causality condition, Eq. 2.32). In the matrix notations it takes the form of the last line of Eq. 4.1), where $\hat{\sigma}_{1}$ is the standard Pauli matrix. The interaction constant is $\sqrt{\gamma}$.

One may now integrate out the degrees of freedom of the Gaussian string to reduce the problem to the particle coordinate only. According to the standard rules of Gaussian integration (see. Appendix A, this leads to the so-called dissipative action for the particle:
$S_{\text {diss }}=-\gamma \iint_{-\infty}^{\infty} d t d t^{\prime} \hat{\Phi}^{T}(t) \underbrace{\left.\hat{\sigma}_{1}^{T} \nabla_{x} \nabla_{x^{\prime}} \hat{D}\left(t-t^{\prime} ; x-x^{\prime}\right)\right|_{x=x^{\prime}=0} \hat{\sigma}_{1}}_{-\hat{L}^{-1}\left(t-t^{\prime}\right)} \hat{\Phi}\left(t^{\prime}\right)$.
The straightforward matrix multiplication shows that the dissipative correlator $\hat{L}^{-1}$ possesses the standard causality structure of the inverse Green function, e.g. Eq. (2.29). Fourier transforming its retarded (advanced) components, one finds:

$$
\begin{equation*}
\left[L^{R(A)}(\epsilon)\right]^{-1}=-\sum_{k} \frac{k^{2}}{(\epsilon \pm i 0)^{2}-k^{2}}= \pm \frac{i}{2} \epsilon+\mathrm{const} \tag{4.3}
\end{equation*}
$$

where we put $v_{s}=1$ for brevity. The $\epsilon$-independent constant (same for $R$ and $A$ components) may be absorbed into the redefinition of the harmonic part of the
potential $U(\Phi)=$ const $\Phi^{2}+\ldots$ and, thus, may be omitted. In equilibrium the Keldysh component of the correlator is set by the FDT:

$$
\begin{equation*}
\left[L^{-1}\right]^{K}(\epsilon)=\operatorname{coth} \frac{\epsilon}{2 T}\left(\left[L^{R}\right]^{-1}-\left[L^{A}\right]^{-1}\right)=i \epsilon \operatorname{coth} \frac{\epsilon}{2 T} \tag{4.4}
\end{equation*}
$$

It is an anti-Hermitian operator with a positive-definite imaginary part, rendering convergence of the functional integral over $\Phi$.

In the time representation the retarded (advanced) component of the correlator takes a simple local form: $\left[L^{R(A)}\right]^{-1}=\mp \frac{1}{2} \delta\left(t-t^{\prime}\right) \partial_{t^{\prime}}$. On the other hand, at low temperatures the Keldysh component is a non-local function, that may be found by the inverse Fourier transform of Eq. (4.4):

$$
\begin{equation*}
\left[L^{-1}\right]^{K}\left(t-t^{\prime}\right)=\frac{i \pi T^{2}}{\sinh ^{2}\left(\pi T\left(t-t^{\prime}\right)\right)} \xrightarrow{T \rightarrow \infty} i 2 T \delta\left(t-t^{\prime}\right) . \tag{4.5}
\end{equation*}
$$

Finally, for the Keldysh action of the particle connected to a string, one obtains:

$$
\begin{align*}
S[\hat{\Phi}] & =\int_{-\infty}^{\infty} d t\left[-2 \Phi_{q}\left(\frac{d^{2} \Phi_{c l}}{d t^{2}}+\frac{\gamma}{2} \frac{d \Phi_{c l}}{d t}\right)-U\left(\Phi_{c l}+\Phi_{q}\right)+U\left(\Phi_{c l}-\Phi_{q}\right)\right] \\
& +i \gamma \iint_{-\infty}^{\infty} d t d t^{\prime} \Phi_{q}(t) \frac{\pi T^{2}}{\sinh ^{2}\left(\pi T\left(t-t^{\prime}\right)\right)} \Phi_{q}\left(t^{\prime}\right) \tag{4.6}
\end{align*}
$$

This action satisfies all the causality criterions listed in section 2.4 Notice, that in the present case the Keldysh $(q-q)$ component is not just a regularization factor, but rather a quantum fluctuations damping term, originating from the coupling to the string. The other manifestation of the string is the presence of the friction term, $\sim \gamma \partial_{t}$ in the $R$ and the $A$ components. In equilibrium the friction coefficient and fluctuations amplitude are rigidly connected by the FDT. The quantum dissipative action, Eq. 4.6, is a convenient playground to demonstrate various approximations and connections to other approaches.

### 4.2. Saddle-point equation

The classical saddle point equation (the one that takes $\Phi_{q}(t)=0$ ) has the form:

$$
\begin{equation*}
-\left.\frac{1}{2} \frac{\delta S[\hat{\Phi}]}{\delta \Phi_{q}}\right|_{\Phi_{q}=0}=\frac{d^{2} \Phi_{c l}}{d t^{2}}+\frac{\gamma}{2} \frac{d \Phi_{c l}}{d t}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}=0 \tag{4.7}
\end{equation*}
$$

This is the deterministic classical equation of motion. In the present case it happens to be the Newton equation with the viscous force: $-(\gamma / 2) \dot{\Phi}_{c l}$. This approximation neglects both quantum and thermal fluctuations.

### 4.3. Classical limit

One may keep the thermal fluctuations, while completely neglecting the quantum ones. To this end it is convenient to restore the Planck constant in the action (4.6) and then take the limit $\hbar \rightarrow 0$. For dimensional reasons, the factor $\hbar^{-1}$ should stand in front of the action. To keep the part of the action responsible for the classical equation of motion 4.7) free from the Planck constant it is convenient to rescale the variables as: $\Phi_{q} \rightarrow \hbar \Phi_{q}$. Finally, to have temperature in energy units, one needs to substitute $T \rightarrow T / \hbar$ in the last term of Eq. (4.6. The limit $\hbar \rightarrow 0$ is now straightforward: (i) one has to expand $U\left(\Phi_{c l} \pm \hbar \Phi_{q}\right)$ to the first order in $\hbar \Phi_{q}$ and neglect all higher order terms; (ii) in the last term of Eq. (4.6) the $\hbar \rightarrow 0$ limit is equivalent to the $T \rightarrow \infty$ limit, see Eq. 4.5. As a result, the classical limit of the dissipative action is:

$$
\begin{equation*}
S[\hat{\Phi}]=2 \int_{-\infty}^{\infty} d t\left[-\Phi_{q}\left(\frac{d^{2} \Phi_{c l}}{d t^{2}}+\frac{\gamma}{2} \frac{d \Phi_{c l}}{d t}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}\right)+i \gamma T \Phi_{q}^{2}\right] \tag{4.8}
\end{equation*}
$$

Physically the limit $\hbar \rightarrow 0$ means that $\hbar \tilde{\Omega} \ll T$, where $\tilde{\Omega}$ is a characteristic classical frequency of the particle. This condition is necessary for the last term of Eq. (4.6) to take the time-local form. The condition for neglecting the higher order derivatives of $U$ is $\hbar \ll \gamma \tilde{\Phi}_{c l}^{2}$, where $\tilde{\Phi}_{c l}$ is a characteristic classical amplitude of the particle motion.

### 4.4. Langevin equations

One way to proceed with the classical action (4.8) is to notice that the exponent of its last term (times $i$ ) may be identically rewritten in the following way:

$$
\begin{equation*}
e^{-2 \gamma T \int d t \Phi_{q}^{2}(t)}=\int \mathcal{D} \xi(t) e^{-\int d t\left[\frac{1}{2 \gamma T} \xi^{2}(t)-2 i \xi(t) \Phi_{q}(t)\right]} \tag{4.9}
\end{equation*}
$$

This identity is called the Hubbard-Stratonovich transformation, while $\xi(t)$ is an auxiliary Hubbard-Stratonovich field. The identity is proved by completing the square in the exponent on the r.h.s., performing the Gaussian integration at every instance of time and multiplying the results. There is a constant multiplicative factor hidden in the integration measure, $\mathcal{D} \xi$.

Exchanging the order of the functional integration over $\xi$ and $\hat{\Phi}$, one finds for the partition function:

$$
\begin{equation*}
Z=\int \mathcal{D} \xi e^{-\frac{1}{2 \gamma T} \int d t \xi^{2}} \int \mathcal{D} \Phi_{c l} \int \mathcal{D} \Phi_{q} e^{-2 i \int d t \Phi_{q}\left(\frac{d^{2} \Phi_{c l}}{d t^{2}}+\frac{\gamma}{2} \frac{d \Phi_{c l}}{d t}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}-\xi\right)} \tag{4.10}
\end{equation*}
$$

Since the last (imaginary) exponent depends only linearly on $\Phi_{q}(t)$, the integration over $\mathcal{D} \Phi_{q}$ results in the $\delta$-function of the expression in the round brackets. This functional $\delta$-function enforces its argument to be zero at every moment of time. Therefore, among all the possible trajectories $\Phi_{c l}(t)$, only those that satisfy the following equation contribute to the partition function:

$$
\begin{equation*}
\frac{d^{2} \Phi_{c l}}{d t^{2}}+\frac{\gamma}{2} \frac{d \Phi_{c l}}{d t}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}=\xi(t) \tag{4.11}
\end{equation*}
$$

This is a Newton equation with a time dependent external force $\xi(t)$. Since, the same arguments are applicable to any correlation function of the classical fields, e.g. $\left\langle\Phi_{c l}(t) \Phi_{c l}\left(t^{\prime}\right)\right\rangle$, a solution strategy is as follows: (i) choose some realization of $\xi(t)$; (ii) solve Eq. 4.11 (e.g. numerically); (iii) having its solution, $\Phi_{c l}(t)$, calculate the correlation function; (iv) average the result over an ensemble of realizations of the force $\xi(t)$. The statistics of the latter are dictated by the weight factor in the $\mathcal{D} \xi$ functional integral. It states that $\xi(t)$ is a Gaussian short-range (white) noise with the correlators:

$$
\begin{equation*}
\langle\xi(t)\rangle=0 ; \quad\left\langle\xi(t) \xi\left(t^{\prime}\right)\right\rangle=\gamma T \delta\left(t-t^{\prime}\right) \tag{4.12}
\end{equation*}
$$

Equation 4.11 with the white noise on the r.h.s. is called the Langevin equation. It describes classical Newtonian dynamics in presence of stochastic thermal fluctuations. The fact that the noise amplitude is related to the friction coefficient, $\gamma$ and to the temperature is a manifestation of the FDT. The latter holds as long as the environment (string) is at thermal equilibrium.

### 4.5. Martin-Siggia-Rose

In section 4.4 one derived the Langevin equation for a classical coordinate, $\Phi_{c l}$, from the action written in terms of $\Phi_{c l}$ and another field, $\Phi_{q}$. An inverse procedure of deriving the effective action from the Langevin equation is known as the Martin-Siggia-Rose (MSR) [5] technique. It is sketched here in the form suggested by De-Dominics [5].

Consider a Langevin equation:

$$
\begin{equation*}
\hat{\mathcal{O}}[\Phi]=\xi(t), \tag{4.13}
\end{equation*}
$$

where $\hat{\mathcal{O}}[\Phi]$ is a (non-linear) differential operator acting on the coordinate $\Phi(t)$ and $\xi(t)$ is a white noise force, specified by Eq. 4.12. Define the "partition function" as:

$$
\begin{equation*}
Z[\xi]=\int \mathcal{D} \Phi \mathcal{J}[\hat{\mathcal{O}}] \delta(\hat{\mathcal{O}}[\Phi]-\xi(t)) \equiv 1 \tag{4.14}
\end{equation*}
$$

It is identically equal to unity by virtue of the integration of the $\delta$-function, provided $\mathcal{J}[\hat{\mathcal{O}}]$ is the Jacobian of the operator $\hat{\mathcal{O}}[\Phi]$. The way to interpret Eq. (4.14) is to discretize the time axis, introducing $N$-dimensional vectors $\Phi_{j}=\Phi\left(t_{j}\right)$ and $\xi_{j}=\xi\left(t_{j}\right)$. The operator takes the form: $\mathcal{O}_{i}=O_{i j} \Phi_{j}+\Gamma_{i j k} \Phi_{j} \Phi_{k}+\ldots$, where a summation is taken over repeated indexes. The Jacobian, $\mathcal{J}$, is given by the absolute value of the determinant of the following $N \times N$ matrix: $J_{i j} \equiv$ $\partial \mathcal{O}_{i} / \partial \Phi_{j}=O_{i j}+2 \Gamma_{i j k} \Phi_{k}+\ldots$. It is possible to choose a proper (retarded) regularization where the $J_{i j}$ matrix is a lower triangular matrix with a unity main diagonal (coming entirely from the $O_{i i}=1$ term). Clearly, in this case, $\mathcal{J}=1$. Indeed, consider, for example, $\hat{\mathcal{O}}[\Phi]=\partial_{t} \Phi-W(\Phi)$. The retarded regularized version of the Langevin equation is: $\Phi_{i}=\Phi_{i-1}+\delta_{t}\left(W\left(\Phi_{i-1}\right)+\xi_{i-1}\right)$. Clearly in this case $J_{i i}=1$ and $J_{i, i-1}=-1-W^{\prime}\left(\Phi_{i-1}\right) \delta_{t}$, while all other components are zero; as a result $\mathcal{J}=1$.

Although the partition function (4.14) is trivial, it is clear that all the meaningful observables and the correlation functions may be obtained by inserting a set of factors: $\Phi(t) \Phi\left(t^{\prime}\right) \ldots$ in the functional integral, Eq. 4.14. Having this in mind, let us proceed with the partition function. Employing the integral representation of the $\delta$-function with the help of an auxiliary field $\Psi(t)$, one obtains:

$$
\begin{equation*}
Z[\xi]=\int \mathcal{D} \Phi \int \mathcal{D} \Psi e^{-2 i \int d t \Psi(t)\left(\hat{\mathcal{O}}^{R}[\Phi(t)]-\xi(t)\right)} \tag{4.15}
\end{equation*}
$$

where $\hat{\mathcal{O}}^{R}$ stands for the retarded regularization of the $\hat{\mathcal{O}}$ operator and thus one takes $\mathcal{J}=1$. One may average now over the white noise, Eq. 4.12, by performing the Gaussian integration over $\xi$ :

$$
\begin{equation*}
Z=\int \mathcal{D} \xi e^{-\frac{1}{2 \gamma T} \int d t \xi^{2}} Z[\xi]=\int \mathcal{D} \Phi \Psi e^{-\int d t\left[2 i \Psi(t) \hat{\mathcal{O}}^{R}[\Phi(t)]+2 \gamma T \Psi^{2}(t)\right]} \tag{4.16}
\end{equation*}
$$

The exponent is exactly the classical limit of the Keldysh action, cf. Eq. 4.8 (including the retarded regularization of the differential operator), where $\Phi=\Phi_{c l}$ and $\Psi=\Phi_{q}$. The message is that the MSR action is nothing, but the classical (high temperature) limit of the Keldysh action. The MSR technique provides a simple way to transform from a classical stochastic problem to its proper functional representation. The latter is useful for an analytical analysis. One example is given below.

### 4.6. Thermal activation

Consider a particle in a meta-stable potential well, plotted in Fig. 4.6. The potential has a meta-stable minimum at $\Phi=0$ and a maximum at $\Phi=1$ with the amplitude $U_{0}$. Let us also assume that the particle's motion is over-damped, i.e.


Fig. 6. a) A potential with a meta-stable minimum. b) The phase portrait of the Hamiltonian system, Eq. 4.19. Thick lines correspond to zero energy, arrows indicate evolution direction.
$\gamma \gg \sqrt{U^{\prime \prime}}$. In this case one may disregard the inertia term, leaving only viscous relaxation dynamics. The classical dissipative action (4.8) takes the form:

$$
\begin{equation*}
S[\hat{\Phi}]=2 \int_{-\infty}^{\infty} d t\left[-\Phi_{q}\left(\frac{\gamma}{2} \frac{d \Phi_{c l}}{d t}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}\right)+i \gamma T \Phi_{q}^{2}\right] \tag{4.17}
\end{equation*}
$$

The corresponding saddle point equations are:

$$
\begin{align*}
\frac{\gamma}{2} \dot{\Phi}_{c l} & =-\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}+2 i \gamma T \Phi_{q}  \tag{4.18}\\
\frac{\gamma}{2} \dot{\Phi}_{q} & =\Phi_{q} \frac{\partial^{2} U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}^{2}}
\end{align*}
$$

These equations possess the classical solution: $\Phi_{q}(t) \equiv 0$ and $\Phi_{c l}(t)$ satisfies the classical equation of motion: $\frac{\gamma}{2} \dot{\Phi}_{c l}=-\partial U\left(\Phi_{c l}\right) / \partial \Phi_{c l}$. For the initial condition $\Phi_{c l}(0)<1$ the latter equation predicts the viscous relaxation towards the minimum at $\Phi_{c l}=0$. According to this equation, there is no possibility to escape from this minimum. Therefore the classical solution of Eqs. 4.18) does not describe thermal activation. Thus one has to look for another possible solution of Eqs. 4.18, the one with $\Phi_{q} \neq 0$.

To this end let us make a simple linear change of variables: $\Phi_{c l}(t)=q(t)$ and $\Phi_{q}(t)=p(t) /(i \gamma)$. Then the dissipative action (4.17) acquires the form of a

Hamiltonian action:

$$
\begin{equation*}
i S=-\int d t(p \dot{q}-H(p, q)) ; \quad H(p, q) \equiv \frac{2}{\gamma}\left[-p \frac{\partial U(q)}{\partial q}+T p^{2}\right] \tag{4.19}
\end{equation*}
$$

where the fictitious Hamiltonian, $H$, is introduced ${ }^{3}$. It is straightforward to see that in terms of the new variables the equations of motion 4.18 take the form of the Hamilton equations: $\dot{q}=\partial H / \partial p$ and $\dot{p}=-\partial H / \partial q$. One needs, thus, to investigate the Hamiltonian system with the Hamiltonian Eq. 4.19. To visualize it, one may plot its phase portrait, consisting of lines of constant energy $E=H(p(t), q(t))$ on the $(p, q)$ plane, Fig. 4.6 . The topology is determined by the two lines of zero energy: $p=0$ and $T p=\partial U(q) / \partial q$, that intersect at the two stationary points of the potential: $q=0$ and $q=1$. The $p=0$ line corresponds to the classical (without Langevin noise) dynamics (notice, that the action is identically zero for motion along this line) and thus $q=0$ is the stable point, while $q=1$ is the unstable one. Due to Liouville theorem, every fixed point must have one stable and one unstable direction. Therefore, along the "non-classical" line: $p=T^{-1} \partial U(q) / \partial q$, the situation is reversed: $q=0$ is unstable, while $q=1$ is stable. It is clear now that, to escape from the bottom of the potential well, $q=0$, the system must move along the non-classical line of zero energy until it reaches the top of the barrier, $q=1$, and then continue to drop according to the classical equation of motion (moving along the classical line $p=0$ ). There is a non-zero action associated with the motion along the non-classical line:

$$
\begin{equation*}
i S=-\int d t p \dot{q}=-\int_{0}^{1} p(q) d q=-\frac{1}{T} \int_{0}^{1} \frac{\partial U(q)}{\partial q} d q=-\frac{U_{0}}{T} \tag{4.20}
\end{equation*}
$$

where one has used that $H=0$ along the integration trajectory. As a result, the thermal escape probability is proportional to $e^{i S}=e^{-U_{0} / T}$, which is nothing but the thermal activation exponent.

### 4.7. Fokker-Planck equation

Another way to approach the action 4.17 is to notice that it is quadratic in $\Phi_{q}$ and therefore the $\mathcal{D} \Phi_{q}$ integration may be explicitly performed. To shorten notations and emphasize the relation to the classical coordinate, we shall follow the

[^2]previous section and denote $\Phi_{c l}(t) \equiv q(t)$. Performing the Gaussian integration over $\Phi_{q}$ of $e^{i S[\hat{\Phi}]}$, with $S\left[\Phi_{c l}, \Phi_{q}\right]$ given by Eq. 4.17], one finds the action, depending on $\Phi_{c l} \equiv q$ only:
\[

$$
\begin{equation*}
i S[q]=-\frac{1}{2 \gamma T} \int_{-\infty}^{\infty} d t\left(\frac{\gamma}{2} \dot{q}+U_{q}^{\prime}\right)^{2} \tag{4.21}
\end{equation*}
$$

\]

One may now employ the same trick, that allows to pass from the Feynman path integral to the Schrödinger equation. Namely, let us introduce the "wave function", $\mathcal{P}(q, t)$, that is a result of the functional integration of $e^{i S[q]}$ over all trajectories that at time $t$ pass through the point $q_{N} \equiv q$. Adding one more time step, $\delta_{t}$, to the trajectory, one may write $\mathcal{P}\left(q_{N}, t+\delta_{t}\right)$ as an integral of $\mathcal{P}\left(q_{N-1}, t\right)=\mathcal{P}\left(q+\delta_{q}, t\right)$ over $\delta_{q} \equiv q_{N-1}-q:$

$$
\begin{align*}
\mathcal{P}\left(q, t+\delta_{t}\right) & =C \int d \delta_{q} e^{-\frac{\delta_{t}}{2 \gamma T}\left(\frac{\gamma}{2} \frac{-\delta_{q}}{\delta_{t}}+U_{q}^{\prime}\left(q+\delta_{q}\right)\right)^{2}} \mathcal{P}\left(q+\delta_{q}, t\right)  \tag{4.22}\\
& =C \int d \delta_{q} e^{-\frac{\gamma}{8 T} \frac{\delta_{q}^{2}}{\delta_{t}}}\left[e^{\frac{\delta_{q}}{2 T} U_{q}^{\prime}\left(q+\delta_{q}\right)-\frac{\delta_{t}}{2 \gamma T}\left(U_{q}^{\prime}\right)^{2}} \mathcal{P}\left(q+\delta_{q}, t\right)\right]
\end{align*}
$$

where the factor $C$ from the integration measure is determined by the condition: $C \int d \delta_{q} \exp \left\{-\gamma \delta_{q}^{2} /\left(8 T \delta_{t}\right)\right\}=1$. Expanding the expression in the square brackets on the r.h.s. of the last equation to the second order in $\delta_{q}$ and the first order in $\delta_{t}$, one finds:

$$
\begin{align*}
\mathcal{P}\left(t+\delta_{t}\right) & =\left(1+\frac{\left\langle\delta_{q}^{2}\right\rangle}{2 T} U_{q q}^{\prime \prime}+\frac{1}{2} \frac{\left\langle\delta_{q}^{2}\right\rangle}{4 T^{2}}\left(U_{q}^{\prime}\right)^{2}-\frac{\delta_{t}}{2 \gamma T}\left(U_{q}^{\prime}\right)^{2}\right) \mathcal{P}+\frac{\left\langle\delta_{q}^{2}\right\rangle}{2 T} U_{q}^{\prime} \mathcal{P}_{q}^{\prime} \\
& +\frac{\left\langle\delta_{q}^{2}\right\rangle}{2} \mathcal{P}_{q q}^{\prime \prime}=\mathcal{P}(t)+\delta_{t}\left(\frac{2}{\gamma} U_{q q}^{\prime \prime} \mathcal{P}+\frac{2}{\gamma} U_{q}^{\prime} \mathcal{P}_{q}^{\prime}+\frac{2 T}{\gamma} \mathcal{P}_{q q}^{\prime \prime}\right) \tag{4.23}
\end{align*}
$$

where $\left\langle\delta_{q}^{2}\right\rangle \equiv C \int d \delta_{q} \exp \left\{-\gamma \delta_{q}^{2} /\left(8 T \delta_{t}\right)\right\} \delta_{q}^{2}=4 T \delta_{t} / \gamma$. Finally, rewriting the last expression in the differential form, one obtains:

$$
\begin{equation*}
\frac{\partial \mathcal{P}}{\partial t}=\frac{2}{\gamma}\left[\frac{\partial}{\partial q} \frac{\partial U}{\partial q}+T \frac{\partial^{2}}{\partial q^{2}}\right] \mathcal{P}=\frac{2}{\gamma} \frac{\partial}{\partial q}\left[\frac{\partial U}{\partial q} \mathcal{P}+T \frac{\partial \mathcal{P}}{\partial q}\right] \tag{4.24}
\end{equation*}
$$

This is the Fokker-Planck (FP) equation for the evolution of the probability distribution function, $\mathcal{P}(q, t)$. The latter describes the probability to find the particle at the point $q(=\Phi)$ at time $t$. If one starts from an initially sharp (deterministic) distribution: $\mathcal{P}(q, 0)=\delta(q-q(0))$, then the first term on the r.h.s. of the FP equation describes the viscous drift of the particle in the potential $U(q)$. Indeed, in the absence of the second term $(T=0)$, the equation is solved by
$\mathcal{P}(q, t)=\delta(q-q(t))$, where $q(t)$ satisfies the deterministic equation of motion $(\gamma / 2) \dot{q}(t)=-\partial U(q(t)) / \partial q^{4}$. The second term describes the diffusion spreading of the probability distribution due to the thermal stochastic noise $\xi(t)$. For a confining potential $U(q)$ (such that $U( \pm \infty) \rightarrow \infty$ ) the stationary solution of the FP equation is the equilibrium Boltzmann distribution: $\mathcal{P}(q) \sim \exp \{-U(q) / T\}$.

The FP equation may be considered as the (imaginary time) Schrödinger equation: $\dot{\mathcal{P}}=\hat{H} \mathcal{P}$, where the "Hamiltonian", $\hat{H}$, is nothing but the "quantized" version of the classical Hamiltonian, introduced in the previous section, Eq. 4.19. The "quantization" rule is $p \rightarrow \hat{p} \equiv-\partial / \partial q$, so the canonical commutation relation: $[q, \hat{p}]_{-}=1$, holds. Notice that before applying this quantization rule, the corresponding classical Hamiltonian must be normally ordered. Namely, the momentum $\hat{p}$ should be to the left of the coordinate $q$, cf. Eq. 4.19. Using the commutation relation, one may rewrite the quantized Hamiltonian as: $\hat{H}=T \hat{p}^{2}-\hat{p} U_{q}^{\prime}=T\left(\hat{p}-U_{q}^{\prime} /(2 T)\right)\left(\hat{p}-U_{q}^{\prime} /(2 T)\right)-\left(U_{q}^{\prime}\right)^{2} /(4 T)+U_{q q}^{\prime \prime} / 2$ (we took $\gamma / 2=1$ ) and perform the canonical transformation: $Q=q$ and $\hat{P}=\hat{p}-U_{q}^{\prime} /(2 T)$. In terms of these new variables the Hamiltonian takes the familiar form: $\hat{H}=T \hat{P}^{2}+V(Q)$, where $V(Q)=-\left(U_{Q}^{\prime}\right)^{2} /(4 T)+U_{Q Q}^{\prime \prime} / 2$, while the "wave function" transforms as $\tilde{\mathcal{P}}(Q, t)=e^{U(Q) /(2 T)} \mathcal{P}$.

### 4.8. From Matsubara to Keldysh

In some applications it may be convenient to derive an action in the equilibrium Matsubara technique and change to the Keldysh representation at a later stage to tackle out-of-equilibrium problems. This section intends to illustrate how such transformation may be carried out. To this end consider the following bosonic Matsubara action:

$$
\begin{equation*}
S\left[\Phi_{m}\right]=\gamma T \sum_{m=-\infty}^{\infty} \frac{1}{2}\left|\epsilon_{m}\right|\left|\Phi_{m}\right|^{2} \tag{4.25}
\end{equation*}
$$

[^3][^4]where $\Phi_{m}=\bar{\Phi}_{-m}$ are the Matsubara components of a real bosonic field, $\Phi(\tau)$. Notice, that due to the absolute value sign: $\left|\epsilon_{m}\right| \neq i \partial_{\tau}$. In fact, in the imaginary time representation the action (4.25) has the non-local form:
\[

$$
\begin{equation*}
S[\Phi]=-\frac{\gamma}{2} \iint_{0}^{\beta} d \tau d \tau^{\prime} \Phi(\tau) \frac{\pi T^{2}}{\sin ^{2}\left(\pi T\left(\tau-\tau^{\prime}\right)\right)} \Phi\left(\tau^{\prime}\right) \tag{4.26}
\end{equation*}
$$

\]

This action is frequently named after Caldeira and Leggett [19], who used it to investigate the influence of dissipation on quantum tunnelling.

To transforn to the Keldysh representation one needs to double the number of degrees of freedom: $\Phi \rightarrow \hat{\Phi}=\left(\Phi_{c l}, \Phi_{q}\right)^{T}$. Then according to the causality structure, section 2.4 the general form of the time translationally invariant Keldysh action is:

$$
S=\gamma \int \frac{d \epsilon}{2 \pi}\left(\Phi_{c l}, \Phi_{q}\right)_{\epsilon}\left(\begin{array}{cc}
0 & {\left[L^{A}(\epsilon)\right]^{-1}}  \tag{4.27}\\
{\left[L^{R}(\epsilon)\right]^{-1}} & {\left[L^{-1}\right]^{K}(\epsilon)}
\end{array}\right)\binom{\Phi_{c l}}{\Phi_{q}}_{\epsilon}
$$

where $\left[L^{R(A)}(\epsilon)\right]^{-1}$ is the analytical continuation of the Matsubara correlator $\left|\epsilon_{m}\right| / 2$ from the upper (lower) half-plane of the imaginary variable $\epsilon_{m}$ to the real axis: $-i \epsilon_{m} \rightarrow \epsilon$. As a result, $\left[L^{R(A)}(\epsilon)\right]^{-1}= \pm i \epsilon / 2$. The Keldysh component follows from the FDT: $\left[L^{-1}\right]^{K}(\epsilon)=i \epsilon \operatorname{coth} \epsilon /(2 T)$, cf. Eqs. 4.3] and 4.4. Therefore the Keldysh counterpart of the Matsubara action, Eqs. 4.25) or 4.26) is the already familiar dissipative action, Eq. (4.6), (without the potential terms, of course). One may now include external fields and allow the system to deviate from the equilibrium.

### 4.9. Dissipative chains and membranes

Instead of dealing with a single particle connected to a bath, let us now consider a chain or lattice of coupled particles, with each one connected to a bath. To this end, one (i) supplies a spatial index, $r$, to the field: $\Phi(t) \rightarrow \Phi(t ; r)$, and (ii) adds the harmonic interaction potential between nearest neighbors particles: $\sim(\Phi(t, r)-\Phi(t, r+1))^{2} \rightarrow\left(\nabla_{r} \Phi\right)^{2}$ in the continuous limit. By changing to the classical-quantum components and performing the spatial integration by parts (cf. Eq. B. 9), the gradient term translates to: $\Phi_{q} \nabla_{r}^{2} \Phi_{c l}+\Phi_{c l} \nabla_{r}^{2} \Phi_{q}$. Thus it modifies the retarded and advanced components of the correlator, but it does not affect the $(q-q)$ Keldysh component:

$$
\begin{equation*}
\left[L^{R(A)}\right]^{-1}=\frac{1}{2} \delta\left(t-t^{\prime}\right) \delta\left(r-r^{\prime}\right)\left(\mp \partial_{t^{\prime}}+D \nabla_{r^{\prime}}^{2}\right) \tag{4.28}
\end{equation*}
$$

where $D$ is the rigidity of the chain or the membrane. In the Fourier representation: $\left[L^{R(A)}(\epsilon ; k)\right]^{-1}=\frac{1}{2}\left( \pm i \epsilon-D k^{2}\right)$. In equilibrium the Keldysh component
is not affected by the gradient terms, and is given by Eq. 4.4 (in the real space representation it acquires the factor $\delta\left(r-r^{\prime}\right)$ ). In particular, its classical limit is (cf. Eq. (4.5)) $\left[L^{-1}\right]^{K}=i 2 T \delta\left(t-t^{\prime}\right) \delta\left(r-r^{\prime}\right)$. As a result, the action of a classical elastic chain in contact with a bath is:

$$
\begin{equation*}
S[\hat{\Phi}]=2 \int d r \int_{-\infty}^{\infty} d t\left[-\Phi_{q}\left(\dot{\Phi}_{c l}-D \nabla_{r}^{2} \Phi_{c l}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}\right)+i 2 T \Phi_{q}^{2}\right] \tag{4.29}
\end{equation*}
$$

where the inertia terms have been neglected and we put $\gamma / 2=1$ for brevity.
One may introduce now an auxiliary Hubbard-Stratonovich field $\xi(t ; r)$ and write the Langevin equation according to section 4.4

$$
\begin{equation*}
\dot{\Phi}_{c l}-D \nabla_{r}^{2} \Phi_{c l}+\frac{\partial U\left(\Phi_{c l}\right)}{\partial \Phi_{c l}}=\xi(t ; r) \tag{4.30}
\end{equation*}
$$

where $\xi$ is a Gaussian noise: $\left\langle\xi(t ; r) \xi\left(t^{\prime} ; r^{\prime}\right)\right\rangle=2 T \delta\left(t-t^{\prime}\right) \delta\left(r-r^{\prime}\right)$ with shortrange correlations.

Let us consider an elastic chain sitting in the bottom of the ( $r$-independent) meta-stable potential well, depicted in Fig. 4.6a. If a sufficiently large piece of the chain thermally escapes from the well, it may find it favorable to slide down the potential, pulling the entire chain out of the well. To find the shape of such an optimally large critical domain and its action, let us change to the Hamiltonian variables of section4.6 $q(t ; r) \equiv \Phi_{c l}(t ; r)$ and $p(t ; r) \equiv 2 i \Phi_{q}(t ; r)$. The action (4.29) takes the Hamiltonian form:

$$
\begin{equation*}
i S=-\iint d r d t(p \dot{q}-H(p, q)) ; \quad H \equiv-p \frac{\partial U(q)}{\partial q}+p D \nabla_{r}^{2} q+T p^{2} \tag{4.31}
\end{equation*}
$$

and the corresponding equations of motion are:

$$
\begin{align*}
\dot{q} & =\frac{\delta H}{\delta p}=D \nabla_{r}^{2} q-U_{q}^{\prime}(q)+2 T p  \tag{4.32}\\
\dot{p} & =-\frac{\delta H}{\delta q}=-D \nabla_{r}^{2} p+p U_{q q}^{\prime \prime}(q)
\end{align*}
$$

These are complicated partial differential equations, that cannot be solved in general. Fortunately, the shape of the optimal critical domain can be found. As was discussed in section 4.6 the minimal action trajectory corresponds to a motion with zero energy, $H=0$. According to Eq. 4.31, this is the case if either $p=0$ (classical zero-action trajectory), or $T p=U_{q}^{\prime}(q)-D \nabla_{r}^{2} q$ (finite-action escape trajectory). In the latter case the equation of motion for $q(t ; r)$ takes the form of the classical equation in the reversed time: $\dot{q}=-D \nabla_{r}^{2} q+U_{q}^{\prime}(q)=T p$. Thanks to the last equality the equation of motion for $p(t ; r)$ is automatically satisfied
${ }^{5}$. In the reversed time dynamics the $q(t ; r)=0$ configuration is unstable and therefore the chain develops a "tongue" that grows until it reaches the stationary shape:

$$
\begin{equation*}
-D \nabla_{r}^{2} q+U_{q}^{\prime}(q)=0 \tag{4.33}
\end{equation*}
$$

The solution of this equation gives the shape of the critical domain. Once it is formed, it may grow further according to the classical equation $\dot{q}=D \nabla_{r}^{2} q-$ $U_{q}^{\prime}(q)$ and $p=0$ with zero action. The action along the non-classical escape trajectory, paid to form the "tongue" is $(H(p, q)=0)$ :
$-i S=\iint d r d t p \dot{q}=\frac{1}{T} \iint d r d t\left(U_{q}^{\prime}(q)-D \nabla_{r}^{2} q\right) \dot{q}=\frac{1}{T} \int d r\left(U(q)+\frac{D}{2}\left(\nabla_{r} q\right)^{2}\right)$,
where in the last equality an explicit integration over time is performed. The escape action is given therefore by the static activation expression that includes both the potential and the elastic energies. The optimal domain, Eq. 4.33, is found by the minimization of this static action 4.34. One arrives, thus, at a thermodynamic Landau-type description of the first-order phase transitions. Notice, that the effective thermodynamic description appears due to the assumption that $H(p, q)=0$ and, therefore, that all the processes take an infinitely long time.

## 5. Fermions

### 5.1. Free fermion Keldysh action

Consider a single quantum state, with the energy $\epsilon_{0}$. This state is populated by spin-less fermions (particles obeying the Pauli exclusion principle). In fact, one may have either zero, or one particle in this state. The secondary quantized Hamiltonian of such a system has the form:

$$
\begin{equation*}
\hat{H}=\epsilon_{0} c^{\dagger} c \tag{5.1}
\end{equation*}
$$

where $c^{\dagger}$ and $c$ are fermion creation and annihilation operators of the state $\epsilon_{0}$. They obey standard anticommutation relations: $\left\{c, c^{\dagger}\right\}_{+}=1$ and $\{c, c\}_{+}=$ $\left\{c^{\dagger}, c^{\dagger}\right\}_{+}=0$, where $\{,\}_{+}$stands for the anti-commutator.

One can now consider the evolution operator along the Keldysh contour, $\mathcal{C}$ and the corresponding "partition function", $Z=1$, defined in exactly the same manner as for bosonic systems: Eq. (2.2). The trace of the equilibrium density

[^5]matrix is $\operatorname{Tr}\left\{\rho_{0}\right\}=1+\rho\left(\epsilon_{0}\right)$, where the two terms stand for the empty and the singly occupied state. One divides the Keldysh contour into $(2 N-2)$ time intervals of length $\delta_{t} \sim 1 / N \rightarrow 0$ and introduces resolutions of unity in $2 N$ points along $\mathcal{C}$, Fig. 1.2. The only difference from the bosonic case in section 2.1 is that now one uses a resolution of unity in the fermionic coherent state basis [17]:
\[

$$
\begin{equation*}
1=\iint d \bar{\psi}_{j} d \psi_{j} e^{-\bar{\psi}_{j} \psi_{j}}\left|\psi_{j}\right\rangle\left\langle\psi_{j}\right| \tag{5.2}
\end{equation*}
$$

\]

where $\bar{\psi}_{j}$ and $\psi_{j}$ are mutually independent Grassmann variables. The rest of the algebra goes through exactly as in the bosonic case, section 2.1 As a result, one arrives at:

$$
\begin{equation*}
Z=\frac{1}{\operatorname{Tr}\left\{\rho_{0}\right\}} \iint \prod_{j=1}^{2 N}\left[d \bar{\psi}_{j} d \psi_{j}\right] e^{i \sum_{j, j^{\prime}=1}^{2 N} \bar{\psi}_{j} \mathcal{G}_{j j^{\prime}}^{-1} \psi_{j^{\prime}}} \tag{5.3}
\end{equation*}
$$

where the $2 N \times 2 N$ matrix $\mathcal{G}_{j j^{\prime}}^{-1}$ stands for:

$$
i \mathcal{G}_{j j^{\prime}}^{-1} \equiv\left[\begin{array}{rrr|rrr}
-1 & & & & & -\rho\left(\epsilon_{0}\right)  \tag{5.4}\\
1-h & -1 & & & \\
& 1-h & -1 & & & \\
\hline & & 1 & -1 & & \\
& & & 1+h & -1 & \\
& & & & 1+h & -1
\end{array}\right]
$$

and $h \equiv i \epsilon_{0} \delta_{t}$. The only difference from the bosonic case is the negative sign before the $\rho\left(\epsilon_{0}\right)$ matrix element, originating from the minus sign in the $\left\langle-\psi_{2 N}\right|$

Reminder: the fermionic coherent state $|\psi\rangle \equiv\left(1-\psi c^{\dagger}\right)|0\rangle$, parameterized by a Grassmann number $\psi$ (such that $\left\{\psi, \psi^{\prime}\right\}_{+}=\{\psi, c\}_{+}=0$ ), is an eigenstate of the annihilation operator: $c|\psi\rangle=\psi|\psi\rangle$. Similarly: $\langle\psi| c^{\dagger}=\langle\psi| \bar{\psi}$, where $\bar{\psi}$ is another Grassmann number, unrelated to $\psi$. The matrix elements of a normally ordered operator, such as e.g. the Hamiltonian, take the form $\langle\psi| \hat{H}\left(c^{\dagger}, c\right)\left|\psi^{\prime}\right\rangle=H\left(\bar{\psi}, \psi^{\prime}\right)\left\langle\psi \mid \psi^{\prime}\right\rangle$. The overlap between any two coherent states is $\left\langle\psi \mid \psi^{\prime}\right\rangle=1+\bar{\psi} \psi^{\prime}=\exp \left\{\bar{\psi} \psi^{\prime}\right\}$. The trace of an operator, $\hat{A}$, is calculated as: $\operatorname{Tr}\{\hat{A}\}=\iint d \bar{\psi} d \psi e^{-\bar{\psi} \psi}\langle-\psi| \hat{A}|\psi\rangle$, where the Grassmann integrals are defined as: $\int d \psi 1=0$ and $\int d \psi \psi=1$.
coherent state in the expression for the fermionic trace. To check the normalization, let us evaluate the determinant of such a matrix:

$$
\begin{equation*}
\operatorname{det}\left[i \mathcal{G}^{-1}\right]=1+\rho\left(\epsilon_{0}\right)\left(1-h^{2}\right)^{N-1} \approx 1+\rho\left(\epsilon_{0}\right) e^{\left(\epsilon_{0} \delta_{t}\right)^{2}(N-1)} \rightarrow 1+\rho\left(\epsilon_{0}\right) \tag{5.5}
\end{equation*}
$$

Employing the fact that the fermionic Gaussian integral is given by the determinant (unlike the inverse determinant for bosons) of the correlation matrix, Appendix A, one finds:

$$
\begin{equation*}
Z=\frac{\operatorname{det}\left[i \mathcal{G}^{-1}\right]}{\operatorname{Tr}\left\{\rho_{0}\right\}}=1 \tag{5.6}
\end{equation*}
$$

as it should be. Once again, the upper-right element of the discrete matrix, Eq. 5.4, is crucial to maintain the correct normalization.

Taking the limit $N \rightarrow \infty$ and introducing the continuous notations, $\psi_{j} \rightarrow$ $\psi(t)$, one obtains:

$$
\begin{equation*}
Z=\int \mathcal{D} \bar{\psi} \psi e^{i S[\bar{\psi}, \psi]}=\int \mathcal{D} \bar{\psi} \psi \exp \left\{i \int_{\mathcal{C}}\left[\bar{\psi}(t) \mathcal{G}^{-1} \psi(t)\right] d t\right\} \tag{5.7}
\end{equation*}
$$

where according to Eqs. (5.3) and (5.4) the action is given by

$$
\begin{equation*}
S[\bar{\psi}, \psi]=\sum_{j=2}^{2 N}\left[i \bar{\psi}_{j} \frac{\psi_{j}-\psi_{j-1}}{\delta t_{j}}-\epsilon_{0} \bar{\psi}_{j} \psi_{j-1}\right] \delta t_{j}+i \bar{\psi}_{1}\left(\psi_{1}+\rho\left(\epsilon_{0}\right) \psi_{2 N}\right) \tag{5.8}
\end{equation*}
$$

where $\delta t_{j} \equiv t_{j}-t_{j-1}= \pm \delta_{t}$. Thus the continuous form of the operator $\mathcal{G}^{-1}$ is the same as for bosons, Eq. (2.13): $\mathcal{G}^{-1}=i \partial_{t}-\epsilon_{0}$. Again the upper-right element of the discrete matrix (the last term in Eq. (5.8), that contains information about the distribution function, is seemingly absent in the continuous notations.

Splitting the Grassmann field $\psi(t)$ into the two components $\psi_{+}(t)$ and $\psi_{-}(t)$ that reside on the forward and the backward parts of the time contour correspondingly, one may rewrite the action as:

$$
\begin{equation*}
S=\int_{-\infty}^{\infty} d t \bar{\psi}_{+}(t)\left[i \partial_{t}-\epsilon_{0}\right] \psi_{+}(t)-\int_{-\infty}^{\infty} d t \bar{\psi}_{-}(t)\left[i \partial_{t}-\epsilon_{0}\right] \psi_{-}(t) \tag{5.9}
\end{equation*}
$$

where the dynamics of $\psi_{+}$and $\psi_{-}$are not independent from each other, due to the presence of non-zero off-diagonal blocks in the discrete matrix, Eq. 5.4.).

The four fermionic Greens functions: $\mathcal{G}^{T(\tilde{T})}$ and $\mathcal{G}^{<(>)}$are defined in the same way as their bosonic counterparts, Eq. (2.16):

$$
\begin{align*}
\left\langle\psi_{+}(t) \bar{\psi}_{-}\left(t^{\prime}\right)\right\rangle & \equiv i \mathcal{G}^{<}\left(t, t^{\prime}\right)=-\frac{\operatorname{Tr}\left\{c^{\dagger}\left(t^{\prime}\right) c(t) \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}}=-n_{F} e^{-i \epsilon_{0}\left(t-t^{\prime}\right)} \\
\left\langle\psi_{-}(t) \bar{\psi}_{+}\left(t^{\prime}\right)\right\rangle & \equiv i \mathcal{G}^{>}\left(t, t^{\prime}\right)=\frac{\operatorname{Tr}\left\{c(t) c^{\dagger}\left(t^{\prime}\right) \hat{\rho}_{0}\right\}}{\operatorname{Tr}\left\{\hat{\rho}_{0}\right\}}=\left(1-n_{F}\right) e^{-i \epsilon_{0}\left(t-t^{\prime}\right)} ;(5.10)  \tag{5.10}\\
\left\langle\psi_{+}(t) \bar{\psi}_{+}\left(t^{\prime}\right)\right\rangle & \equiv i \mathcal{G}^{T}\left(t, t^{\prime}\right)=\theta\left(t-t^{\prime}\right) i \mathcal{G}^{>}\left(t, t^{\prime}\right)+\theta\left(t^{\prime}-t\right) i \mathcal{G}^{<}\left(t, t^{\prime}\right) \\
\left\langle\psi_{-}(t) \bar{\psi}_{-}\left(t^{\prime}\right)\right\rangle & \equiv i \mathcal{G}^{\tilde{T}}\left(t, t^{\prime}\right)=\theta\left(t^{\prime}-t\right) i \mathcal{G}^{>}\left(t, t^{\prime}\right)+\theta\left(t-t^{\prime}\right) i \mathcal{G}^{<}\left(t, t^{\prime}\right)
\end{align*}
$$

The difference is in the minus sign in the expression for $\mathcal{G}^{<}$, due to the anticommutation relations, and the Planck occupation number is exchanged for the Fermi one: $n \rightarrow n_{F} \equiv \rho\left(\epsilon_{0}\right) /\left(1+\rho\left(\epsilon_{0}\right)\right)$. Equations 2.17) and 2.18 hold for the fermionic Green functions as well.

### 5.2. Keldysh rotation

It is customary to perform the Keldysh rotation in the fermionic case in a different manner from the bosonic one. Define the new fields as:

$$
\begin{equation*}
\psi_{1}(t)=\frac{1}{\sqrt{2}}\left(\psi_{+}(t)+\psi_{-}(t)\right) ; \quad \psi_{2}(t)=\frac{1}{\sqrt{2}}\left(\psi_{+}(t)-\psi_{-}(t)\right) \tag{5.11}
\end{equation*}
$$

This line is exactly parallel to the bosonic one, Eq. 2.19. However, following Larkin and Ovchinnikov [20], it is agreed that the "bar" fields transform in a different way:

$$
\begin{equation*}
\bar{\psi}_{1}(t)=\frac{1}{\sqrt{2}}\left(\bar{\psi}_{+}(t)-\bar{\psi}_{-}(t)\right) ; \quad \bar{\psi}_{2}(t)=\frac{1}{\sqrt{2}}\left(\bar{\psi}_{+}(t)+\bar{\psi}_{-}(t)\right) \tag{5.12}
\end{equation*}
$$

The point is that the Grassmann fields $\bar{\psi}$ are not conjugated to $\psi$, but rather are completely independent fields, that may be chosen to transform in an arbitrary manner (as long as the transformation matrix has a non-zero determinant). Notice, that there is no issue regarding the convergence of the integrals, since the Grassmann integrals are always convergent. We also avoid the subscripts $c l$ and $q$, because the Grassmann variables never have a classical meaning. Indeed, one can never write a saddle-point or any other equation in terms of $\bar{\psi}, \psi$ rather they must always be integrated out in some stage of the calculations.

Employing Eqs. (5.11, (5.12) along with Eq. 5.10, one finds:

$$
-i\left\langle\psi_{a}(t) \bar{\psi}_{b}\left(t^{\prime}\right)\right\rangle \equiv \hat{\mathcal{G}}^{a b}=\left(\begin{array}{cc}
\mathcal{G}^{R}\left(t, t^{\prime}\right) & \mathcal{G}^{K}\left(t, t^{\prime}\right)  \tag{5.13}\\
0 & \mathcal{G}^{A}\left(t, t^{\prime}\right)
\end{array}\right)
$$

where hereafter $a, b=(1,2)$. The presence of zero in the $(2,1)$ element of this matrix is a manifestation of identity 2.18. The retarded, advanced and Keldysh components of the Green function are expressed in terms of $\mathcal{G}^{T(\tilde{T})}$ and $\mathcal{G}^{<(>)}$in exactly the same way as their bosonic analogs, Eq. 2.21, and therefore posses the same symmetry properties: Eqs. (2.22-(2.24). An important consequence of Eqs. 2.23, 2.24 is:

$$
\begin{equation*}
\operatorname{Tr}\left\{\hat{\mathcal{G}}_{1} \circ \hat{\mathcal{G}}_{2} \circ \ldots \circ \hat{\mathcal{G}}_{k}\right\}(t, t)=0 \tag{5.14}
\end{equation*}
$$

where the circular multiplication sign involves integration over the intermediate times along with the $2 \times 2$ matrix multiplication. The argument $(t, t)$ states that the first time argument of $\hat{\mathcal{G}}_{1}$ and the last argument of $\hat{\mathcal{G}}_{k}$ are the same.

Notice that the fermionic Green function has a different structure from its bosonic counterpart, Eq. 2.20): the positions of the $R, A$ and $K$ components in the matrix are exchanged. The reason, of course, is the different convention for transformation of the "bar" fields. One could choose the fermionic convention to be the same as the bosonic (but not the other way around!), thus having the same structure, Eq. 2.20, for fermions as for bosons. The rationale for the Larkin-Ovchinnikov choice, Eq. [5.13], is that the inverse Green function, $\mathcal{G}^{-1}$ and fermionic self energy $\hat{\Sigma}_{F}$ have the same appearance as $\hat{\mathcal{G}}$ :

$$
\hat{\mathcal{G}}^{-1}=\left(\begin{array}{cc}
{\left[\mathcal{G}^{R}\right]^{-1}} & {\left[\mathcal{G}^{-1}\right]^{K}}  \tag{5.15}\\
0 & {\left[\mathcal{G}^{A}\right]^{-1}}
\end{array}\right) ; \quad \hat{\Sigma}_{F}=\left(\begin{array}{cc}
\Sigma_{F}^{R} & \Sigma_{F}^{K} \\
0 & \Sigma_{F}^{A}
\end{array}\right)
$$

whereas in the case of bosons $\hat{G}^{-1}$, Eq. 2.29, and $\hat{\Sigma}$, Eq. (3.10, look differently from $\hat{G}$, Eq. 2.20. This fact gives the form Eq. 5.13, (5.15) a certain technical advantage.

For the single fermionic state (see. Eq. (5.10):

$$
\begin{align*}
\mathcal{G}^{R}\left(t, t^{\prime}\right) & =-i \theta\left(t-t^{\prime}\right) e^{-i \epsilon_{0}\left(t-t^{\prime}\right)} \rightarrow\left(\epsilon-\epsilon_{0}+i 0\right)^{-1} \\
\mathcal{G}^{A}\left(t, t^{\prime}\right) & =i \theta\left(t^{\prime}-t\right) e^{-i \epsilon_{0}\left(t-t^{\prime}\right)} \rightarrow\left(\epsilon-\epsilon_{0}-i 0\right)^{-1}  \tag{5.16}\\
\mathcal{G}^{K}\left(t, t^{\prime}\right) & =-i\left(1-2 n_{F}\right) e^{-i \epsilon_{0}\left(t-t^{\prime}\right)} \rightarrow\left(1-2 n_{F}(\epsilon)\right)(-2 \pi i) \delta\left(\epsilon-\epsilon_{0}\right)
\end{align*}
$$

where the r.h.s. provides also the Fourier transforms. In thermal equilibrium, one obtains:

$$
\begin{equation*}
\mathcal{G}^{K}(\epsilon)=\tanh \frac{\epsilon}{2 T}\left(\mathcal{G}^{R}(\epsilon)-\mathcal{G}^{A}(\epsilon)\right) . \tag{5.17}
\end{equation*}
$$

This is the FDT for fermions. As in the case of bosons, the FDT statement is a generic feature of an equilibrium system, not restricted to the toy model.

In general, it is convenient to parameterize the anti-Hermitian Keldysh Green function by a Hermitian matrix $\mathcal{F}=\mathcal{F}^{\dagger}$ as:

$$
\begin{equation*}
\mathcal{G}^{K}=\mathcal{G}^{R} \circ \mathcal{F}-\mathcal{F} \circ \mathcal{G}^{A}, \tag{5.18}
\end{equation*}
$$

The Wigner transform of $\mathcal{F}\left(t, t^{\prime}\right)$ plays the role of the fermionic distribution function.

One may continue now to a system with many degrees of freedom, counted by an index $k$. To this end, one simply changes: $\epsilon_{0} \rightarrow \epsilon_{k}$ and perform summations over $k$. If $k$ is a momentum and $\epsilon_{k}=k^{2} /(2 m)$, it is instructive to transform to the real space representation: $\psi(t ; k) \rightarrow \psi(t ; r)$ and $\epsilon_{k}=k^{2} /(2 m)=-(2 m)^{-1} \nabla_{r}^{2}$. Finally, the Keldysh action for a non-interacting gas of fermions takes the form:

$$
\begin{equation*}
S_{0}[\bar{\psi}, \psi]=\iint d x d x^{\prime} \sum_{a, b=1}^{2} \bar{\psi}_{a}(x)\left[\hat{\mathcal{G}}^{-1}\left(x, x^{\prime}\right)\right]_{a b} \psi_{b}\left(x^{\prime}\right) \tag{5.19}
\end{equation*}
$$

where $x=(t ; r)$ and the matrix correlator $\left[\hat{\mathcal{G}}^{-1}\right]_{a b}$ has the structure of Eq. (5.15) with

$$
\begin{equation*}
\left[\mathcal{G}^{R(A)}\left(x, x^{\prime}\right)\right]^{-1}=\delta\left(x-x^{\prime}\right)\left(i \partial_{t^{\prime}}+\frac{1}{2 m} \nabla_{r^{\prime}}^{2}\right) \tag{5.20}
\end{equation*}
$$

Although in continuous notations the $R$ and the $A$ components look seemingly the same, one has to remember that in the discrete time representation, they are matrices with the structure below and above the main diagonal correspondingly. The Keldysh component is a pure regularization, in the sense that it does not have a continuum limit (the self-energy Keldysh component does have a nonzero continuum representation). All this information is already properly taken into account, however, in the structure of the Green function, Eq. (5.13).

### 5.3. External fields and sources

Let us introduce an external time-dependent scalar potential $-V(t)$ defined along the contour. It interacts with the fermions as: $S_{V}=\int_{\mathcal{C}} d t V(t) \psi(t) \psi(t)$. Expressing it via the field components, one finds:

$$
\begin{align*}
S_{V} & =\int_{-\infty}^{\infty} d t\left[V_{+} \bar{\psi}_{+} \psi_{+}-V_{-} \bar{\psi}_{-} \psi_{-}\right]=\int_{-\infty}^{\infty} d t\left[V_{c l}\left(\bar{\psi}_{+} \psi_{+}-\bar{\psi}_{-} \psi_{-}\right)+V_{q}\left(\bar{\psi}_{+} \psi_{+}+\bar{\psi}_{-} \psi_{-}\right)\right] \\
& =\int_{-\infty}^{\infty} d t\left[V_{c l}\left(\bar{\psi}_{1} \psi_{1}+\bar{\psi}_{2} \psi_{2}\right)+V_{q}\left(\bar{\psi}_{1} \psi_{2}+\bar{\psi}_{2} \psi_{1}\right)\right] \tag{5.21}
\end{align*}
$$

where the $V_{c l}$ and the $V_{q}$ components are defined in the standard way for real bosonic fields: $V_{c l(q)}=\left(V_{+} \pm V_{-}\right) / 2$. Notice that the physical fermionic density (symmetrized over the two branches of the contour): $\varrho=\frac{1}{2}\left(\bar{\psi}_{+} \psi_{+}+\bar{\psi}_{-} \psi_{-}\right)$is coupled to the quantum component of the source field, $V_{q}$. On the other hand, the classical source component, $V_{c l}$, is nothing but an external physical scalar potential, the same at the two branches.

Notations may be substantially compactified by introducing vertex gammamatrices:

$$
\hat{\gamma}^{c l} \equiv\left(\begin{array}{cc}
1 & 0  \tag{5.22}\\
0 & 1
\end{array}\right) ; \quad \hat{\gamma}^{q} \equiv\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)
$$

With the help of these definitions, the source action 5.21 may be written as:

$$
\begin{equation*}
S_{V}=\int_{-\infty}^{\infty} d t \sum_{a, b=1}^{2}\left[V_{c l} \bar{\psi}_{a} \gamma_{a b}^{c l} \psi_{b}+V_{q} \bar{\psi}_{a} \gamma_{a b}^{q} \psi_{b}\right]=\int_{-\infty}^{\infty} d t \hat{\bar{\psi}} V_{\alpha} \hat{\gamma}^{\alpha} \hat{\psi} \tag{5.23}
\end{equation*}
$$

where the summation index is $\alpha=(c l, q)$.
Let us define now the "generating" function as:

$$
\begin{equation*}
Z\left[V_{c l}, V_{q}\right] \equiv\left\langle e^{i S_{V}}\right\rangle \tag{5.24}
\end{equation*}
$$

where the angular brackets denote the functional integration over the Grassmann fields $\bar{\psi}$ and $\psi$ with the weight of $e^{i S_{0}}$ specified by the fermionic action 5.19. In the absence of the quantum component, $V_{q}=0$, the source field is the same at both branches of the contour. Therefore, the evolution along the contour still brings the system back to its exact initial state. Thus one expects that the classical component alone does not change the fundamental normalization, $Z=1$. As a result:

$$
\begin{equation*}
Z\left[V_{c l}, V_{q}=0\right]=1 \tag{5.25}
\end{equation*}
$$

One may verify this statement explicitly by expanding the action in powers of $V_{c l}$ and employing the Wick theorem. For example, in the first order one finds: $Z\left[V_{c l}, 0\right]=1+\int d t V_{c l}(t) \operatorname{Tr}\left\{\hat{\gamma}^{c l} \hat{\mathcal{G}}(t, t)\right\}=1$, where one uses that $\hat{\gamma}^{c l}=\hat{1}$ along with Eq. (5.14). It is straightforward to see that for exactly the same reasons all higher order terms in $V_{c l}$ vanish as well.

A lesson from Eq. 5.25) is that one necessarily has to introduce quantum sources (that change sign between the forward and the backward branches of the contour). The presence of such source fields explicitly violates causality, and thus changes the generating function. On the other hand, these fields usually do not have a physical meaning and play only an auxiliary role. In most cases one uses them only to generate observables by an appropriate differentiation. Indeed, as
was mentioned above, the physical density is coupled to the quantum component of the source. In the end, one takes the quantum sources to be zero, restoring the causality of the action. Notice that the classical component, $V_{c l}$, does not have to be taken to zero.

Let us see how it works. Suppose we are interested in the average fermion density $\varrho$ at time $t$ in the presence of a certain physical scalar potential $V_{c l}(t)$. According to Eqs. 5.21 and 5.24 it is given by:

$$
\begin{equation*}
\varrho\left(t ; V_{c l}\right)=-\left.\frac{i}{2} \frac{\delta}{\delta V_{q}(t)} Z\left[V_{c l}, V_{q}\right]\right|_{V_{q}=0} . \tag{5.26}
\end{equation*}
$$

The problem is simplified if the external field, $V_{c l}$, is weak in some sense. Then one may restrict oneself to the linear response, by defining the susceptibility:

$$
\begin{equation*}
\left.\Pi^{R}\left(t, t^{\prime}\right) \equiv \frac{\delta}{\delta V_{c l}\left(t^{\prime}\right)} \varrho\left(t ; V_{c l}\right)\right|_{V_{c l}=0}=-\left.\frac{i}{2} \frac{\delta^{2} Z\left[V_{c l}, V_{q}\right]}{\delta V_{c l}\left(t^{\prime}\right) \delta V_{q}(t)}\right|_{V_{q}=V_{c l}=0} \tag{5.27}
\end{equation*}
$$

We add the subscript $R$ anticipating on physical grounds that the response function must be retarded (causality). We shall demonstrate it momentarily. First, let us introduce the polarization matrix as:

$$
\hat{\Pi}^{\alpha \beta}\left(t, t^{\prime}\right) \equiv-\left.\frac{i}{2} \frac{\delta^{2} \ln Z[\hat{V}]}{\delta V_{\beta}\left(t^{\prime}\right) \delta V_{\alpha}(t)}\right|_{\hat{V}=0}=\left(\begin{array}{cc}
0 & \Pi^{A}\left(t, t^{\prime}\right)  \tag{5.28}\\
\Pi^{R}\left(t, t^{\prime}\right) & \Pi^{K}\left(t, t^{\prime}\right)
\end{array}\right)
$$

Due to the fundamental normalization, Eq. (5.25), the logarithm is redundant for the $R$ and the $A$ components and therefore the two definitions (5.27) and (5.28) are not in contradiction. The fact that $\Pi^{c l, c l}=0$ is obvious from Eq. 5.25. To evaluate the polarization matrix, $\hat{\Pi}$, consider the Gaussian action, Eq. 5.19 . Adding the source term, Eq. (5.23), one finds: $S_{0}+S_{V}=\int d t \hat{\bar{\psi}}\left[\hat{\mathcal{G}}^{-1}+V_{\alpha} \hat{\gamma}^{\alpha}\right] \psi$. Integrating out the fermion fields $\hat{\bar{\psi}}, \hat{\psi}$ according to the rules of fermionic Gaussian integration, Appendix A one obtains:

$$
\begin{equation*}
Z[\hat{V}]=\frac{1}{\operatorname{Tr} \hat{\rho}_{0}} \operatorname{det}\left\{i \hat{\mathcal{G}}^{-1}+i V_{\alpha} \hat{\gamma}^{\alpha}\right\}=\operatorname{det}\left\{1+\hat{\mathcal{G}} V_{\alpha} \hat{\gamma}^{\alpha}\right\}=e^{\operatorname{Tr} \ln \left(1+\hat{\mathcal{G}} V_{\alpha} \hat{\gamma}^{\alpha}\right)} \tag{5.29}
\end{equation*}
$$

where one used normalization, Eq. [5.6. Notice, that the normalization is exactly right, since $Z[0]=1$. One may now expand $\ln \left(1+\hat{\mathcal{G}} V_{\alpha} \hat{\gamma}^{\alpha}\right)$ to the second order in $V$ and then differentiate twice. As a result, one finds for the polarization matrix:

$$
\begin{equation*}
\Pi^{\alpha \beta}\left(t, t^{\prime}\right)=-\frac{i}{2} \operatorname{Tr}\left\{\hat{\gamma}^{\alpha} \hat{\mathcal{G}}\left(t, t^{\prime}\right) \hat{\gamma}^{\beta} \hat{\mathcal{G}}\left(t^{\prime}, t\right)\right\} \tag{5.30}
\end{equation*}
$$

Substituting the explicit form of the gamma-matrices, Eq. (5.22), and the Green functions, Eq. 5.13, one obtains for the response and the correlation components:

$$
\begin{align*}
& \Pi^{R(A)}\left(t, t^{\prime}\right)=-\frac{i}{2}\left[\mathcal{G}^{R(A)}\left(t, t^{\prime}\right) \mathcal{G}^{K}\left(t^{\prime}, t\right)+\mathcal{G}^{K}\left(t, t^{\prime}\right) \mathcal{G}^{A(R)}\left(t^{\prime}, t\right)\right]  \tag{5.31}\\
& \Pi^{K}\left(t, t^{\prime}\right)=-\frac{i}{2}\left[\mathcal{G}^{K}\left(t, t^{\prime}\right) \mathcal{G}^{K}\left(t^{\prime}, t\right)+\mathcal{G}^{R}\left(t, t^{\prime}\right) \mathcal{G}^{A}\left(t^{\prime}, t\right)+\mathcal{G}^{A}\left(t, t^{\prime}\right) \mathcal{G}^{R}\left(t^{\prime}, t\right)\right]
\end{align*}
$$

From the first line it is obvious that $\Pi^{R(A)}\left(t, t^{\prime}\right)$ is indeed a lower (upper) triangular matrix in the time domain, justifying their superscripts. Moreover, from the symmetry properties of the fermionic Green functions (same as Eq. (2.22) one finds: $\Pi^{R}=\left[\Pi^{A}\right]^{\dagger}$ and $\Pi^{K}=-\left[\Pi^{K}\right]^{\dagger}$. As a result, the polarization matrix, $\hat{\Pi}$, possesses all the symmetry properties of the bosonic self-energy $\hat{\Sigma}$, Eq. 3.10.

Exercise: In the stationary case: $\hat{\mathcal{G}}\left(t, t^{\prime}\right)=\hat{\mathcal{G}}\left(t-t^{\prime}\right)$. Fourier transform to the energy domain and write down expressions for $\hat{\Pi}(\omega)$. Assume thermal equilibrium and, using Eq. 5.17, rewrite your results in terms of $\mathcal{G}^{R(A)}$ and the equilibrium distribution function. Show that in equilibrium, the response, $\Pi^{R(A)}(\omega)$, and the correlation, $\Pi^{K}(\omega)$, functions are related by the bosonic FDT:

$$
\begin{equation*}
\Pi^{K}(\omega)=\operatorname{coth} \frac{\omega}{2 T}\left(\Pi^{R}(\omega)-\Pi^{A}(\omega)\right) \tag{5.32}
\end{equation*}
$$

Equation 5.31) for $\Pi^{R}$ constitutes the Kubo formula for the density-density response function. In equilibrium it may be derived using the Matsubara technique. The Matsubara routine involves, however, the analytical continuation from discrete imaginary frequency $\omega_{m}$ to real frequency $\omega$. This procedure may prove to be cumbersome in specific applications. The purpose of the above discussion is to demonstrate how the linear response problems may be compactly formulated in the Keldysh language. The latter allows to circumvent the analytical continuation and yields results directly in the real frequency domain.

### 5.4. Tunnelling current

As a simple application of the technique, let us derive the expression for the tunnelling conductance. Our starting point is the tunnelling Hamiltonian:

$$
\begin{equation*}
\hat{H}=\sum_{k}\left[\epsilon_{k}^{(c)} c_{k}^{\dagger} c_{k}+\epsilon_{k}^{(d)} d_{k}^{\dagger} d_{k}\right]+\sum_{k k^{\prime}}\left[T_{k k^{\prime}} c_{k}^{\dagger} d_{k^{\prime}}+T_{k k^{\prime}}^{*} d_{k^{\prime}}^{\dagger} c_{k}\right] \tag{5.33}
\end{equation*}
$$

where the operators $c_{k}$ and $d_{k^{\prime}}$ describe fermions in the left and right leads, while $T_{k k^{\prime}}$ are tunnelling matrix elements between the two. The current operator is: $\hat{J}=\frac{d}{d t} \sum_{k} c_{k}^{\dagger} c_{k}=i\left[\hat{H}, \sum_{k} c_{k}^{\dagger} c_{k}\right]_{-}=-i \sum_{k k^{\prime}}\left[T_{k k^{\prime}} c_{k}^{\dagger} d_{k^{\prime}}-T_{k k^{\prime}}^{*} d_{k^{\prime}}^{\dagger} c_{k}\right]$.

To describe the system in the Keldysh formalism, one introduces the the fourcomponent spinor: $\hat{\bar{\psi}}_{k}=\left(\bar{\psi}_{1 k}^{(c)}, \bar{\psi}_{2 k}^{(c)}, \bar{\psi}_{1 k}^{(d)}, \bar{\psi}_{2 k}^{(d)}\right)$, a similarly one for the fields without the bar, and the $4 \times 4$ matrices:
$\hat{\mathcal{G}}_{k}=\left(\begin{array}{cc}\hat{\mathcal{G}}_{k}^{(c)} & 0 \\ 0 & \hat{\mathcal{G}}_{k}^{(d)}\end{array}\right) ; \quad \hat{T}_{k, k^{\prime}}=\left(\begin{array}{cc}0 & T_{k k^{\prime}} \hat{\gamma}^{c l} \\ T_{k k^{\prime}}^{*} \hat{\gamma}^{c l} & 0\end{array}\right) ; \quad \hat{J}_{k k^{\prime}}=\left(\begin{array}{cc}0 & i T_{k k^{\prime}} \hat{\gamma}^{q} \\ -i T_{k k^{\prime}}^{*} \hat{\gamma}^{q} & 0\end{array}\right)$.
In addition to the already familiar Keldysh structure the spinors and matrices above possess the structure of the left-right space. In terms of these objects the action and the current operator take the form:

$$
\begin{equation*}
S=\int_{-\infty}^{\infty} d t \sum_{k k^{\prime}} \hat{\bar{\psi}}_{k}\left[\delta_{k k^{\prime}} \hat{\mathcal{G}}_{k}^{-1}-\hat{T}_{k, k^{\prime}}\right] \hat{\psi}_{k^{\prime}} ; \quad \hat{J}(t)=-\sum_{k k^{\prime}} \hat{\bar{\psi}}_{k} \hat{J}_{k, k^{\prime}} \hat{\psi}_{k^{\prime}} \tag{5.35}
\end{equation*}
$$

The current is expressed through the $\gamma^{q}$ vertex matrix in the Keldysh space because any observable is generated by differentiation over the quantum component of the source field (the classical component of the source does not change the normalization, Eq. (5.25).

One is now in a position to calculate the average tunnelling current up to the second order in the matrix elements $T_{k, k^{\prime}}$. To this end one expands the action up to the first order in $T_{k, k^{\prime}}$, and applies the Wick theorem:

$$
\begin{gather*}
J(t)=i \int_{-\infty}^{\infty} d t^{\prime} \sum_{k k^{\prime}} \operatorname{Tr}\left\{\hat{J}_{k k^{\prime}} \hat{\mathcal{G}}_{k^{\prime}}\left(t, t^{\prime}\right) \hat{T}_{k^{\prime} k} \hat{\mathcal{G}}_{k}\left(t^{\prime}, t\right)\right\}  \tag{5.36}\\
=\int_{-\infty}^{\infty} d t^{\prime} \sum_{k k^{\prime}}\left|T_{k k^{\prime}}\right|^{2} \operatorname{Tr}\left\{\hat{\gamma}^{q} \hat{\mathcal{G}}_{k}^{(c)}\left(t, t^{\prime}\right) \hat{\gamma}^{c l} \hat{\mathcal{G}}_{k^{\prime}}^{(d)}\left(t^{\prime}, t\right)-\hat{\gamma}^{q} \hat{\mathcal{G}}_{k^{\prime}}^{(d)}\left(t, t^{\prime}\right) \hat{\gamma}^{c l} \hat{\mathcal{G}}_{k}^{(c)}\left(t^{\prime}, t\right)\right\}= \\
\int_{-\infty}^{\infty} d t^{\prime} \sum_{k k^{\prime}}\left|T_{k k^{\prime}}\right|^{2}\left[\mathcal{G}_{k\left(t, t^{\prime}\right)}^{(c) R} \mathcal{G}_{k^{\prime}\left(t^{\prime}, t\right)}^{(d) K}+\mathcal{G}_{k\left(t, t^{\prime}\right)}^{(c) K} \mathcal{G}_{k^{\prime}\left(t^{\prime}, t\right)}^{(d) A}-\mathcal{G}_{k^{\prime}\left(t, t^{\prime}\right)}^{(d) R} \mathcal{G}_{k\left(t^{\prime}, t\right)}^{(c) K}-\mathcal{G}_{k^{\prime}\left(t, t^{\prime}\right)}^{(d) K} \mathcal{G}_{k\left(t^{\prime}, t\right)}^{(c) A}\right] .
\end{gather*}
$$

Now let us assume a stationary situation so that all the Green functions depend on the time difference only. We shall also assume that each lead is in local thermal equilibrium and thus its Green functions are related to each other via the FDT: $\mathcal{G}_{k}^{(c) K}(\epsilon)=\left(1-2 n_{F}^{(c)}(\epsilon)\right)\left[\mathcal{G}_{k}^{(c) R}(\epsilon)-\mathcal{G}_{k}^{(c) A}(\epsilon)\right]$. Similarly for the "d"-lead with a different occupation function $n_{F}^{(d)}(\epsilon)$. As a result, one finds for the tunnelling current:

$$
\begin{equation*}
J=\int_{-\infty}^{\infty} \frac{d \epsilon}{\pi}\left[n_{F}^{(d)}(\epsilon)-n_{F}^{(c)}(\epsilon)\right] \sum_{k k^{\prime}}\left|T_{k k^{\prime}}\right|^{2}\left[\mathcal{G}_{k}^{(c) R}-\mathcal{G}_{k}^{(c) A}\right]\left[\mathcal{G}_{k^{\prime}}^{(d) R}-\mathcal{G}_{k^{\prime}}^{(d) A}\right] \tag{5.37}
\end{equation*}
$$

If the current matrix elements may be considered as approximately momentum independent: $\left|T_{k k^{\prime}}\right|^{2} \approx|T|^{2}$, the last expression is reduced to:

$$
\begin{equation*}
J=4 \pi|T|^{2} \int_{-\infty}^{\infty} d \epsilon\left[n_{F}^{(c)}(\epsilon)-n_{F}^{(d)}(\epsilon)\right] \nu^{(c)}(\epsilon) \nu^{(d)}(\epsilon) \tag{5.38}
\end{equation*}
$$

where the density of states (DOS) is defined as:

$$
\begin{equation*}
\nu(\epsilon) \equiv \frac{i}{2 \pi} \sum_{k}\left[\mathcal{G}_{k}^{R}(\epsilon)-\mathcal{G}_{k}^{A}(\epsilon)\right] \tag{5.39}
\end{equation*}
$$

### 5.5. Interactions

Consider a liquid of fermions that interact through instantaneous density-density interactions: $\hat{H}_{\text {int }}=-\frac{1}{2} \iint d r d r^{\prime}: \hat{\varrho}(r) U\left(r-r^{\prime}\right) \hat{\varrho}\left(r^{\prime}\right):$, where $\hat{\varrho}(r)=c_{r}^{\dagger} c_{r}$ is the local density operator and : . . : stands for normal ordering. The corresponding Keldysh contour action has the form: $S_{\text {int }}=\frac{1}{2} \int_{\mathcal{C}} d t \iint d r d r^{\prime} U(r-$ $\left.r^{\prime}\right) \bar{\psi}_{r} \bar{\psi}_{r^{\prime}} \psi_{r} \psi_{r^{\prime}}$. One may now perform the Hubbard-Stratonovich transformation with the help of a real boson field $\varphi(t ; r)$, defined along the contour:

$$
\begin{equation*}
e^{\frac{i}{2} \int d t \iint d r d r^{\prime} U\left(r-r^{\prime}\right) \bar{\psi}_{r} \bar{\psi}_{r^{\prime}} \psi_{r} \psi_{r^{\prime}}}=\int \mathcal{D} \varphi e^{i \int \mathcal{C}^{i} d t\left[\frac{1}{2} \iint d r d r^{\prime} \varphi_{r} U_{r r^{\prime}}^{-1} \varphi_{r^{\prime}}+\int d r \varphi_{r} \bar{\psi}_{r} \psi_{r}\right]} \tag{5.40}
\end{equation*}
$$

where $U^{-1}$ is a kernel, that is inverse to the interaction potential: $U^{-1} \circ U=$ 1. One notices that the auxiliary bosonic field, $\varphi$, enters the fermionic action in exactly the same manner as a scalar source field. Following Eq. 5.21, one introduces $\varphi_{c l(q)} \equiv\left(\varphi_{+} \pm \varphi_{-}\right) / 2$ and rewrites the fermion-boson interaction term as $\bar{\psi}_{a} \varphi_{\alpha} \gamma_{a b}^{\alpha} \psi_{b}$, where summations are assumed over $a, b=(1,2)$ and $\alpha=$ $(c l, q)$. The free bosonic term takes the form of: $\frac{1}{2} \varphi U^{-1} \varphi \rightarrow \varphi_{\alpha} U^{-1} \hat{\sigma}_{1}^{\alpha \beta} \varphi_{\beta}$.

At this stage the fermionic action is Gaussian and one may integrate out the Grassmann variables in the same way it was done in Eq. 5.29. As a result, one finds for the generating function, Eq. (5.24), of the interacting fermionic liquid:

$$
\begin{equation*}
Z[\hat{V}]=\int \mathcal{D} \varphi e^{i \int_{-\infty}^{\infty} d t \iint d r d r^{\prime} \hat{\varphi} U^{-1} \hat{\sigma}_{1} \hat{\varphi}+\operatorname{Tr} \ln \left[1+\hat{\mathcal{G}}\left(V_{\alpha}+\varphi_{\alpha}\right) \hat{\gamma}^{\alpha}\right]} \tag{5.41}
\end{equation*}
$$

Quite generally, thus, one may reduce an interacting fermionic problem to a theory of an effective non-linear bosonic field (longitudinal photons). Let us demonstrate that this bosonic theory possesses the causality structure. To this end, one formally expands the logarithm on the r.h.s. of Eq. (5.41). Employing Eq. (5.14)
and recalling that $\hat{\gamma}^{c l}=\hat{1}$, one notices that for $\varphi_{q}=V_{q}=0$ the bosonic action is zero. As a result, Eq. 2.32 holds.

To proceed we shall restrict ourselves to the, so called, random phase approximation (RPA). It neglects all terms in the expansion of the logarithm beyond the second order. The second order term in the expansion is conveniently expressed through the (bare) polarization matrix $\Pi^{\alpha, \beta}$ (see Eq. (5.30) of the noninteracting fermions. The resulting effective bosonic theory is Gaussian with the action:

$$
\begin{equation*}
S_{R P A}[\hat{\varphi}, \hat{V}]=\iint_{-\infty}^{\infty} d t d t^{\prime} \iint d r d r^{\prime}\left[\hat{\varphi}\left(U^{-1} \hat{\sigma}_{1}-\hat{\Pi}\right) \hat{\varphi}-2 \hat{\varphi} \hat{\Pi} \hat{V}-\hat{V} \hat{\Pi} \hat{V}\right] \tag{5.42}
\end{equation*}
$$

One notices that the bare polarization matrix plays exactly the same role as of the self-energy, $\hat{\Sigma}$, cf. Eqs. (3.9), (3.10), in the effective bosonic theory. As a result, the full bosonic correlator $\left(U^{-1} \hat{\sigma}_{1}-\hat{\Pi}\right)$ possesses all the causality properties, listed in section 2.4

Finally, let us evaluate the dressed polarization matrix of the interacting fermiliquid in the RPA. To this end one may perform the bosonic Gaussian integration in the RPA action 5.42 to find the logarithm of the generating function: $i \ln Z_{R P A}[\hat{V}]=\hat{V}\left(\hat{\Pi}+\hat{\Pi}\left(U^{-1} \hat{\sigma}_{1}-\hat{\Pi}\right)^{-1} \hat{\Pi}\right) \hat{V}$. Finally, employing the definition of the polarization matrix, Eq. 5.28, and performing simple matrix algebra, one finds:

$$
\begin{equation*}
\hat{\Pi}_{R P A}=\hat{\Pi} \circ\left(1-\hat{\sigma}_{1} U \circ \hat{\Pi}\right)^{-1} \tag{5.43}
\end{equation*}
$$

It is straightforward to demonstrate that the dressed polarization matrix possesses the same causality structure as the bare one, Eq. (5.28). For the response component of the dressed polarization, $\hat{\Pi}_{R P A}^{R}$, the second factor on the r.h.s. of Eq. (5.43) may be considered as a modification of the applied field, $V_{c l}$. Indeed, cf. Eq. 5.27, $\varrho=\hat{\Pi}_{R P A}^{R} \circ V_{c l}=\hat{\Pi}^{R} \circ V_{c l}^{s c r}$, where the screened external potential $V_{c l}^{s c r}$ is given by: $V_{c l}^{s c r}=\left(1-\hat{\sigma}_{1} U \circ \hat{\Pi}^{R}\right)^{-1} \circ V_{c l}$. This is the RPA result for the screening of an external scalar potential.

### 5.6. Kinetic equation

According to Eq. (5.30) to evaluate the bare (and thus RPA dressed, Eq. (5.43)) polarization matrix, one needs to know the fermionic Green function, $\mathcal{\mathcal { G }}$. While it is known in equilibrium, it has to be determined self-consistently in an out-of-equilibrium situation. To this end one employs the same idea that was used
in the bosonic theory of chapter 3 Namely, one writes down the Dyson equation for the dressed fermionic Green function:

$$
\begin{equation*}
\left(\hat{\mathcal{G}}_{0}^{-1}-\hat{\Sigma}_{F}\right) \circ \hat{\mathcal{G}}=\hat{1} \tag{5.44}
\end{equation*}
$$

where the subscript " 0 " indicates the bare Green function. The fermionic selfenergy, $\Sigma_{F}$ turns out to have the same structure as $\hat{\mathcal{G}}^{-1}$, Eq. (5.15). Thus the $R$ and $A$ components of the Dyson equation take a simple form:

$$
\begin{equation*}
\left(i \partial_{t}+\frac{1}{2 m} \nabla_{r}^{2}\right) \mathcal{G}^{\mathbf{R}(\mathbf{A})}=\Sigma_{F}^{R(A)} \circ \mathcal{G}^{\mathbf{R}(\mathbf{A})} \tag{5.45}
\end{equation*}
$$

Employing the parameterization $\mathcal{G}^{\mathbf{K}}=\mathcal{G}^{\mathbf{R}} \circ \mathcal{F}-\mathcal{F} \circ \mathcal{G}^{\mathbf{A}}$, where $\mathcal{F}$ is a Hermitian matrix, along with Eq. [5.45], one rewrites the Keldysh component of the Dyson equation as:

$$
\begin{equation*}
\left[\mathcal{F},\left(i \partial_{t}+\frac{1}{2 m} \nabla_{r}^{2}\right)\right]_{-}=\Sigma_{F}^{K}-\left(\Sigma_{F}^{R} \circ \mathcal{F}-\mathcal{F} \circ \Sigma_{F}^{A}\right)=-i I_{\text {col }}[\mathcal{F}] \tag{5.46}
\end{equation*}
$$

This equation is the quantum kinetic equation for the distribution matrix $\mathcal{F}$. Its 1.h.s. is the kinetic term, while the r.h.s. is the collision integral with $\Sigma_{F}^{K}$ having the meaning of an "incoming" term and $\Sigma_{F}^{R} \circ \mathcal{F}-\mathcal{F} \circ \Sigma_{F}^{A}$ that of an "outgoing" term.

The simplest diagram for the fermionic self-energy matrix, $\hat{\Sigma}_{F}^{a b}$, is obtained by expanding the Hubbard-Stratonovich transformed action, Eq. [5.40, to the second order in the fermion-boson interaction vertex, $\hat{\bar{\psi}}_{a} \varphi_{\alpha} \hat{\gamma}_{a b}^{\alpha} \hat{\psi}_{b}$, and applying the Wick theorem for both fermion and boson fields. As a result, one finds:

$$
\begin{align*}
& \hat{\Sigma}_{F}^{a b}\left(t, t^{\prime}\right)=\left(\hat{\gamma}_{a c}^{\alpha} \hat{\mathcal{G}}^{c d}\left(t, t^{\prime}\right) \hat{\gamma}_{d b}^{\beta}\right)\left\langle\varphi_{\alpha}(t) \varphi_{\beta}\left(t^{\prime}\right)\right\rangle=\left(\hat{\gamma}^{c l} \hat{\mathcal{G}}\left(t, t^{\prime}\right) \hat{\gamma}^{c l}\right)^{a b} i D^{K}\left(t, t^{\prime}\right) \\
& +\left(\hat{\gamma}^{c l} \hat{\mathcal{G}}\left(t, t^{\prime}\right) \hat{\gamma}^{q}\right)^{a b} i D^{R}\left(t, t^{\prime}\right)+\left(\hat{\gamma}^{q} \hat{\mathcal{G}}\left(t, t^{\prime}\right) \hat{\gamma}^{c l}\right)^{a b} i D^{A}\left(t, t^{\prime}\right), \tag{5.47}
\end{align*}
$$

where summations over all repeated indexes are understood and the spatial arguments have the same general structure as the time ones. The boson Green function is denoted as $\left\langle\varphi_{\alpha}(t) \varphi_{\beta}\left(t^{\prime}\right)\right\rangle=i D^{\alpha \beta}\left(t, t^{\prime}\right)$. Finally one finds for the $R, A$ (i.e. $(1,1)$ and $(2,2))$ and $K$ (i.e $(1,2))$ components of the fermionic self-energy:

$$
\begin{align*}
& \Sigma_{F}^{R(A)}\left(t, t^{\prime}\right)=i\left(\mathcal{G}^{R(A)}\left(t, t^{\prime}\right) D^{K}\left(t, t^{\prime}\right)+\mathcal{G}^{K}\left(t, t^{\prime}\right) D^{R(A)}\left(t, t^{\prime}\right)\right)  \tag{5.48}\\
& \Sigma_{F}^{K}\left(t, t^{\prime}\right)=i\left(\mathcal{G}^{K}\left(t, t^{\prime}\right) D^{K}\left(t, t^{\prime}\right)+\mathcal{G}^{R}\left(t, t^{\prime}\right) D^{R}\left(t, t^{\prime}\right)+\mathcal{G}^{A}\left(t, t^{\prime}\right) D^{A}\left(t, t^{\prime}\right)\right) \\
& \quad=i\left(\mathcal{G}^{K}\left(t, t^{\prime}\right) D^{K}\left(t^{\prime}, t\right)+\left(\mathcal{G}^{R}\left(t, t^{\prime}\right)-\mathcal{G}^{A}\left(t, t^{\prime}\right)\right)\left(D^{R}\left(t, t^{\prime}\right)-D^{A}\left(t, t^{\prime}\right)\right)\right)
\end{align*}
$$

where in the last equality one had used that $\mathcal{G}^{R(A)}\left(t, t^{\prime}\right) D^{A(R)}\left(t, t^{\prime}\right)=0$, since these expressions have no support in the time domain (see, however, the footnote in section 3.1. For the same reason: $\Sigma_{F}^{21}\left(t, t^{\prime}\right)=i\left(\mathcal{G}^{A}\left(t, t^{\prime}\right) D^{R}\left(t, t^{\prime}\right)+\right.$ $\left.\mathcal{G}^{R}\left(t, t^{\prime}\right) D^{A}\left(t, t^{\prime}\right)\right)=0$. As expected, the retarded and advanced components are lower and upper triangular matrices correspondingly, with $\Sigma^{R}=\left[\Sigma^{A}\right]^{\dagger}$, while $\Sigma^{K}=-\left[\Sigma^{K}\right]^{\dagger}$. Notice the close resemblance of expressions 5.48 to their bosonic counterparts, Eqs. (3.14)-3.16.

If one understands the bosonic Green function, $\hat{D}$, as the bare instantaneous interaction potential (i.e. $D^{R}=D^{A}=U\left(r-r^{\prime}\right) \delta\left(t-t^{\prime}\right)$ and $D^{K}=0$ ), one finds: $\Sigma_{F}^{R}=\Sigma_{F}^{A}=i U \mathcal{G}^{K}(t, t) \delta\left(t-t^{\prime}\right)$ and $\Sigma_{F}^{K}=0$. In this approximation the r.h.s. of the kinetic equation (5.46 vanishes (since $\mathcal{F}$ is a symmetric matrix) and so there is no collisional relaxation. Thus one has to employ an approximation for $\hat{D}$ that contains some retardation. The simplest and most convenient one is the RPA, where $\hat{D}=\left(U^{-1} \hat{\sigma}_{1}-\hat{\Pi}\right)^{-1}$, cf. Eq. [5.42, with a matrix $\hat{\Pi}$ that is non-local in time. This relation may be rewritten as the Dyson equation for $\hat{D}$, namely $\left(U^{-1} \hat{\sigma}_{1}-\hat{\Pi}\right) \circ \hat{D}=\hat{1}$. One may easily solve it for the three components of $\hat{D}$ and write them in the following way:

$$
\begin{equation*}
D^{R(A)}=D^{R} \circ\left(U^{-1}-\Pi^{A(R)}\right) \circ D^{A} ; \quad D^{K}=D^{R} \circ \Pi^{K} \circ D^{A} \tag{5.49}
\end{equation*}
$$

Performing the Wigner transform following sections 3.5 3.6 the kinetic term (the l.h.s. of Eq. (5.46) is exactly the same as for the complex boson case (one has to take into account the gradient terms to obtain a non-zero result for the WT of the commutator). The result is (cf. Eq. 3.17):

$$
\begin{equation*}
\left(\partial_{\tau}-v_{k} \nabla_{\rho}-E \nabla_{k}\right) \mathbf{f}_{\mathbf{F}}(\tau, \rho, k)=I_{c o l}\left[\mathbf{f}_{\mathbf{F}}\right] \tag{5.50}
\end{equation*}
$$

where $v_{k}=\partial_{k} \epsilon_{k}, E$ is an external electric field and the collision integral, $I_{c o l}$, is $i$ times the WT of the r.h.s. of Eq. 5.46. On the r.h.s. one may keep only the leading terms (without the gradients). One also employs a parameterization of the Keldysh component of the fermionic Green function through the corresponding distribution function: $\mathcal{G}^{K} \rightarrow \mathrm{~g}^{K}=\mathbf{f}_{\mathbf{F}}\left(\mathrm{g}^{R}-\mathrm{g}^{A}\right)$, where $\mathbf{f}_{\mathbf{F}}(\tau, \rho, k)$ is the WT of $\mathcal{F}$. Assuming, for brevity, a spatially uniform and momentum isotropic case, one may restrict oneself to $\mathbf{f}_{\mathbf{F}}\left(\tau, \epsilon_{k}\right)=\mathbf{f}_{\mathbf{F}}(\tau, \epsilon)$. As a result, one finds for the collision integral:

$$
\begin{align*}
& I_{c o l}\left[\mathbf{f}_{\mathbf{F}}(\epsilon)\right]=i \int d \omega \sum_{q} D^{R}(\omega, q) D^{A}(\omega, q) \boldsymbol{\Delta}_{\mathrm{g}}(\epsilon-\omega, k-q)  \tag{5.51}\\
& \times\left[\left(\Pi^{R}-\Pi^{A}\right)\left(1-\mathbf{f}_{\mathbf{F}}(\epsilon-\omega) \mathbf{f}_{\mathbf{F}}(\epsilon)\right)-\Pi^{K}\left(\mathbf{f}_{\mathbf{F}}(\epsilon)-\mathbf{f}_{\mathbf{F}}(\epsilon-\omega)\right)\right]
\end{align*}
$$

where $\Pi^{\alpha \beta}=\Pi^{\alpha \beta}(\omega, q)$, while the time index, $\tau$, is suppressed for brevity and the notation

$$
\begin{equation*}
\boldsymbol{\Delta}_{\mathrm{g}}(\epsilon, k) \equiv \frac{i}{2 \pi}\left(\mathrm{~g}^{R}(\epsilon, k)-\mathrm{g}^{A}(\epsilon, k)\right) \tag{5.52}
\end{equation*}
$$

is introduced. For free fermions $\boldsymbol{\Delta}_{\mathrm{g}}(\epsilon, k)=\delta\left(\epsilon-\epsilon_{k}\right)$. At this stage one may observe that if the bosonic system is at equilibrium: $\Pi^{K}=\operatorname{coth}(\omega / 2 T)\left[\Pi^{R}-\Pi^{A}\right]$, then the fermionic collision integral is nullified by:

$$
\begin{equation*}
\mathbf{f}_{\mathbf{F}}(\epsilon)=\tanh \frac{\epsilon}{2 T} . \tag{5.53}
\end{equation*}
$$

Indeed, $1-\tanh (b-a) \tanh (b)=\operatorname{coth}(a)(\tanh (b)-\tanh (b-a))$. One should take into account, however, that the bosonic degrees of freedom are not independent from the fermionic ones. Namely, components of the polarization matrix $\hat{\Pi}$ are expressed through the fermionic Green functions according to Eq. [5.31. In the WT representation these relations take the form:
$\Pi^{R}-\Pi^{A}=i \pi \int d \epsilon^{\prime} \sum_{k^{\prime}} \boldsymbol{\Delta}_{\mathbf{g}}\left(\epsilon^{\prime}, k^{\prime}\right) \boldsymbol{\Delta}_{\mathbf{g}}\left(\epsilon^{\prime}-\omega, k^{\prime}-q\right)\left[\mathbf{f}_{\mathbf{F}}\left(\epsilon^{\prime}-\omega\right)-\mathbf{f}_{\mathbf{F}}\left(\epsilon^{\prime}\right)\right] ;$
$\Pi^{K}(\omega, q)=i \pi \int d \epsilon^{\prime} \sum_{k^{\prime}} \boldsymbol{\Delta}_{\mathbf{g}}\left(\epsilon^{\prime}, k^{\prime}\right) \boldsymbol{\Delta}_{\mathbf{g}}\left(\epsilon^{\prime}-\omega, k^{\prime}-q\right)\left[\mathbf{f}_{\mathbf{F}}\left(\epsilon^{\prime}-\omega\right) \mathbf{f}_{\mathbf{F}}\left(\epsilon^{\prime}\right)-1\right] .(5.54$
Due to the same trigonometric identity the equilibrium argument can be made self-consistent: if the fermionic system is in equilibrium, Eq. (5.53), then components of $\hat{\Pi}$ satisfy the bosonic FDT, Eq. 3.22).

One may substitute now Eqs. (5.54) into Eq. (5.51) to write down the closed kinetic equation for the fermionic distribution function. Most conveniently it is done in terms of the occupation numbers, defined as $\mathbf{f}_{\mathbf{F}} \equiv 1-2 \mathbf{n}^{6}$ :
$\frac{\partial \mathbf{n}_{\epsilon}}{\partial \tau}=\iint d \omega d \epsilon^{\prime} M\left[\mathbf{n}_{\epsilon^{\prime}} \mathbf{n}_{\epsilon-\omega}\left(1-\mathbf{n}_{\epsilon}\right)\left(1-\mathbf{n}_{\epsilon^{\prime}-\omega}\right)-\mathbf{n}_{\epsilon} \mathbf{n}_{\epsilon^{\prime}-\omega}\left(1-\mathbf{n}_{\epsilon^{\prime}}\right)\left(1-\mathbf{n}_{\epsilon-\omega}\right)\right]$,
where the transition probability is given by:
$M(\epsilon, \omega)=4 \pi \sum_{q, k^{\prime}}\left|D^{R}(\omega, q)\right|^{2} \boldsymbol{\Delta}_{\mathrm{g}}(\epsilon-\omega, k-q) \boldsymbol{\Delta}_{\mathrm{g}}\left(\epsilon^{\prime}, k^{\prime}\right) \boldsymbol{\Delta}_{\mathrm{g}}\left(\epsilon^{\prime}-\omega, k^{\prime}-q\right) .(5.56)$
Equation (5.55) is a generic kinetic equation with a "four-fermion" collisional relaxation. The first term in the square brackets on its r.h.s. may be identified as "in", while the second one as "out". Each of these terms consists of the product of four occupation numbers, giving a probability of having two initial states

[^6]

Fig. 7. Structure of the four-fermion collision integral. The full lines are fermionic Green functions; the wavy lines are the RPA screened interaction potential. The fermionic loop represents the polarization matrix, $\hat{\Pi}(\omega, q)$.
occupied and two final states empty. For $\mathbf{n}(\epsilon)$ given by the Fermi function the "in" and the "out" terms cancel each other. Therefore in thermal equilibrium the components of the dressed fermionic Green function must satisfy the FDT:

$$
\begin{equation*}
\mathcal{G}^{\mathbf{K}}=\tanh \frac{\epsilon}{2 T}\left(\mathcal{G}^{\mathbf{R}}-\mathcal{G}^{\mathbf{A}}\right) \tag{5.57}
\end{equation*}
$$

The structure of the transmission probability $M$ is illustrated in Fig. 5.6 The three factors of $\Delta_{\mathrm{g}}$ enforce that all three intermediate fermionic particles must satisfy energy-momentum conservation (stand on mass-shell), up to the quasiparticle life-time. The real factor $\left|D^{R}\right|^{2}$ is associated with the square of the matrix element of the screened interaction potential (in the RPA).

## 6. Disordered fermionic systems

### 6.1. Disorder averaging

We consider fermions in the field of a static (quenched) space-dependent scalar potential $U_{d i s}(r)$. The potential is meant to model the effect of random static impurities, dislocations, etc. Since one does not know the exact form of the
potential, the best one can hope for is to evaluate the statistical properties of various observables, assuming some statistics for $U_{d i s}(r)$. It is usually a reasonable guess to prescribe a Gaussian distribution for the potential. Namely, one assumes that the relative probability for a realization of the potential to appear in nature is given by:

$$
\begin{equation*}
\mathcal{P}\left[U_{d i s}\right] \sim e^{-\pi \nu \tau \int d r U_{d i s}^{2}(r)} \tag{6.1}
\end{equation*}
$$

where $\nu$ is the bare fermionic DOS at the Fermi level and $\tau$, called the mean-free time, measures the strength of the random potential.

In this chapter we concentrate on non-interacting fermions. We would like to evaluate, say, the response function, $\Pi^{R}$, in presence of the random potential and average it over the realizations of $U_{d i s}$ with the weight given by Eq. 6.1. The crucial observation is that the response function, $\Pi^{R}$, may be defined as variation of the generating function, Eq. (5.27), and not the logarithm of the generating function. More precisely, the two definitions with, Eq. 5.28), and without, Eq. 5.27, the logarithm coincide due to the fundamental normalization, Eq. (5.25). This is not the case in the equilibrium formalism, where the presence of the logarithm (leading to the factor $Z^{-1}$ after differentiation) is unavoidable in order to have the correct normalization. Such a factor $Z^{-1}=Z^{-1}\left[U_{d i s}\right]$ formidably complicates the averaging over $U_{\text {dis }}$. Two techniques were invented to perform the averaging: the replica trick [21] and the super-symmetry (SUSY) [22]. The first one utilizes the observation that $\ln Z=\lim _{n \rightarrow 0}\left(Z^{n}-1\right) / n$, to perform calculations for an integer number, $n$, replicas of the same system and take $n \rightarrow 0$ at the end of the calculations. The second one is based on the fact that $Z^{-1}$ of the non-interacting fermionic system equals to $Z$ of a bosonic system in the same random potential. One thus introduces an additional bosonic replica of the fermionic system at hand. Both of these ideas have serious drawbacks: the replica technique requires analytical continuation, while the SUSY is not applicable to interacting systems.

The Keldysh formalism provides an alternative to these two methods by insuring that $Z=1$ by construction. One may thus directly perform the averaging of the generating function, Eq. [5.24, over realizations of $U_{d i s}$. Since the disorder potential possesses only the classical component (it is exactly the same on both branches of the contour), it is coupled only to $\hat{\gamma}^{c l}=\hat{1}$. The disorder-dependent term in the averaged generating function has the form:

$$
\begin{equation*}
\left.\int \mathcal{D} U_{d i s} e^{-\int d r\left[\pi \nu \tau U_{d i s}^{2}(r)-i U_{d i s}(r) \int_{-\infty}^{\infty} d t \hat{\bar{\psi}}_{t} \hat{\gamma}^{c l} \hat{\psi}_{t}\right.}\right] \tag{6.2}
\end{equation*}
$$

$$
-\frac{1}{4 \pi \nu \tau} \int d r \iint_{-\infty}^{\infty} d t d t^{\prime}\left(\bar{\psi}_{t}^{a} \psi_{t}^{a}\right)\left(\bar{\psi}_{t^{\prime}}^{b} \psi_{t^{\prime}}^{b}\right)
$$

where $a, b=1,2$, and there is a summation over repeated indexes. One can rearrange the expression in the exponent on the r.h.s. of the last equation as $\left(\bar{\psi}_{t}^{a} \psi_{t}^{a}\right)\left(\bar{\psi}_{t^{\prime}}^{b} \psi_{t^{\prime}}^{b}\right)=-\left(\bar{\psi}_{t}^{a} \psi_{t^{\prime}}^{b}\right)\left(\bar{\psi}_{t^{\prime}}^{b} \psi_{t}^{a}\right)^{7}$ and then use the Hubbard-Stratonovich matrix field, $\hat{Q}=Q_{t, t^{\prime}}^{a b}(r)$ :

$$
\begin{equation*}
e^{\frac{1}{4 \pi \nu \tau} \int d r \iint_{-\infty}^{\infty} d t d t^{\prime}\left(\bar{\psi}_{t}^{a} \psi_{t^{\prime}}^{b}\right)\left(\bar{\psi}_{t^{\prime}}^{b} \psi_{t}^{a}\right)}=\int \mathcal{D} \hat{Q} e^{-\int d r\left[\frac{\pi \nu}{4 \tau} \operatorname{Tr}\{\hat{Q} \circ \hat{Q}\}-\frac{i}{2 \tau} \iint_{-\infty}^{\infty} d t d t^{\prime} Q_{t, t^{\prime}}^{a b} \bar{\psi}_{t^{\prime}}^{b} \psi_{t}^{a}\right]} \tag{6.3}
\end{equation*}
$$

where the spatial coordinate, $r$, is suppressed in both $\hat{Q}$ and $\hat{\psi}$. At this stage the average action becomes quadratic in the Grassmann variables and they may be integrated out leading to the determinant of the corresponding quadratic form: $\hat{\mathcal{G}}_{0}^{-1}+V_{\alpha} \hat{\gamma}^{\alpha}+\frac{i}{2 \tau} \hat{Q}$. All the matrices here should be understood as having a $2 \times 2$ Keldysh structure along with an $N \times N$ structure in discrete time. One thus finds for the disorder averaged generating function:

$$
\begin{equation*}
Z[\hat{V}]=\int \mathcal{D} \hat{Q} e^{i S[\hat{Q} ; \hat{V}]} \tag{6.4}
\end{equation*}
$$

where

$$
\begin{equation*}
i S[\hat{Q} ; \hat{V}]=-\frac{\pi \nu}{4 \tau} \int d r \operatorname{Tr}\left\{\hat{Q}^{2}\right\}+\operatorname{Tr} \ln \left[\hat{\mathcal{G}}_{0}^{-1}+\frac{i}{2 \tau} \hat{Q}+V_{\alpha} \hat{\gamma}^{\alpha}\right] \tag{6.5}
\end{equation*}
$$

As a result, one has traded the initial functional integral over the static field $U_{\text {dis }}(r)$ for the functional integral over the dynamic matrix field $\hat{Q}_{t, t^{\prime}}(r)$. At first glance, it does not strike as a terribly bright idea. Nevertheless, there is a great simplification hidden in this procedure. The point is that the disorder potential, being $\delta$-correlated, is a rapidly oscillating function. On the other hand, as one will see below, the $\hat{Q}$-matrix field is a slow (both in space and time) function. Thus it represents the true macroscopic (or hydrodynamic) degrees of freedom of the system, that happen to be the diffusively propagating modes.

### 6.2. Non-linear $\sigma$-model

To execute this program, one first looks for the stationary configurations of the action (6.5). Taking the variation over $\hat{Q}_{t, t^{\prime}}(r)$, one obtains:

$$
\begin{equation*}
\underline{\hat{Q}}_{t, t^{\prime}}(r)=\left.\frac{i}{\pi \nu}\left[\hat{\mathcal{G}}_{0}^{-1}+\frac{i}{2 \tau} \underline{\hat{Q}}\right]^{-1}\right|_{t, t^{\prime} ; r, r} \tag{6.6}
\end{equation*}
$$

[^7]where $\underline{\hat{Q}}$ denotes a stationary configuration of the fluctuating field $\hat{Q}$. For the purpose of finding the stationary configurations one has omitted the small source field, $\hat{V}$. It is important to notice that the spatially non-local operator $\left[\hat{\mathcal{G}}_{0}^{-1}+\right.$ $\left.\frac{i}{2 \tau} \underline{Q}\right]^{-1}\left(t, t^{\prime} ; r, r^{\prime}\right)$ on the r.h.s. is taken at coinciding spatial points $r^{\prime}=r$.

The strategy is to find first a spatially uniform and time-translationally invariant solution of Eq. (6.6): $\underline{\underline{Q}}_{t-t^{\prime}}$, and then consider space and time-dependent deviations from such a solution. This strategy is adopted from the theory of magnetic systems, where one first finds the uniform static magnetized configurations and then treats spin-waves as smooth perturbations on top of such static uniform solutions. From the structure of Eq. 6.6 one expects that the saddlepoint configuration $\underline{\hat{Q}}$ possesses the same structure as the fermionic self-energy, Eq. 5.15) (more accurately, one expects that among the possible saddle points there is a "classical" one, that satisfies the causality structure, Eq. (5.15). One looks, therefore, for a solution of Eq. 6.6 in the form of:

$$
\underline{\hat{Q}}_{\epsilon} \equiv \hat{\Lambda}_{\epsilon}=\left(\begin{array}{cc}
\Lambda_{\epsilon}^{R} & \Lambda_{\epsilon}^{K}  \tag{6.7}\\
0 & \Lambda_{\epsilon}^{A}
\end{array}\right)
$$

Substituting this expression in Eq. 6.6, one finds
$\Lambda_{\epsilon}^{R(A)}=\left.\frac{i}{\pi \nu} \frac{1}{\left[\mathcal{G}_{0}^{R(A)}\right]^{-1}+\frac{i}{2 \tau} \Lambda_{\epsilon}^{R(A)}}\right|_{r, r}=\frac{i}{\pi \nu} \sum_{k} \frac{1}{\epsilon-\xi_{k}+\frac{i}{2 \tau} \Lambda_{\epsilon}^{R(A)}}= \pm 1$,
where $\xi_{k} \equiv k^{2} /(2 m)-\mu$ and one adopts $\sum_{k} \ldots=\nu \int d \xi_{k} \ldots$, where $\nu$ is the DOS at the Fermi surface. The summation over momentum appears because the matrix on the r.h.s. is taken at coinciding spatial points. The signs are chosen so as to respect causality: the retarded (advanced) Green function is analytic in the entire upper (lower) half-plane of complex energy $\epsilon$. One has also assumed that $1 /(2 \tau) \ll \mu$. The Keldysh component, as always, may be parameterized through a Hermitian distribution function matrix: $\Lambda^{K}=\Lambda^{R} \circ \mathcal{F}-\mathcal{F} \circ \Lambda^{A}=2 \mathcal{F}_{\epsilon}$, where the distribution function $\mathcal{F}_{\epsilon}$ is not fixed by the saddle point equation 6.6 and must be determined through the boundary conditions. As a result one obtains:

$$
\hat{\Lambda}_{\epsilon}=\left(\begin{array}{cc}
1 & 2 \mathcal{F}_{\epsilon}  \tag{6.9}\\
0 & -1
\end{array}\right)
$$

Transforming back to the time representation, one obtains $\Lambda_{t-t^{\prime}}^{R(A)}= \pm \delta\left(t-t^{\prime} \mp\right.$ 0 ), where $\mp 0$ indicates that the $\delta$-function is shifted below (above) the main diagonal, $t=t^{\prime}$. As a result, $\operatorname{Tr} \hat{\Lambda}_{\epsilon}=0$ and $S[\hat{\Lambda}]=0$, as it should be, of course, for any purely classical field configuration, Eq. 6.7. There is, however, a wider
class of configurations, that leave the action (6.5) invariant (zero). Indeed, any field configuration of the form:

$$
\begin{equation*}
\hat{Q}=\hat{\mathcal{T}} \circ \hat{\Lambda} \circ \hat{\mathcal{T}}^{-1} \tag{6.10}
\end{equation*}
$$

where $\hat{\mathcal{T}}_{t, t^{\prime}}(r)=\hat{\mathcal{T}}_{t-t^{\prime}}$, and thus commutes with $\hat{\mathcal{G}}_{0}$, obviously does not change the action (6.5). This is the zero-mode Goldstone manifold. The standard way to introduce the massless modes ("spin-waves") is to allow the deformation matrices $\hat{\mathcal{T}}$ to be slow functions of $t+t^{\prime}$ and $r$. Thus the expression (6.10 parameterizes the soft modes manifold of the field $\hat{Q}$. One may thus restrict oneself only to the field configurations given by Eq. 6.10) and disregard all others (massive modes). An equivalent way to characterize this manifold is by the condition (cf. Eq. 6.9):

$$
\begin{equation*}
\hat{Q}^{2}=\hat{1} \tag{6.11}
\end{equation*}
$$

Our goal now is to derive an action for the soft-mode field configurations given by Eqs. 6.10 or 6.11). To this end one substitutes $\hat{Q}=\hat{\mathcal{T}} \circ \hat{\Lambda} \circ \hat{\mathcal{T}}^{-1}$ into Eq. (6.5) and cyclically permutes the $\hat{\mathcal{T}}$ matrices under the trace sign. This way one arrives at $\hat{\mathcal{T}}^{-1} \circ \hat{\mathcal{G}}_{0}^{-1} \circ \hat{\mathcal{T}}=\hat{\mathcal{G}}_{0}^{-1}+\hat{\mathcal{T}}^{-1} \circ\left[\hat{\mathcal{G}}_{0}^{-1}, \hat{\mathcal{T}}\right]_{-}=\hat{\mathcal{G}}_{0}^{-1}+i \hat{\mathcal{T}}^{-1} \circ$ $\left[\partial_{t}+v_{F} \nabla_{r}, \hat{\mathcal{T}}\right]_{-}$, where one has linearized the dispersion relation near the Fermi surface $k^{2} /(2 m)-\mu \approx v_{F} k \rightarrow i v_{F} \nabla_{r}$. As a result, the desired action has the form:

$$
\begin{equation*}
i S[\hat{Q}]=\operatorname{Tr} \ln \left[1+i \hat{\mathcal{G}} \hat{\mathcal{T}}^{-1}\left[\partial_{t}, \hat{\mathcal{T}}\right]_{-}+i \hat{\mathcal{G}} \hat{\mathcal{T}}^{-1}\left[v_{F} \nabla_{r}, \hat{\mathcal{T}}\right]_{-}\right] \tag{6.12}
\end{equation*}
$$

where $\hat{\mathcal{G}}$ is the impurity dressed Green function, defined as: $\left(\hat{\mathcal{G}}_{0}^{-1}+\frac{i}{2 \tau} \hat{\Lambda}\right) \hat{\mathcal{G}}=\hat{1}$. For practical calculations it is convenient to write it as:

$$
\hat{\mathcal{G}}_{\epsilon}(k)=\left(\begin{array}{cc}
\mathcal{G}_{\epsilon}^{R}(k) & \mathcal{G}_{\epsilon}^{K}(k)  \tag{6.13}\\
0 & \mathcal{G}_{\epsilon}^{A}(k)
\end{array}\right)=\frac{1}{2}\left(\mathcal{G}_{\epsilon}^{R}(k)\left[\hat{1}+\hat{\Lambda}_{\epsilon}\right]+\mathcal{G}_{\epsilon}^{A}(k)\left[\hat{1}-\hat{\Lambda}_{\epsilon}\right]\right),
$$

with

$$
\begin{align*}
\mathcal{G}_{\epsilon}^{R(A)}(k) & =\left(\epsilon-\xi_{k} \pm i /(2 \tau)\right)^{-1}  \tag{6.14}\\
\mathcal{G}_{\epsilon}^{K}(k) & =\mathcal{G}_{\epsilon}^{R}(k) \mathcal{F}_{\epsilon}-\mathcal{F}_{\epsilon} \mathcal{G}_{\epsilon}^{A}(k)
\end{align*}
$$

Notice that $\sum_{k} \hat{\mathcal{G}}_{\epsilon}(k)=-i \pi \nu \hat{\Lambda}_{\epsilon}$ and $\sum_{k} \mathcal{G}_{\epsilon}^{R}(k) \mathcal{G}_{\epsilon}^{A}(k)=2 \pi \nu \tau$, while the other combinations vanish: $\sum_{k} \mathcal{G}_{\epsilon}^{R}(k) \mathcal{G}_{\epsilon}^{R}(k)=\sum_{k} \mathcal{G}_{\epsilon}^{A}(k) \mathcal{G}_{\epsilon}^{A}(k)=0$, due to the complex $\xi_{k}$-plane integration.

One can now expand the logarithm in Eq. (6.12) to the first order in the $\partial_{t}$ term and to the second order in the $\nabla_{r}$ term (the first order term in $\nabla_{r}$ vanishes due to the angular integration) and evaluate traces using Eq. 6.14. For the $\partial_{t}$ term one
finds: $\pi \nu \operatorname{Tr}\left\{\hat{\Lambda} \hat{\mathcal{T}}^{-1}\left[\partial_{t}, \hat{\mathcal{T}}\right]_{-}\right\}=\pi \nu \operatorname{Tr}\left\{\partial_{t} \hat{Q}\right\}$, where one used that $\operatorname{Tr}\left\{\partial_{t} \hat{\Lambda}\right\}=0$. For the $\nabla_{r}$ term, one finds: $-\frac{1}{4} \pi \nu D \operatorname{Tr}\left\{\left(\nabla_{r} \hat{Q}\right)^{2}\right\}$, where $D \equiv v_{F}^{2} \tau / d$ is the diffusion constant and $d$ is the spatial dimensionality ${ }^{8}$. Finally, one finds for the action of the soft-mode configurations [14]:
$S[\hat{Q}]=i \pi \nu \int d r \operatorname{Tr}\left\{\frac{1}{4} D\left(\nabla_{r} \hat{Q}(r)\right)^{2}-\partial_{t} \hat{Q}(r)-i V_{\alpha} \hat{\gamma}^{\alpha} \hat{Q}(r)-\frac{i}{\pi} \hat{V}^{T} \hat{\sigma}_{1} V\right\}$,
where the trace is performed over the $2 \times 2$ Keldysh structure as well as over the $N \times N$ time structure. In the last expression we have restored the source term from Eq. 6.5. The last term, $\hat{V}^{T} \hat{\sigma}_{1} V$ is the static compressibility of the electron gas. It originates from the second order expansion of Eq. 6.5 in $\hat{V}$, while keeping the high energy part of the $\mathcal{G}^{R} \mathcal{G}^{R}$ and $\mathcal{G}^{A} \mathcal{G}^{A}$ terms. Despite of the simple appearance, the action (6.15) is highly non-linear due to the condition $\hat{Q}^{2}=1$. The theory specified by Eqs. 6.11) and 6.15) is called the matrix nonlinear $\sigma$-model ( $\mathrm{NL} \sigma \mathrm{M}$ ). The name came from the theory of magnetism, where the unit-length vector, $\sigma(r)$, represents a local (classical) spin, that may rotate over the sphere $\sigma^{2}=1$.

### 6.3. Usadel equation

Our goal is to investigate the physical consequences of the $\mathrm{NL} \sigma \mathrm{M}$. As a first step, one wants to determine the most probable (stationary) configuration, $\hat{Q}_{t, t^{\prime}}(r)$, on the soft-modes manifold, Eq. 6.11. To this end one parameterizes deviations from $\underline{\hat{Q}}_{t, t^{\prime}}(r)$ as $\hat{Q}=\hat{\mathcal{T}} \circ \underline{\hat{Q}} \circ \hat{\mathcal{T}}^{-1}$ and chooses $\hat{\mathcal{T}}=e^{\hat{\mathcal{W}}}$, where $\hat{\mathcal{W}}_{t, t^{\prime}}(r)$ is the generator of rotations. Expanding to the first order in $\hat{\mathcal{W}}$, one finds: $\hat{Q}=\underline{\hat{Q}}+$ $[\hat{\mathcal{W}} \circ, \underline{\hat{Q}}]_{-}$. One may now substitute such a $\hat{Q}$-matrix into the action 6.15) and require that the term linear in $\hat{\mathcal{W}}$ vanishes. This leads to the saddle-point equation for $\hat{Q}$. For the first term in the curly brackets on the r.h.s. of Eq. 6.15) one obtains: $\frac{1}{2} \operatorname{Tr}\left\{\hat{\mathcal{W}} \circ \nabla_{r} D\left(\nabla_{r} \underline{\hat{Q}} \circ \underline{\hat{Q}}-\underline{\hat{Q}} \circ \nabla_{r} \underline{\hat{Q}}\right)\right\}=-\operatorname{Tr}\left\{\hat{\mathcal{W}} \circ \nabla_{r} D\left(\underline{\hat{Q}} \circ \nabla_{r} \underline{\hat{Q}}\right)\right\}$, where one employed that $\nabla_{r} \underline{\hat{Q}} \circ \underline{\hat{Q}}+\underline{\hat{Q}} \circ \nabla_{r} \underline{\hat{Q}}=0$, since $\underline{\hat{Q}}^{2}=\hat{1}$. For the second term one finds: $\operatorname{Tr}\left\{\hat{\mathcal{W}}_{t, t^{\prime}}\left(\bar{\partial}_{t^{\prime}}+\partial_{t}\right) \hat{Q}_{t^{\prime}, t}\right\}$. It is written more compactly in the energy representation, where $\partial_{t} \rightarrow-i \epsilon$, and thus the second term is: $-i \operatorname{Tr}\left\{\hat{\mathcal{W}} \circ[\epsilon, \underline{Q}]_{-}\right\}$. Demanding that the linear term in $\hat{\mathcal{W}}$ vanish, one finds:

$$
\begin{equation*}
\nabla_{r}\left(D \underline{\hat{Q}} \circ \nabla_{r} \underline{\hat{Q}}\right)+i[\epsilon, \underline{\hat{Q}}]_{-}=0 \tag{6.16}
\end{equation*}
$$

[^8]This is the Usadel equation for the stationary $\underline{\hat{Q}}$-matrix, that must also satisfy $\underline{\hat{Q}}^{2}=\hat{1}$. In the time representation $i[\epsilon, \underline{\hat{Q}}]_{-} \rightarrow-\left\{\partial_{t}, \underline{\hat{Q}}\right\}_{+}$.

If one looks for a solution of the Usadel equation 6.16 in the subspace of "classical" (having the causality structure) configurations, then the condition $\underline{\hat{Q}}^{2}=\hat{1}$ restricts the possible solutions to $\hat{\Lambda}$, Eq. 6.9 (with a yet unspecified distribution matrix $\mathcal{F}_{t, t^{\prime}}(r)$ ). Therefore, in the non-superconducting case the Usadel equation is reduced to a single equation for the distribution matrix $\mathcal{F}_{t, t^{\prime}}(r)$. It contains much more information for the superconducting case (i.e. it also determines the local energy spectrum and superconducting phase). Substituting Eq. (6.9) into the Usadel equation (6.16), one finds:

$$
\begin{equation*}
\nabla_{r}\left(D \nabla_{r} \mathcal{F}\right)+i[\epsilon, \mathcal{F}]_{-}=0 \tag{6.17}
\end{equation*}
$$

Finally, performing the time Wigner transform, $\mathcal{F}_{t, t^{\prime}}(r) \rightarrow \mathbf{f}_{\mathbf{F}}(\tau, \epsilon ; r)$, as explained in section 3.5 one obtains:

$$
\begin{equation*}
\nabla_{r}\left(D \nabla_{r} \mathbf{f}_{\mathbf{F}}\right)-\partial_{\tau} \mathbf{f}_{\mathbf{F}}=0 \tag{6.18}
\end{equation*}
$$

This is the kinetic equation for the fermionic distribution function $\mathbf{f}_{\mathbf{F}}(\tau, \epsilon ; r)$ of the disordered system. It happens to be the diffusion equation. Notice, that it is the same equation for any energy $\epsilon$ and different energies do not "talk" to each other (in the adiabatic case, where the WT works). This is a feature of noninteracting systems. In the presence of interactions, the equation acquires the collision integral on the r.h.s. that mixes different energies between themselves. It is worth mentioning that elastic scattering does not show up in the collision integral. It was already fully taken into account in the derivation of the Usadel equation and went into the diffusion term, $D \nabla_{r}^{2}$.

As an example, let us consider a disordered one-dimensional wire of length $L$ [23], attached to two leads, that are kept at different voltages. There is a stationary current passing through the wire. We look for the space dependent distribution function, $\mathbf{f}_{\mathbf{F}}(\epsilon ; r)$, that satisfies $D \nabla_{r}^{2} \mathbf{f}_{\mathbf{F}}=0$ in a stationary setup (for a space independent diffusion constant, $D$ ). As a result,

$$
\begin{equation*}
\mathbf{f}_{\mathbf{F}}(\epsilon ; r)=f_{L}(\epsilon)+\left(f_{R}(\epsilon)-f_{L}(\epsilon)\right) \frac{r}{L} \tag{6.19}
\end{equation*}
$$

where $f_{L(R)}(\epsilon)$ are the distribution functions of the left and right leads. The distribution function inside the wire interpolates the two distributions linearly. At low temperatures it looks like a two-step function, where the energy separation between the steps is the applied voltage, $e V$, while the height depends on position. Such a distribution was measured in a beautiful experiment [23]. Comparing equation 6.18 with the continuity equation, one notices that the current (at a given energy $\epsilon$ ) is given by $J(\epsilon)=D \nabla_{r} \mathbf{f}_{\mathbf{F}}=D\left(f_{R}(\epsilon)-f_{L}(\epsilon)\right) / L$. And
thus the total current is $J=e \sum_{k} J(\epsilon)=e \frac{\nu D}{L} \int d \epsilon\left(f_{R}(\epsilon)-f_{L}(\epsilon)\right)=e^{2} \frac{\nu D}{L} V$. This is the Drude conductivity: $\sigma_{D}=e^{2} \nu D$.

### 6.4. Fluctuations

Our next goal is to consider fluctuations near the stationary solution, $\underline{Q}_{t, t^{\prime}}(r)$. We restrict ourselves to the soft-mode fluctuations that satisfy $\hat{Q}^{2}=1$ only, and neglect all massive modes that stay outside this manifold. As was already stated above these fluctuations of the $\hat{Q}$-matrix may be parameterized as

$$
\begin{equation*}
\hat{Q}=e^{-\mathcal{W}} \circ \underline{\hat{Q}} \circ e^{\mathcal{W}} \tag{6.20}
\end{equation*}
$$

The part of $\mathcal{W}$ that commutes with $\hat{Q}$ does not generate any fluctuations, therefore one restricts $\mathcal{W}$ to satisfy: $\mathcal{W} \circ \underline{\hat{Q}}+\underline{\hat{Q}} \circ \mathcal{W}=0$. Since $\underline{\hat{Q}}$ may be diagonalized according to:

$$
\underline{\hat{Q}}=\left(\begin{array}{cc}
1 & 2 \mathcal{F}  \tag{6.21}\\
0 & -1
\end{array}\right)=\left(\begin{array}{cc}
1 & \mathcal{F} \\
0 & -1
\end{array}\right) \circ\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) \circ\left(\begin{array}{cc}
1 & \mathcal{F} \\
0 & -1
\end{array}\right)
$$

any generator $\mathcal{W}$ that anticommutes with $\underline{\hat{Q}}$ may be parameterized as
$\mathcal{W}=\left(\begin{array}{cc}1 & \mathcal{F} \\ 0 & -1\end{array}\right) \circ\left(\begin{array}{cc}0 & w \\ \bar{w} & 0\end{array}\right) \circ\left(\begin{array}{cc}1 & \mathcal{F} \\ 0 & -1\end{array}\right)=\left(\begin{array}{cc}\mathcal{F} \circ \overline{\mathcal{F}} & \mathcal{F} \circ \bar{w} \circ \mathcal{F}-w \\ -\bar{w} & -\bar{w} \circ \mathcal{F}\end{array}\right)$,
where $\bar{w}_{t, t^{\prime}}(r)$ and $w_{t, t^{\prime}}(r)$ are arbitrary Hermitian matrices in time space. One, thus, understands the functional integration over $\hat{Q}$ as an integration over Hermitian $\bar{w}$ and $w$. The physical meaning of $w$ is a deviation of the fermionic distribution function, $\mathcal{F}$, from its stationary value. At the same time, $\bar{w}$ has no classical interpretation. To a large extent it plays the role of the quantum counterpart of $w$, that appears only as the internal line in the diagrams.

One may now expand the action, Eqs. (6.15), in powers of $\bar{w}$ and $w$. Since $\hat{Q}$ was chosen to be a stationary point, the expansion starts from the second order. In a spatially uniform case one obtains:
$i S^{(2)}[\mathcal{W}]=2 \pi \nu \int d r \iint \frac{d \epsilon_{1} d \epsilon_{2}}{4 \pi^{2}} \bar{w}_{\epsilon_{1} \epsilon_{2}}(r)\left[-D \nabla_{r}^{2}+i\left(\epsilon_{1}-\epsilon_{2}\right)\right] w_{\epsilon_{2} \epsilon_{1}}(r)$.
The quadratic form is diagonalized by transforming to the momentum representation. As a result, the propagator of small $\hat{Q}$-matrix fluctuations is given by:

$$
\begin{equation*}
\left\langle w_{\epsilon_{2} \epsilon_{1}}(q) \bar{w}_{\epsilon_{3} \epsilon_{4}}(-q)\right\rangle_{\mathcal{W}}=-\frac{1}{2 \pi \nu} \frac{\delta_{\epsilon_{1} \epsilon_{3}} \delta_{\epsilon_{2} \epsilon_{4}}}{D q^{2}+i \omega} \equiv-\frac{\delta_{\epsilon_{1} \epsilon_{3}} \delta_{\epsilon_{2} \epsilon_{4}}}{2 \pi \nu} D(\omega, q) \tag{6.24}
\end{equation*}
$$

where $\omega \equiv \epsilon_{1}-\epsilon_{2}$ and the object $D(\omega, q)=D\left(\epsilon_{1}-\epsilon_{2}, q\right)=\left(D q^{2}+i\left(\epsilon_{1}-\epsilon_{2}\right)\right)^{-1}$ is called a diffuson. It is an advanced (retarded) function of its first (second) energy argument, $\epsilon_{1(2)}$, (or correspondingly $t_{1(2)}$ ). The higher order terms of the action's expansion describe non-linear interactions of the diffusons with vertices called Hikami boxes. These non-linear terms are responsible for the localization corrections. If the distribution function $\mathcal{F}$ is spatially non-uniform, there is an additional term in the quadratic action $i \tilde{S}^{(2)}[\mathcal{W}]=-2 \pi \nu D \operatorname{Tr}\left\{\bar{w} \nabla_{r} \mathcal{F} \bar{w} \nabla_{r} \mathcal{F}\right\}$. This term generates non-zero correlations of the type $\langle\overline{w w}\rangle$ and is actually necessary for the convergence of the functional integral over $\bar{w}$ and $w$. In the spatially uniform case, such a convergence term is pure regularization (the situation that was already encountered before).

One can now derive the linear density response to the applied scalar potential. According to the general expression, Eq. 5.28, the retarded response is given by

$$
\begin{align*}
& \Pi^{R}\left(t, t^{\prime} ; r, r^{\prime}\right)=-\left.\frac{i}{2} \frac{\delta^{2} Z[\hat{V}]}{\delta V_{c l}\left(t^{\prime} ; r^{\prime}\right) \delta V_{q}(t ; r)}\right|_{\hat{V}=0} \\
& =\nu \delta_{t, t^{\prime}} \delta_{r, r^{\prime}}+\frac{i}{2}(\pi \nu)^{2}\left\langle\operatorname{Tr}\left\{\hat{\gamma}^{q} Q_{t, t}(r)\right\} \operatorname{Tr}\left\{\hat{\gamma}^{c l} Q_{t^{\prime}, t^{\prime}}\left(r^{\prime}\right)\right\}\right\rangle \tag{6.25}
\end{align*}
$$

where the angular brackets stand for the averaging over the action 6.15. In the Fourier representation the last expression takes the form:

$$
\begin{equation*}
\Pi^{R}(\omega ; q)=\nu+\frac{i}{2}(\pi \nu)^{2} \iint \frac{d \epsilon d \epsilon^{\prime}}{(2 \pi)^{2}}\left\langle\operatorname{Tr}\left\{\hat{\gamma}^{q} Q_{\epsilon, \epsilon+\omega}(q)\right\} \operatorname{Tr}\left\{\hat{\gamma}^{c l} Q_{\epsilon^{\prime}+\omega, \epsilon^{\prime}}(-q)\right\}\right\rangle .(6 \tag{6.26}
\end{equation*}
$$

Employing Eq. 6.22, one finds the linear in $\mathcal{W}$ terms:

$$
\begin{align*}
& \operatorname{Tr}\left\{\hat{\gamma}^{q} Q_{\epsilon, \epsilon+\omega}(q)\right\} \sim 2\left(\mathcal{F}_{\epsilon} \bar{w}_{\epsilon, \epsilon+\omega}(q)-\bar{w}_{\epsilon, \epsilon+\omega}(q) \mathcal{F}_{\epsilon+\omega}\right) ;  \tag{6.27}\\
& \operatorname{Tr}\left\{\hat{\gamma}^{c l} Q_{\epsilon^{\prime}+\omega, \epsilon^{\prime}}(q)\right\} \sim 2\left(\mathcal{F}_{\epsilon^{\prime}+\omega} \bar{w}_{\epsilon^{\prime}+\omega, \epsilon^{\prime}}(q) \mathcal{F}_{\epsilon^{\prime}}-\bar{w}_{\epsilon^{\prime}+\omega, \epsilon^{\prime}}(q)+w_{\epsilon^{\prime}+\omega, \epsilon^{\prime}}(q)\right) .
\end{align*}
$$

For a spatially uniform distribution $\langle\overline{w w}\rangle=0$ and only the last term of the last expression contributes to the correlator. The result is:

$$
\begin{align*}
\Pi^{R}(\omega ; q) & =\nu+\frac{i}{2}(\pi \nu)^{2} 4 \int \frac{d \epsilon}{2 \pi}\left(\mathcal{F}_{\epsilon}-\mathcal{F}_{\epsilon+\omega}\right)\left\langle w_{\epsilon^{\prime}+\omega, \epsilon^{\prime}}(-q) \bar{w}_{\epsilon, \epsilon+\omega}(q)\right\rangle \\
& =\nu\left[1+\frac{i \omega}{D q^{2}-i \omega}\right]=\nu \frac{D q^{2}}{D q^{2}-i \omega} \tag{6.28}
\end{align*}
$$

where we have used the fact that for any reasonable fermionic distribution $\mathcal{F}_{ \pm \infty}=$ $\pm 1$ and therefore $\int d \epsilon\left(\mathcal{F}_{\epsilon}-\mathcal{F}_{\epsilon+\omega}\right)=-2 \omega$. The fact that $\Pi(\omega, 0)=0$ is a consequence of the particle number conservation. One has obtained the diffusion form of the density-density response function. Also notice that this function is indeed
retarded (analytic in the upper half-plane of complex $\omega$ ), as it should be. The current-current response function, $K^{R}(\omega ; q)$ may be obtained using the continuity equation $q j+\omega \varrho=0$ and is $K^{R}(\omega ; q)=\omega^{2} \Pi^{R}(\omega ; q) / q^{2}$. As a result the conductivity is given by

$$
\begin{equation*}
\sigma(\omega ; q)=\frac{e^{2}}{i \omega} K^{R}(\omega ; q)=e^{2} \nu D \frac{-i \omega}{D q^{2}-i \omega} \tag{6.29}
\end{equation*}
$$

In the uniform limit $q \rightarrow 0$, one obtains the Drude result: $\sigma(\omega ; 0)=e^{2} \nu D$.

### 6.5. Spectral statistics

Consider a piece of disordered metal of size $L$ such that $L \gg l$, where $l \equiv v_{F} \tau$ is the elastic mean free path. The spectrum of the Schrödinger equation consists of a discrete set of levels, $\epsilon_{n}$, that may be characterized by the sample-specific DOS, $\nu(\epsilon) \sim \sum_{n} \delta\left(\epsilon-\epsilon_{n}\right)$. This quantity fluctuates wildly and usually cannot (and need not) be calculated analytically. One may average it over the realizations of disorder to obtain a mean DOS: $\overline{\nu(\epsilon)}$. The latter is a smooth function of energy on the scale of the Fermi energy and thus at low temperature may be taken as a constant $\overline{\nu\left(\epsilon_{F}\right)} \equiv \nu$. This is exactly the DOS that was used in the previous sections.

One may wonder how to sense fluctuations of the sample-specific DOS $\nu(\epsilon)$ and, in particular, a given spectrum at one energy $\epsilon$ is correlated with itself at another energy $\epsilon^{\prime}$. To answer this question one may calculate the spectral correlation function:

$$
\begin{equation*}
R\left(\epsilon, \epsilon^{\prime}\right) \equiv \overline{\nu(\epsilon) \nu\left(\epsilon^{\prime}\right)}-\nu^{2} \tag{6.30}
\end{equation*}
$$

This function was calculated in the seminal paper of Altshuler and Shklovskii [24] in 1986. Here we derive it using the Keldysh NL $\sigma$ M.

The DOS is $\nu(\epsilon)=i \sum_{k}\left(\mathcal{G}_{k}^{R}(\epsilon)-\mathcal{G}_{k}^{A}(\epsilon)\right) /(2 \pi)=\left(\left\langle\psi_{1} \bar{\psi}_{1}\right\rangle-\left\langle\psi_{2} \bar{\psi}_{2}\right\rangle\right) /(2 \pi)=$ $-\hat{\bar{\psi}} \hat{\sigma}_{3} \hat{\psi} /(2 \pi)$, where the angular brackets denote quantum (as opposed to disorder) averaging and the indexes are in Keldysh space. To generate the DOS at any given energy one adds a source term $-\int d \epsilon /(2 \pi) J_{\epsilon} \int d r \hat{\bar{\psi}}_{\epsilon}(r) \hat{\sigma}_{3} \hat{\psi}_{\epsilon}(r)=$ $-\iint d t d t^{\prime} \int d r \hat{\bar{\psi}}_{t}(r) J_{t-t^{\prime}} \hat{\sigma}_{3} \hat{\psi}_{t^{\prime}}(r)$ to the fermionic action. Then the DOS is obtained by $\nu(\epsilon)=\delta Z[J] / \delta J_{\epsilon}$. After averaging over disorder and changing to the $\hat{Q}$-matrix representation in exactly the same manner as above, the source term is translated to $\pi \nu \int d \epsilon /(2 \pi) J_{\epsilon} \int d r \operatorname{Tr}\left\{\hat{Q}_{\epsilon, \epsilon}(r) \hat{\sigma}_{3}\right\}$. The derivation is the same as the derivation of Eq. 6.15. It in now clear that $\overline{\nu(\epsilon)}=\frac{1}{2} \nu\left\langle\operatorname{Tr}\left\{\hat{Q}_{\epsilon, \epsilon} \hat{\sigma}_{3}\right\}\right\rangle_{Q}$. Substituting $\hat{Q}_{\epsilon, \epsilon}=\hat{\Lambda}_{\epsilon}$ one finds $\overline{\nu(\epsilon)}=\nu$, as it should be, of course. It is also easy to check that the fluctuations around $\hat{\Lambda}$ do not change the result (all the
fluctuation diagrams cancel due to the causality constraints). We are now in the position to calculate the correlation function:

$$
\begin{equation*}
R\left(\epsilon, \epsilon^{\prime}\right) \equiv \frac{\delta^{2} Z[J]}{\delta J_{\epsilon} \delta J_{\epsilon^{\prime}}}-\nu^{2}=\nu^{2}\left[\frac{1}{4}\left\langle\operatorname{Tr}\left\{\hat{Q}_{\epsilon, \epsilon} \hat{\sigma}_{3}\right\} \operatorname{Tr}\left\{\hat{Q}_{\epsilon^{\prime}, \epsilon^{\prime}} \hat{\sigma}_{3}\right\}\right\rangle_{Q}-1\right] \tag{6.31}
\end{equation*}
$$

Employing the parameterization of Eqs. 6.20 - 6.22), one finds up to the second order in $\mathcal{W}$ :

$$
\begin{equation*}
\operatorname{Tr}\left\{\hat{Q} \hat{\sigma}_{3}\right\}=2[1+\mathcal{F} \circ \bar{w}+\bar{w} \circ \mathcal{F}+w \circ \bar{w}+\bar{w} \circ w] . \tag{6.32}
\end{equation*}
$$

Since $\langle\overline{w w}\rangle=0$, the only non-vanishing terms contributing to Eq. 6.31 are those with no $w$ and $\bar{w}$ at all (they cancel $\nu^{2}$ term) and those of the type $\langle w \bar{w} w \bar{w}\rangle$. Collecting the latter terms one finds:

$$
\begin{align*}
& R\left(\epsilon, \epsilon^{\prime}\right)  \tag{6.33}\\
& =\nu^{2} \int d r \iint \frac{d \epsilon_{1} d \epsilon_{2}}{4 \pi^{2}}\left\langle\left(w_{\epsilon, \epsilon_{1}} \bar{w}_{\epsilon_{1}, \epsilon}+\bar{w}_{\epsilon, \epsilon_{1}} w_{\epsilon_{1}, \epsilon}\right)\left(w_{\epsilon^{\prime}, \epsilon_{2}} \bar{w}_{\epsilon_{2}, \epsilon^{\prime}}+\bar{w}_{\epsilon^{\prime}, \epsilon_{2}} w_{\epsilon_{2}, \epsilon^{\prime}}\right)\right\rangle_{Q}
\end{align*}
$$

Finally, performing the Wick contractions according to Eq. 6.24 and taking into account that $\int d \epsilon_{1} D^{2}\left(\epsilon-\epsilon_{1} ; q\right)=0$, due to the integration of a function that is analytic in the entire upper half-plane of $\epsilon_{1}$, one finds:

$$
\begin{equation*}
R\left(\epsilon, \epsilon^{\prime}\right)=\frac{1}{(2 \pi)^{2}} \sum_{q}\left[D^{2}\left(\epsilon-\epsilon^{\prime} ; q\right)+D^{2}\left(\epsilon^{\prime}-\epsilon ; q\right)\right] \tag{6.34}
\end{equation*}
$$

where the $q$-summation stands for a summation over the discrete modes of the diffusion operator $D \nabla_{r}^{2}$ with the zero current (zero derivative) at the boundary of the metal. This is the result of Altshuler and Shklovskii for the unitary symmetry class. Notice that the correlation function depends on the energy difference $\omega=$ $\epsilon-\epsilon^{\prime}$ only.

For a small energy difference $\omega<E_{\text {Thouless }} \equiv D / L^{2}$ only the lowest homogenous mode, $q=0$, of the diffusion operator (the so called zero-mode) may be retained and thus: $R(\omega)=-1 /\left(2 \pi^{2} \omega^{2}\right)$. This is the universal random matrix result. The negative correlations mean energy levels' repulsion. Notice that the correlations decay very slowly - as the inverse square of the energy distance. One may notice that the true random matrix result $R_{R M T}(\omega)=$ $-(1-\cos (2 \pi \omega / \delta)) /\left(2 \pi^{2} \omega^{2}\right)$, where $\delta$ is the mean level spacing, contains also an oscillatory function of the energy difference. These oscillations reflect discreteness of the underlying energy spectrum. They cannot be found by the perturbation theory in small fluctuations near the $\hat{\Lambda}$ "point". However, they may be recovered once additional stationary points (not possessing the causality structure) are taken into account [25]. The saddle-point method and perturbation
theory work as long as $\omega>\delta$. Currently it is not known how to work with the Keldysh $\mathrm{NL} \sigma \mathrm{M}$ at $\omega<\delta$.

In the opposite limit, $\omega>E_{\text {Thouless }}$, the summation over modes may be replaced by an integration and thus $R(\omega)=-c_{d} / \omega^{2-d / 2}$, where $c_{d}$ is a positive dimensionality dependent constant. This algebraic decay of the correlations is reflected by many experimentally observable phenomena generally known as mesoscopic fluctuations.

The purpose of these notes is to give the reader a general perspective of the Keldysh formalism, its structure, guiding principles, its strength and its limitations. Due to space limitations, I could not include many topics of contemporary research interests into this introductory course. I hope to fulfill some of the gaps on future occasions.

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## Appendix A. Gaussian integration

For any complex $N \times N$ matrix $A_{i j}$, where $i, j=1, \ldots N$, such that all its eigenvalues, $\lambda_{i}$, have a positive real part: $\Re \lambda_{i}>0$, the following statement holds:
$Z[J]=\iint_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{d \Re z_{j} d \Im z_{j}}{\pi} e^{-\sum_{i j}^{N} \bar{z}_{i} A_{i j} z_{j}+\sum_{j}^{N}\left[\bar{z}_{j} J_{j}+\bar{J}_{j} z_{j}\right]}=\frac{e^{\sum_{i j}^{N} \bar{J}_{i}\left(A^{-1}\right)_{i j} J_{j}}}{\operatorname{det} A}$, (A. 1)
where $J_{j}$ is an arbitrary complex vector. To prove it, one may start from a Hermitian matrix, that is diagonalized by a unitary transformation: $A=U^{\dagger} \Lambda U$, where $\Lambda=\operatorname{diag}\left\{\lambda_{j}\right\}$. The identity is then easily proven by a change of variables (with unit Jacobian) to $w_{i}=U_{i j} z_{j}$. Finally, one notices that the r.h.s. of Eq. A. 1 is an analytic function of both $\Re A_{i j}$ and $\Im A_{i j}$. Therefore, one may continue them analytically to the complex plane to reach an arbitrary complex matrix $A_{i j}$. The identity A. 1 is thus valid as long as the integral on its l.h.s. is well defined, that is all the eigenvalues of $A_{i j}$ have a positive real part.

The Wick theorem deals with the average value of a string $z_{a_{1}} \ldots z_{a_{k}} \bar{z}_{b_{1}} \ldots \bar{z}_{b_{k}}$ weighted with the factor $\exp \left\{-\sum_{i j}^{N} \bar{z}_{i} A_{i j} z_{j}\right\}$. The theorem states that this av-
erage is given by the sum of all possible products of pair-wise averages. For example,

$$
\begin{align*}
& \left.\left\langle z_{a} \bar{z}_{b}\right\rangle \equiv \frac{1}{Z[0]} \frac{\delta^{2} Z[J]}{\delta \bar{J}_{a} \delta J_{b}}\right|_{J=0}=\left(A^{-1}\right)_{a b} ;  \tag{A.2}\\
& \left.\left\langle z_{a_{1}} z_{a_{2}} \bar{z}_{b_{1}} \bar{z}_{b_{2}}\right\rangle \equiv \frac{1}{Z[0]} \frac{\delta^{4} Z[J]}{\delta \bar{J}_{a_{1}} \bar{J}_{a_{2}} \delta J_{b_{1}} J_{b_{2}}}\right|_{J=0}=A_{a_{1} b_{1}}^{-1} A_{a_{2} b_{2}}^{-1}+A_{a_{1} b_{2}}^{-1} A_{a_{2} b_{1}}^{-1}, \text { (A. 3) } \tag{A.3}
\end{align*}
$$

etc.
The Gaussian identity for integration over real variables has the form:

$$
\begin{equation*}
Z[J]=\int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{d x_{j}}{\sqrt{\pi}} e^{-\sum_{i j}^{N} x_{i} A_{i j} x_{j}+2 \sum_{j}^{N} x_{j} J_{j}}=\frac{e^{\sum_{i j}^{N} J_{i}\left(A^{-1}\right)_{i j} J_{j}}}{\sqrt{\operatorname{det} A}} \tag{A.4}
\end{equation*}
$$

where $A$ is a symmetric complex matrix with all its eigenvalues having a positive real part. The proof is similar to the proof in the case of complex variables: one starts from a real symmetric matrix, that may be diagonalized by an orthogonal transformation. The identity A .4 is then easily proved by the change of variables. Finally, one may analytically continue the r.h.s. (as long as the integral is well defined) from a real symmetric matrix $A_{i j}$, to a complex symmetric one.

For an integration over two sets of independent Grassmann variables, $\bar{\xi}_{j}$ and $\xi_{j}$, where $j=1,2, \ldots, N$, the Gaussian identity is valid for any invertible complex matrix $A$ :

$$
\begin{align*}
& Z[\bar{\chi}, \chi]  \tag{A.5}\\
& =\iint \prod_{j=1}^{N} d \bar{\xi}_{j} d \xi_{j} e^{-\sum_{i j}^{N} \bar{\xi}_{i} A_{i j} \xi_{j}+\sum_{j}^{N}\left[\bar{\xi}_{j} \chi_{j}+\bar{\chi}_{j} \xi_{j}\right]}=\operatorname{det} A e^{\sum_{i j}^{N} \bar{\chi}_{i}\left(A^{-1}\right)_{i j} \chi_{j}} .
\end{align*}
$$

Here $\bar{\chi}_{j}$ and $\chi_{j}$ are two additional mutually independent (and independent from $\bar{\xi}_{j}$ and $\xi_{j}$ ) sets of Grassmann numbers. The proof may be obtained by e.g. brute force expansion of the exponential factors, while noticing that only terms that are linear in all $2 N$ variables $\bar{\xi}_{j}$ and $\xi_{j}$ are non-zero. The Wick theorem is formulated in the same manner as for the bosonic case, with the exception that every combination is multiplied by the parity of the corresponding permutation. E.g. the first term on the r.h.s. of Eq. A. 3 comes with a minus sign.

## Appendix B. Single particle quantum mechanics

The simplest many-body system of a single bosonic state (considered in Chapter 2) is of course equivalent to a single-particle harmonic oscillator. To make this connection explicit, consider the Keldysh contour action Eq. (2.11) with the correlator Eq. 2.13) written in terms of the complex field $\phi(t)$. The latter may be parameterized by its real and imaginary parts as:

$$
\begin{align*}
\phi(t) & =\frac{1}{\sqrt{2 \omega_{0}}}\left(p(t)-i \omega_{0} q(t)\right) \\
\bar{\phi}(t) & =\frac{1}{\sqrt{2 \omega_{0}}}\left(p(t)+i \omega_{0} q(t)\right) \tag{B.1}
\end{align*}
$$

In terms of the real fields $p(t)$ and $q(t)$ the action, Eq. 2.11, takes the form:

$$
\begin{equation*}
S[p, q]=\int_{\mathcal{C}} d t\left[p \dot{q}-\frac{1}{2}\left(p^{2}+\omega_{0}^{2} q^{2}\right)\right] \tag{B.2}
\end{equation*}
$$

where the full time derivatives of $p^{2}, q^{2}$ and $p q$ were omitted, since they contribute only to the boundary terms, not written explicitly in the continuous notation (they have to be kept for proper regularization). Equation B. 2 is nothing but the action of the quantum harmonic oscillator in the Hamiltonian form. One may perform the Gaussian integration over the $p(t)$ field to obtain:

$$
\begin{equation*}
S[q]=\int_{\mathcal{C}} d t\left[\frac{1}{2} \dot{q}^{2}-\frac{\omega_{0}^{2}}{2} q^{2}\right] \tag{B.3}
\end{equation*}
$$

This is the Feynman Lagrangian action of the harmonic oscillator, written on the Keldysh contour. It may be generalized for an arbitrary single particle potential $U(q)$ :

$$
\begin{equation*}
S[q(t)]=\int_{\mathcal{C}} d t\left[\frac{1}{2}(\dot{q}(t))^{2}-U(q(t))\right] \tag{B.4}
\end{equation*}
$$

One may split the $q(t)$ field into two components: $q_{+}(t)$ and $q_{-}(t)$, residing on the forward and backward branches of the contour, and then perform the Keldysh rotation: $q_{ \pm}=q_{c l} \pm q_{q}$. In terms of these fields the action takes the form:

$$
\begin{equation*}
S\left[q_{c l}, q_{q}\right]=\int_{-\infty}^{\infty} d t\left[-2 q_{q} \frac{d^{2} q_{c l}}{d t^{2}}-U\left(q_{c l}+q_{q}\right)+U\left(q_{c l}-q_{q}\right)\right] \tag{B.5}
\end{equation*}
$$

where integration by parts was performed in the term $\dot{q}_{q} \dot{q}_{c l}$. This is the Keldysh form of the Feynman path integral. The omitted boundary terms provide a convergence factor of the form $\sim i 0 q_{q}^{2}$.

If the fluctuations of the quantum component $q_{q}(t)$ are regarded as small, one may expand the potential to the first order and find for the action:

$$
\begin{equation*}
S\left[q_{c l}, q_{q}\right]=\int_{-\infty}^{\infty} d t\left[-2 q_{q}\left(\frac{d^{2} q_{c l}}{d t^{2}}+\frac{\partial U\left(q_{c l}\right)}{\partial q_{c l}}\right)+i 0 q_{q}^{2}+O\left(q_{q}^{3}\right)\right] \tag{B.6}
\end{equation*}
$$

In this limit the integration over the quantum component, $q_{q}$, may be explicitly performed, leading to a functional $\delta$-function of the expression in the round brackets. This $\delta$-function enforces the classical Newtonian dynamics of $q_{c l}$ :

$$
\begin{equation*}
\frac{d^{2} q_{c l}}{d t^{2}}=-\frac{\partial U\left(q_{c l}\right)}{\partial q_{c l}} \tag{B.7}
\end{equation*}
$$

For this reason the symmetric (over forward and backward branches) part of the Keldysh field is called the classical component. The quantum mechanical information is contained in the higher order terms in $q_{q}$, omitted in Eq. B. 6. Notice, that for the harmonic oscillator potential the terms denoted as $O\left(q_{q}^{3}\right)$ are absent identically. The quantum (semiclassical) information resides, thus, in the convergence term, $i 0 q_{q}^{2}$, as well as in the retarded regularization of the $d^{2} /\left(d t^{2}\right)$ operator in Eq. B. 6.

One may generalize the single particle quantum mechanics onto a chain (or lattice) of harmonically coupled particles by assigning an index $r$ to particle coordinates: $q_{r}(t)$, and adding the spring potential energy: $\frac{v_{s}^{2}}{2}\left(q_{r+1}(t)-q_{r}(t)\right)^{2}$. Changing to spatially continuous notations: $\varphi(t ; r) \equiv q_{r}(t)$, one finds for the Keldysh action of the real (phonon) field:

$$
\begin{equation*}
S[\varphi]=\int d r \int_{\mathcal{C}} d t\left[\frac{1}{2} \dot{\varphi}^{2}-\frac{v_{s}^{2}}{2}\left(\nabla_{r} \varphi\right)^{2}-U(\varphi)\right] \tag{B.8}
\end{equation*}
$$

where the constant $v_{s}$ has the meaning of the sound velocity. Finally, splitting the field into $\left(\varphi_{+}, \varphi_{-}\right)$components and performing the Keldysh transformation: $\varphi_{ \pm}=\varphi_{c l} \pm \varphi_{q}$, and integrating by parts, one obtains:

$$
\begin{equation*}
S\left[\varphi_{+}, \varphi_{-}\right]=\int d r \int_{-\infty}^{\infty} d t\left[2 \varphi_{q}\left(v_{s}^{2} \nabla_{r}^{2}-\partial_{t}^{2}\right) \varphi_{c l}-U\left(\varphi_{c l}+\varphi_{q}\right)+U\left(\varphi_{c l}-\varphi_{q}\right)\right] \tag{B.9}
\end{equation*}
$$

According to the general structure of the Keldysh theory the differential operator on the r.h.s., $\left(-\partial_{t}^{2}+v_{s}^{2} \nabla_{r}^{2}\right)$, should be understood as the retarded one. This
means it is a lower triangular matrix in the time domain. Actually, one may symmetrize the action by performing the integration by parts, and write it as: $\varphi_{q}\left(-\partial_{t}^{2}+v_{s}^{2} \nabla_{r}^{2}\right)^{R} \varphi_{c l}+\varphi_{c l}\left(-\partial_{t}^{2}+v_{s}^{2} \nabla_{r}^{2}\right)^{A} \varphi_{q}$.

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[^0]:    ${ }^{1}$ Strictly speaking, $G^{R}\left(t^{\prime}, t\right)$ and $G^{A}\left(t^{\prime}, t\right)$ are both simultaneously non-zero at the diagonal: $t=t^{\prime}$. The contribution of the diagonal to the integrals is, however, $\sim \delta_{t}^{2} N \rightarrow 0$, when $N \rightarrow \infty$.

[^1]:    ${ }^{2}$ Only products of WT's are retained, while all the gradient terms are neglected, in particular $\mathbf{D}^{\mathbf{K}} \rightarrow \mathbf{f}\left(\mathbf{d}^{\mathbf{R}}-\mathbf{d}^{\mathbf{A}}\right)$. The energy-momentum representation is used, instead of the time-space representation as in Eqs. 3.14-3.16, and in the equation for $\Sigma^{R} \circ \mathbf{F}-\mathbf{F} \circ \Sigma^{A}$ one performs a symmetrization between the $\omega$ and $\epsilon-\omega$ arguments.

[^2]:    ${ }^{3}$ Amazingly, this trick of rewriting viscous (or diffusive) dynamics as a Hamiltonian one, works in a wide class of problems. The price, one has to pay, is the doubling of the number of degrees of freedom: $q$ and $p$ in the Hamiltonian language, or "classical" and "quantum" components in the Keldysh language.

[^3]:    ${ }^{4}$ To check this statement one may substitute $\mathcal{P}(q, t)=\delta(q-q(t))$ into the $T=0$ FP equation: $\delta_{q}^{\prime}(q-q(t))(-\dot{q}(t))=(2 / \gamma)\left[U_{q q}^{\prime \prime} \delta(q-q(t))+U_{q}^{\prime} \delta_{q}^{\prime}(q-q(t))\right]$. Then multiplying both parts of this equation by $q$ and integrating over $d q$ (by performing integration by parts), one finds: $\dot{q}(t)=$ $-(2 / \gamma) U_{q}^{\prime}(q(t))$.

[^4]:    Reminder: The Matsubara technique deals with the imaginary time $\tau$ confined to the interval $\tau \in[0, \beta[$, where $\beta=1 / T$. All bosonic fields must be periodic in this interval: $\phi(\tau+\beta)=\phi(\tau)$, while the fermionic fields are antiperiodic: $\psi(\tau+\beta)=-\psi(\tau)$. It is convenient to introduce the
    discrete Fourier (Matsubara) transform, e.g. $\phi_{m}=\int_{0}^{\beta} d \tau \phi(\tau) e^{i \epsilon_{m} \tau}$, where for bosons
    $\epsilon_{m} \equiv 2 \pi m T$, while for fermions $\epsilon_{m} \equiv \pi(2 m+1) T$ and $m=0, \pm 1, \ldots$.

[^5]:    ${ }^{5}$ Indeed, $T \dot{p}=\partial_{t} \dot{q}=-D \nabla_{r}^{2} \dot{q}+\dot{q} U_{q q}^{\prime \prime}=T\left(-D \nabla_{r}^{2} p+p U_{q q}^{\prime \prime}\right)$. This non-trivial fact reflects the existence of an accidental conservation law: $H(p(t ; r), q(t ; r))=0$ - locally! While from the general principles only the total global energy has to be conserved.

[^6]:    ${ }^{6}$ To derive this expression one should add and subtract $\mathbf{n}_{\epsilon} \mathbf{n}_{\epsilon^{\prime}-\omega} \mathbf{n}_{\epsilon} \mathbf{n}_{\epsilon-\omega}$ in the square brackets.

[^7]:    ${ }^{7}$ The minus sign originates from commuting the Grassmann numbers.

[^8]:    ${ }^{8}$ One uses that $v_{F}=k / m$ and $\sum_{k} \mathcal{G}_{\epsilon}^{R}(k) \frac{k}{m} \mathcal{G}_{\epsilon}^{A}(k) \frac{k}{m}=2 \pi \nu \tau v_{F}^{2} / d=2 \pi \nu D$, while the corresponding $R-R$ and $A-A$ terms vanish. Employing Eq. 6.13], one then arrives at $\frac{1}{4} \operatorname{Tr}\{[\hat{1}+$ $\left.\left.\hat{\Lambda}_{\epsilon}\right]\left(\hat{\mathcal{T}}^{-1} \nabla_{r} \hat{\mathcal{T}}\right)\left[\hat{1}-\hat{\Lambda}_{\epsilon}\right]\left(\hat{\mathcal{T}}^{-1} \nabla_{r} \hat{\mathcal{T}}\right)\right\}=-\frac{1}{8} \operatorname{Tr}\left\{\left(\nabla_{r}\left(\hat{\mathcal{T}} \hat{\Lambda}_{\epsilon} \hat{\mathcal{T}}^{-1}\right)\right)^{2}\right\}=-\frac{1}{8} \operatorname{Tr}\left\{\left(\nabla_{r} \hat{Q}\right)^{2}\right\}$.

