
GREEN'S FUNCTIONS ON FRACTALS

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Abstract

For a regular harmonic structure on a post-critically finite (p.c.f.) self-similar fractal, the Dirichlet problem for the Laplacian can be solved by integrating against an explicitly given Green's function. We give a recursive formula for computing the values of the Green's function near the diagonal, and use it to give sharp estimates for the decay of the Green's function near the boundary. We present data from computer experiments searching for the absolute maximum of the Green's function for two different examples, and we formulate two radically different conjectures for where the maximum occurs. We also investigate a local Green's function that can be used to solve an initial value problem for the Laplacian, giving an explicit formula for the case of the Sierpinski gasket. The local Green's function turns out to be unbounded, and in fact not even integrable, but because of cancelation, it is still possible to form a singular integral to solve the initial value problem if the given function satisfies a Hölder condition.

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1. INTRODUCTION

At the same time that scientists have been using fractals to model objects in the real world, mathematicians have been developing a theory of analysis on a special class of fractals, to provide a framework in which to describe the fractal analogs of the differential equations of mathematical physics. The self-similar fractals considered in this mathematical theory are too regular to serve as models for natural objects, but they provide a first step in the direction of understanding more realistic examples. In addition, it has been recently suggested by Hohlfeld and Cohen (1992)¹ that it might be advantageous to construct antennae in the shape of these self-similar fractals. Thus the mathematical theory may prove useful in understanding the physical properties of man-made objects. In this paper we will deal only with mathematical developments, presenting both theoretical and experimental results.

For a large class of self-similar fractals K , called post-critically finite (p.c.f.), the first author has developed a theory of Laplacians.^{2–9} The essential ingredients are a harmonic structure, which gives rise to a Dirichlet form $\mathcal{E}(u, v)$, and a measure μ . The existence of harmonic structures on p.c.f. self-similar fractals is still an open question (see Refs. 10–12 for some partial results). In the following discussion, we assume that a harmonic structure is given. The Laplacian Δ_μ may then be defined by the identity

$$-\mathcal{E}(u, v) = \int_K v \Delta_\mu u d\mu \quad (1.1)$$

for v in the domain of \mathcal{E} and vanishing on the boundary of K (denoted V_0). The theory of these Laplacians has been extensively developed. The references give a sampling of some papers in this area.^{2–25}

One of the key results in Kigami (1993)³ is the existence of an explicit Green's function $G(x, y)$, depending only on the harmonic structure, that can be used to solve the Poisson's problem with Dirichlet boundary conditions

$$-\Delta_\mu u = f \quad \text{on } K \quad (1.2)$$

$$u|_{V_0} = 0 \quad (1.3)$$

for continuous f via the integral

$$u(x) = \int_K G(x, y) f(y) d\mu(y). \quad (1.4)$$

For a regular harmonic structure, which is the case

we will be concerned with, G is a bounded continuous function and points have positive capacity, so the theory is more akin to the second derivative on an interval than a Laplacian on a manifold of dimension greater than one. The unit interval is the simplest example of a p.c.f. fractal, generated by the contractions $F_1x = \frac{1}{2}x$ and $F_2x = \frac{1}{2}x + \frac{1}{2}$. For a general p.c.f. self-similar fractal K , we have contractions F_1, \dots, F_N such that the images F_jK have only finite intersections. The boundary of K is V_0 , the set of all pre-images of these intersection points, also assumed to be finite. We form a sequence of graphs $\{\Gamma_j\}$ with vertices $\{V_j\}$ where each V_j is the union of the images of V_{j-1} with edges that describe the connectivity of the images of K . From a single Dirichlet form \mathcal{E}_0 on V_0 and positive constants r_1, \dots, r_N with each $r_j < 1$ (this is the regularity assumption), we generate a sequence of Dirichlet forms \mathcal{E}_j on V_j which satisfy a compatibility condition (this is a hypothesis), and in the limit generate a Dirichlet form \mathcal{E} on K which is self-similar:

$$\mathcal{E}(u, v) = \sum_{j=1}^N r_j^{-1} \mathcal{E}(u \circ F_j, v \circ F_j). \quad (1.5)$$

In the case of the unit interval for the choice $r_1 = r_2 = 1/2$, we obtain the usual Dirichlet form

$$\int_0^1 u'(x)v'(x) dx.$$

The reader should consult Refs. 3, 9, 13 or 14 for the details of the construction.

The construction of the Green's function in Ref. 3 goes as follows. We take the Dirichlet form \mathcal{E}_1 on V_1 and restrict it to the new points $V_1 \setminus V_0$ to obtain a matrix X_1 which turns out to be invertible, and we define a matrix G by $G = -X_1^{-1}$. The entries of G are denoted $G_{p,q}$ for $p, q \in V_1 \setminus V_0$, and it turns out that they are nonnegative. For any particular harmonic structure it is not difficult to compute these entries explicitly. We use these entries to construct a function

$$\Psi(x, y) = \sum_{p,q} G_{p,q} \psi_p(x) \psi_q(y) \quad (1.6)$$

where $\psi_p(x)$ denotes the continuous function on K whose restrictions to $F_j(K \setminus V_0)$ are harmonic satisfying $\psi_p(q) = \delta_{pq}$ for all $q \in V_1$ (in particular ψ_p vanishes on all points in V_0). The Green's function will be a weighted sum of scaled copies of the function $\Psi(x, y)$.

It is convenient to use notation for iterated products of the mappings $\{F_j\}$ based on words w on the letters $\{1, \dots, N\}$. If $w = (w_1, \dots, w_m)$ is a word of length m , we write $F_w = F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_m}$ and $r_w = r_{w_1} r_{w_2} \dots r_{w_m}$. We also write $K_w = F_w K$. The Green's function is given by

$$G(x, y) = \sum r_w \Psi(F_w^{-1}x, F_w^{-1}y) \quad (1.7)$$

where the sum extends over all words (including the empty word) such that x and y both belong to K_w . Note that for $x \neq y$ the sum is finite. It is also finite on the diagonal if $x = y$ is a point in V_m for some m , for then $F_w^{-1}x$ is a boundary point where Ψ vanishes, when the length of w exceeds m . In any case, the regularity assumption implies that the sum converges uniformly to a continuous function.

The main purpose of this paper is to extract as much information as possible from the representation (1.7). We note that this is a completely explicit algorithmic formula. We use it to generate numerical data for two basic examples, the Sierpinski gasket and the pentagasket (or pentakun). We use this data to experimentally explore the question of finding the absolute maximum value of $G(x, x)$ (the maximum of $G(x, y)$ is easily seen to lie on the diagonal) and the location of the points where it is attained. We formulate two conjectures of radically different content: for the pentagasket the maximum appears to occur at points in V_1 , while for the Sierpinski gasket it appears to occur at points not in any V_m , but which have periodic addresses. The data and these conjectures are reported in Sec. 3.

We also obtain a different representation of $G(x, y)$ for x and y in V_m which are nearby (belonging to the same K_w for a word w of length m). We call this the *near diagonal formula*. This is a recursive algorithm similar in spirit to some of the algorithms in Dalrymple et al. (1999)¹⁵ for computing harmonic and biharmonic functions and eigenfunctions of the Laplacians, although it is a bilinear rather than linear algorithm. We use this result to obtain sharp estimates on the decay of the Green's function in a neighborhood of a boundary point. These results are in Sec. 2.

Section 4 is devoted to a related function, the *local Green's function* $G_z(x, y)$, that can be used to solve (1.2) with vanishing "initial conditions" at $x = z$. The local Green's function was introduced in Strichartz (2000)¹⁶. Here we compute $G_z(x, y)$ explicitly for the Sierpinski gasket with z as the boundary point, and we discover that it becomes unbounded as y approaches z . In fact it

is not even integrable as a function of y , but it does have cancellation properties that enable us to define the analog of (1.4) as a singular integral when f is assumed to be Hölder continuous of any order. We then prove that this integral does provide a solution (the uniqueness was proved in Ref. 16) to the initial value problem.

One of the basic results of the theory is that a harmonic function on K is determined from its values on the boundary by a recursive algorithm that is local in nature. There exist matrices A_i for $i = 1, \dots, N$, easily determined from the harmonic structure, such that

$$h|_{F_i V_0} = A_i h|_{V_0} \quad (1.8)$$

for any harmonic function h , and more generally,

$$h|_{F_w F_i V_0} = A_i h|_{F_w V_0} \quad (1.9)$$

for any word w . Of course (1.9) is an abbreviation for

$$h(F_w F_i v_j) = \sum_k (A_i)_{jk} h(F_w v_k). \quad (1.10)$$

The matrix A_i has non-negative entries and row sums equal to 1 (since constants are harmonic functions). We need more specific information in the case when v_i is the fixed point of F_i (we arrange the labeling so the fixed point and the mapping have the same index). In that case the i -th row of A_i is δ_{ji} , and we denote by \tilde{A}_i the submatrix obtained by deleting row and column i . We assume that $K \setminus V_0$ is connected which implies that harmonic functions satisfy the strong maximum principle. It follows that \tilde{A}_i is irreducible. The argument is simply that the k -th row of \tilde{A}_i^m just gives the values of a certain harmonic function (the one with $h(v_j) = \delta_{jk}$) on $F_i^m V_0$, and for large enough m the only boundary point in $F_i^m V_0$ is v_i .

The matrix A_i has the trivial eigenvalue 1, and all other eigenvalues of A_i are eigenvalues of \tilde{A}_i . The following result was used in Ref. 16, but we give a brief proof here since it does not appear explicitly in the literature.

Lemma 1.1. *If $F_i v_i = v_i$, then the largest eigenvalue of \tilde{A}_i (hence the second largest eigenvalue of A_i) is r_i .*

Proof. Let h be the harmonic function such that $\{h(v_j)\}$ is the eigenvector associated to the second

largest eigenvalue λ of A_i . Then $h(v_i) = 0$ and $h(v_j) > 0$ for $j \neq i$. Now the normal derivative of h at v_i exists:³

$$\partial_n h(v_i) = \lim_{m \rightarrow \infty} r_i^{-m} \sum_{j \neq i} c_j (h(v_i) - h(F_i^m v_j)) \tag{1.11}$$

for certain positive coefficients c_j , and for harmonic functions it is not necessary to take the limit. Now $h(v_i) - h(F_i^m v_j) = \lambda^m (h(v_i) - h(v_j))$ so if $\lambda > r_i$ the limit cannot exist, while if $\lambda < r_i$ the limit is 0. But the individual terms are all negative, so the limit cannot be 0. \square

Aside from the trivial example of the unit interval, the simplest example of a harmonic structure on a p.c.f. fractal is the standard harmonic structure on the Sierpinski gasket. We take $V_0 = \{v_1, v_2, v_3\}$ to be the vertices of an equilateral triangle in the plane, and $F_i x = \frac{1}{2}(x + v_i)$, $i = 1, 2, 3$, and the Sierpinski gasket is the fractal they generate. To get the standard harmonic structure we take $\mathcal{E}_0(u, u) = \sum_{j < k} (u(v_j) - u(v_k))^2$ and $r_1 = r_2 = r_3 = 5/3$. Then

$$\mathcal{E}(u, u) = \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m \sum_{x \sim_m y} (u(x) - u(y))^2 \tag{1.12}$$

where $x \sim_m y$ means there is an edge joining x and y in Γ_m . In this example,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

and the other A_i 's are obtained by permutation from A_1 . Also,

$$G_{p,q} = \begin{cases} \frac{9}{50} & p = q \\ \frac{3}{50} & p \neq q \end{cases}$$

for $p, q \in V_1 \setminus V_0$. The standard Laplacian is obtained by using the standard harmonic structure and the normalized Hausdorff measure. It may be given by the pointwise formula²

$$\Delta u(x) = \frac{3}{2} \lim_{m \rightarrow \infty} 5^m \sum_{y \sim_m x} (u(y) - u(x)) \tag{1.13}$$

for any vertex point x (if x is not a boundary point there are exactly four neighbors in each Γ_m for m

large enough). This example is studied in detail in Refs. 2, 15 and 17

In the trivial case of the unit interval, the Green's function is well-known:

$$G(x, y) = \begin{cases} x(1 - y) & \text{if } x \leq y \\ y(1 - x) & \text{if } y \leq x. \end{cases} \tag{1.14}$$

But even in this case, (1.7) gives a new perspective. Let $H(t)$ denote the "hat function" $H(t) = (\frac{1}{2} - |t - \frac{1}{2}|)_+$ that is piecewise linear and continuous with $H(1/2) = 1/2$, $H(0) = H(1) = 0$ and support in the unit interval. Then (1.7) takes the form

$$\begin{aligned} G(x, y) &= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} 2^{-k} H(2^k x - n) H(2^k y - n) \\ &= \sum_{k \in A(x,y)} 2^{-k} H(\langle 2^k x \rangle) H(\langle 2^k y \rangle) \end{aligned} \tag{1.15}$$

where $A(x, y) = \{k \geq 0 : \lfloor 2^k x \rfloor = \lfloor 2^k y \rfloor\}$. Fix a value of y , say with $y \geq \frac{1}{2}$. Then $H(x)H(y) = G(x, y)$ when $x \leq \frac{1}{2}$, so $G(x, y) - H(x)H(y)$ is supported on $[\frac{1}{2}, 1]$. In particular, (1.15) holds for $x \leq \frac{1}{2}$ with just one term in the sum. For $x > \frac{1}{2}$, we rescale and iterate the argument.

On the diagonal we have $G(x, x) = x(1 - x)$, a smooth parabolic arch. There are now an infinite number of terms in (1.15) for nondyadic x , but each of them has nondifferentiable corner points;

$$H(x)^2 = x^2 - x + \frac{1}{2} - 2 \left| x - \frac{1}{2} \right| \quad \text{on } [0, 1].$$

We have somewhat similar behavior on the Sierpinski gasket.

2. THE NEAR DIAGONAL FORMULA

In this section, we derive a formula for $G(x, y)$ when x and y are vertices on the boundary of a cell $F_w K$. Since this requires x and y to be close together, we call it the *near diagonal formula*. For each word w , we let G_w denote the $N_0 \times N_0$ matrix

$$(G_w)_{jk} = G(F_w v_j, F_w v_k). \tag{2.1}$$

Note that for the empty word the corresponding matrix is zero. The near diagonal formula gives an inductive scheme for computing G_w . For this it suffices to show how to compute G_{wi} in terms of

G_w . For this we will need two $N_0 \times N_0$ matrices, A_i already defined by (1.8) and B_i defined by

$$(B_i)_{jk} = G_{F_i v_j, F_i v_k} \tag{2.2}$$

for $1 \leq i \leq N$.

Theorem 2.1. (Near diagonal formula) For every word w and every $i, 1 \leq i \leq N$,

$$G_{wi} = A_i G_w {}^t A_i + r_w B_i. \tag{2.3}$$

Proof. By (1.7) we have

$$\begin{aligned} &G(F_{w_m} F_{w_{m-1}} \cdots F_{w_1} v_j, F_{w_m} F_{w_{m-1}} \cdots F_{w_1} v_k) \\ &= \Psi(F_{w_m} F_{w_{m-1}} \cdots F_{w_1} v_j, F_{w_m} F_{w_{m-1}} \cdots F_{w_1} v_k) \\ &\quad + r_{w_m} \Psi(F_{w_{m-1}} \cdots F_{w_1} v_j, F_{w_{m-1}} \cdots F_{w_1} v_k) \\ &\quad + \cdots + r_{w_m} \cdots r_{w_2} \Psi(F_{w_1} v_j, F_{w_1} v_k) \end{aligned} \tag{2.4}$$

and similarly

$$\begin{aligned} &G(F_{w_m} F_{w_{m-1}} \cdots F_{w_1} F_i v_j, F_{w_m} F_{w_{m-1}} \cdots F_{w_1} F_i v_k) \\ &= \Psi(F_{w_m} F_{w_{m-1}} \cdots F_{w_1} F_i v_j, F_{w_m} F_{w_{m-1}} \cdots F_{w_1} F_i v_k) \\ &\quad + \cdots + r_{w_m} \cdots r_{w_2} \Psi(F_{w_1} F_i v_j, F_{w_1} F_i v_k) \\ &\quad + r_{w_m} \cdots r_{w_1} \Psi(F_i v_j, F_i v_k). \end{aligned} \tag{2.5}$$

Now we claim the last summand in (2.5) is $r_w B_i$, and for this it suffices to show that

$$\Psi(F_i v_j, F_i v_k) = G_{F_i v_j, F_i v_k}. \tag{2.6}$$

But by (1.6)

$$\Psi(F_i v_j, F_i v_k) = \sum G_{pq} \psi_p(F_i v_j) \psi_q(F_i v_k) \tag{2.7}$$

where p, q range over V_1 . But $\psi_p(q) = \delta_{pq}$ for $p, q \in V_1$, so the only nonzero summand in (2.7) corresponds to the choice $p = F_i v_j$ and $q = F_i v_k$, and this establishes (2.6). Comparing the remaining terms in (2.4) and (2.5), the only difference is the occurrence of F_i before v_j and v_k in (2.5). Thus to complete the proof of (2.3) it suffices to show that for any word w ,

$$\begin{aligned} &\Psi(F_w F_i v_j, F_w F_i v_k) \\ &= \sum_{\ell} \sum_n (A_i)_{j\ell} (A_i)_{kn} \Psi(F_w v_\ell, F_w v_n). \end{aligned} \tag{2.8}$$

However, from (1.6)

$$\Psi(F_w x, F_w y) = \sum_{p,q} G_{pq} \psi_p(F_w x) \psi_p(F_w y) \tag{2.9}$$

and $\psi_p(F_w x)$ is a harmonic function of x for x in $F_i K$. Thus

$$\psi_p(F_w F_i v_j) = \sum_{\ell} (A_i)_{j\ell} \psi_p(F_w v_\ell) \text{ and} \tag{2.10}$$

$$\psi_p(F_w F_i v_k) = \sum_n (A_i)_{kn} \psi_p(F_w v_n)$$

by (1.10). Now substitute (2.10) in (2.9) with $x = F_i v_j$ and $y = F_i v_k$ to obtain (2.8). \square

We can iterate (2.3) to obtain the explicit formula

$$\begin{aligned} G_w &= \sum_{k=1}^m r_{w_1} \cdots r_{w_{k-1}} A_{w_m} \\ &\quad \cdots A_{w_{k+1}} B_{w_k} {}^t (A_{w_m} \cdots A_{w_{k+1}}) \end{aligned} \tag{2.11}$$

for $w = (w_1, \dots, w_m)$, where it is understood in (2.11) that the $k = 1$ term has no r factors, and the $k = m$ term has no A factors. A special case is when $w = (j, \dots, j)$; then

$$G_w = \sum_{k=1}^m r_j^{k-1} A_j^{m-k} B_j ({}^t A_j)^{m-k}. \tag{2.12}$$

We can use (2.12) to understand the boundary behavior of the Green's function.

We first make the mild assumption that every boundary point is the fixed point of one of the mappings F_j (see Remark 2.4 for the general case). Let us label the mappings so that $F_j v_j = v_j$ for $1 \leq j \leq N_0$. For simplicity of notation we take $j = 1$. Then the matrix A_1 has first row $(1, 0, \dots, 0)$, and B_1 has zeros in the first row and column since $F_1 v_1$ is a boundary point. Let \tilde{A}_1 and \tilde{B}_1 denote the $(N_0 - 1) \times (N_0 - 1)$ submatrices obtained by deleting the first row and column. Similarly, let \tilde{G}_m denote the matrix $(G_w)_{jk}$ with $w = (1, \dots, 1)$ (m ones) and $2 \leq j, k \leq N_0$. Then (2.12) becomes

$$\tilde{G}_m = \sum_{k=1}^m r_1^{k-1} \tilde{A}_1^{m-k} \tilde{B}_1 ({}^t \tilde{A}_1)^{m-k}. \tag{2.13}$$

Theorem 2.2. For $1 \leq j \leq N_0$, we have

$$G|_{F_j^m K \times F_j^m k} = O(r_j^m) \text{ as } m \rightarrow \infty. \tag{2.14}$$

Proof. Without loss of generality we may take $j = 1$. First we prove the estimate $\tilde{G}_m = O(r_1^m)$,

which is (2.14) for boundary points on $F_j^m K$. This follows easily from (2.13) if we have the estimate

$$\|\tilde{A}_1^k\|_l \leq cr_1^k \tag{2.15}$$

for then each summand is bounded by a multiple of r_1^{2m-k} . Now \tilde{A}_1 is a matrix with strictly positive entries by the maximal theorem for harmonic functions. By Lemma 1.1, the largest eigenvalue is exactly r_1 . By the Perron-Frobenius theorem, this is a simple eigenvalue and all other eigenvalues have strictly smaller modulus; this implies (2.15).

To obtain the estimate (2.14) at interior points, it suffices to estimate $G(x, x)$ for x any vertex point in $F_j^m K$. That means we need an $O(r_1^m)$ bound for the matrices G_w for all words w with $w_j = 1$ for $1 \leq j \leq m$. This is an easy consequence of repeated use of (2.3) together with the observation that there is a uniform bound on the norms of all products of the A_j matrices (every entry in such a product lies in $[0, 1]$ by the maximal theorem for harmonic functions). Thus

$$\|G_w\| \leq cr_1^m(1 + r_{w_{m+1}} + r_{w_{m+1}}r_{w_{m+2}} + \dots)$$

and the expression in parenthesis is bounded by a convergent geometric series. \square

It is easy to see that the estimate (2.14) is sharp. For the case of the standard harmonic structure on the Sierpinski gasket, we can give a more precise statement.

Theorem 2.3. *For the standard harmonic structure on the Sierpinski gasket,*

$$\begin{aligned} G(F_1^m v_2, F_1^m v_2) &= G(F_1^m v_3, F_1^m v_3) \\ &= c_1 \left(\frac{3}{5}\right)^m - c_2 \left(\frac{3}{5}\right)^{2m} - c_3 \left(\frac{1}{5}\right)^{2m} \end{aligned} \tag{2.16}$$

and

$$G(F_1^m v_2, F_1^m v_3) = c_4 \left(\frac{3}{5}\right)^m - c_2 \left(\frac{3}{5}\right)^{2m} + c_3 \left(\frac{1}{5}\right)^{2m} \tag{2.17}$$

for $c_1 = \frac{51}{140}$, $c_2 = \frac{3}{10}$, $c_3 = \frac{9}{140}$ and $c_4 = \frac{33}{140}$.

Proof. In this case $r_1 = 3/5$,

$$\tilde{A}_1 = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \tilde{B}_1 = \begin{pmatrix} \frac{9}{50} & \frac{3}{50} \\ \frac{3}{50} & \frac{9}{50} \end{pmatrix}.$$

We easily compute

$$\tilde{A}_1^k = \frac{1}{2} 5^{-n} \begin{pmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{pmatrix}$$

hence

$$\tilde{A}_1^k \tilde{B}_1^t \tilde{A}_1^k = \frac{3}{50} 5^{-2k} \begin{pmatrix} 2 \cdot 3^{2k} + 1 & 2 \cdot 3^{2k} - 1 \\ 2 \cdot 3^{2k} - 1 & 2 \cdot 3^{2k} + 1 \end{pmatrix}.$$

Thus (2.13) yields

$$\frac{3}{50} \left(2 \sum_{k=1}^m \left(\frac{3}{5}\right)^{2m-k} \pm \sum_{k=1}^m \frac{3^k}{5^{2m-k}} \right) \tag{2.18}$$

for the left sides of (2.16) and (2.17), respectively. By evaluating these geometric progressions, we obtain the right sides of (2.16) and (2.17). \square

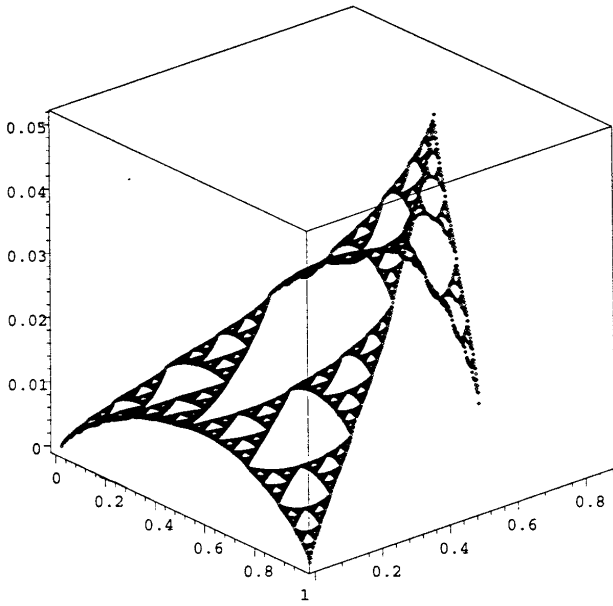
Remark 2.4. We can also obtain a version of Theorem 2.2 without the assumption that every vertex is a fixed point of a mapping. Under the p.c.f. assumption, the most general situation would be a boundary point, call it v_1 , such that there exist words w and w' and another boundary point, call it v_2 , with $v_1 = F_{w'}v_2$ and $v_2 = F_wv_2$. Then $F_{w'}F_w^m K$ gives a system of neighborhoods of v_1 , and in place of (2.14) we have

$$G|_{F_{w'}F_w^m K \times F_{w'}F_w^m K} = O(r_w^m). \tag{2.19}$$

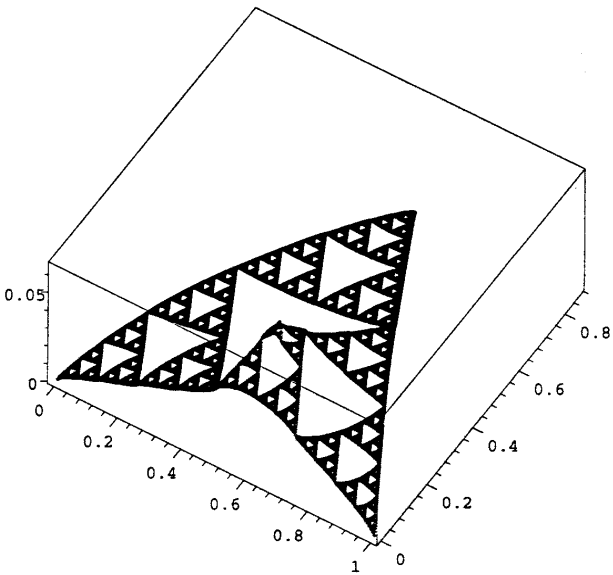
The proof is essentially the same, so we omit the details.

3. MAXIMUM VALUES (EXPERIMENTAL RESULTS)

We have calculated the Green's function for two examples using a computer implementation of (1.7). In this section we present some of this data in graphical form, and we address the problem: for which points x does $G(x, x)$ attain its maximum value? The first example is the standard harmonic structure on the Sierpinski gasket. In Fig. 3.1, we show the graph of $G(\cdot, y)$ for two different points y . The first is $y = F_1^2 v_2$, a point in V_2 , while the second is a point in V_8 ($y = F_3 F_2 F_1 F_2 F_3 F_1 F_2 F_3 v_1$) but not in V_7 . We know that $G(\cdot, y)$ is harmonic in the complement of y and assumes its maximum value at the point y . This is evident from the graphs. We notice a sharper peak in the first case. The second case



(a)



(b)

Fig. 3.1 The graph of $G(\cdot, y)$ on the Sierpinski gasket. (a) $y = F_1^2 v_1$ (in V_2) (b) $y = F_3 F_2 F_1 F_2 F_3 F_1 F_2 F_3 v_1$ (in V_8).

is close to the behavior for y equal to a nonvertex point, and we do expect a qualitative difference for nonvertex as opposed to vertex points.

Figure 3.2 shows the graph of $G(x, x)$ calculated for all points in V_9 [this requires the sum over words of length up to nine in (1.7)]. In Fig. 3.3, we show the first 4 partial sums of (1.7) over words of length $\leq k$. It is quite striking that the sharp cusps

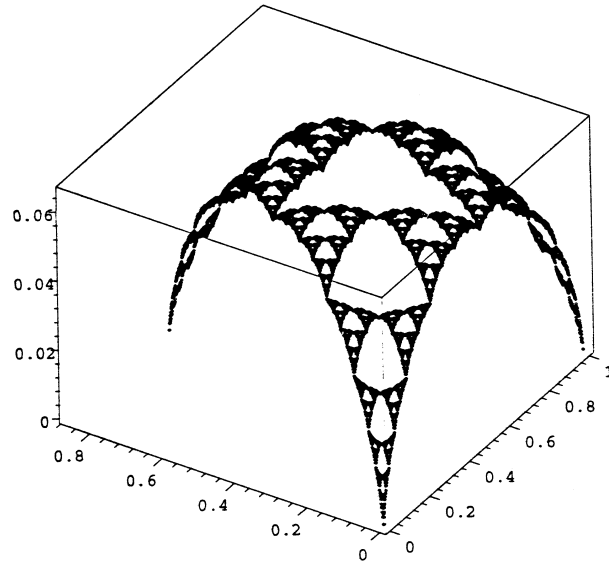


Fig. 3.2 The graph of $G(x, x)$ on the Sierpinski gasket (calculated at all points in V_9).

that appear in the partial sums become masked in the infinite sum (of course our “infinity” is only nine). Figure 3.4 shows the restriction of $G(x, x)$ to the line segment joining two boundary points. This suggests that all the upward pointing cusps have been smoothed out, but there are many downward pointing cusps. At present we have no explanation for this observed behavior.

To find the maximum of $G(x, x)$, we compute the locations of the maxima when x is restricted to V_m for values $m = 1, 2, \dots, 10$. The results show a striking pattern. For $m \geq 3$, there are exactly six maxima, obtained from any one of them by the dihedral-3 symmetry group. The location of each maximum follows a spiral pattern, as m varies, one being the sequence

$$\begin{aligned}
 &F_3 F_2 F_1 v_2, \quad F_3 F_2 F_1 F_2 v_3, \\
 &F_3 F_2 F_1 F_2 F_3 v_1, \quad F_3 F_2 F_1 F_2 F_3 F_1 v_2, \\
 &F_3 F_2 F_1 F_2 F_3 F_1 F_2 v_3, \quad F_3 F_2 F_1 F_2 F_3 F_1 F_2 F_3 v_1, \\
 &F_3 F_2 F_1 F_2 F_3 F_1 F_2 F_3 F_1 v_2, \\
 &F_3 F_2 F_1 F_2 F_3 F_1 F_2 F_3 F_1 F_2 v_3,
 \end{aligned}$$

the others being obtained by a permutation of the indices. Note that after the mappings $F_3 F_2$, the remaining indices follow the sequence 1, 2, 3 repeatedly. This leads to the following conjecture.

Conjecture 3.1. *The maximum value of $G(x, x)$ on the Sierpinski gasket is attained at the point*

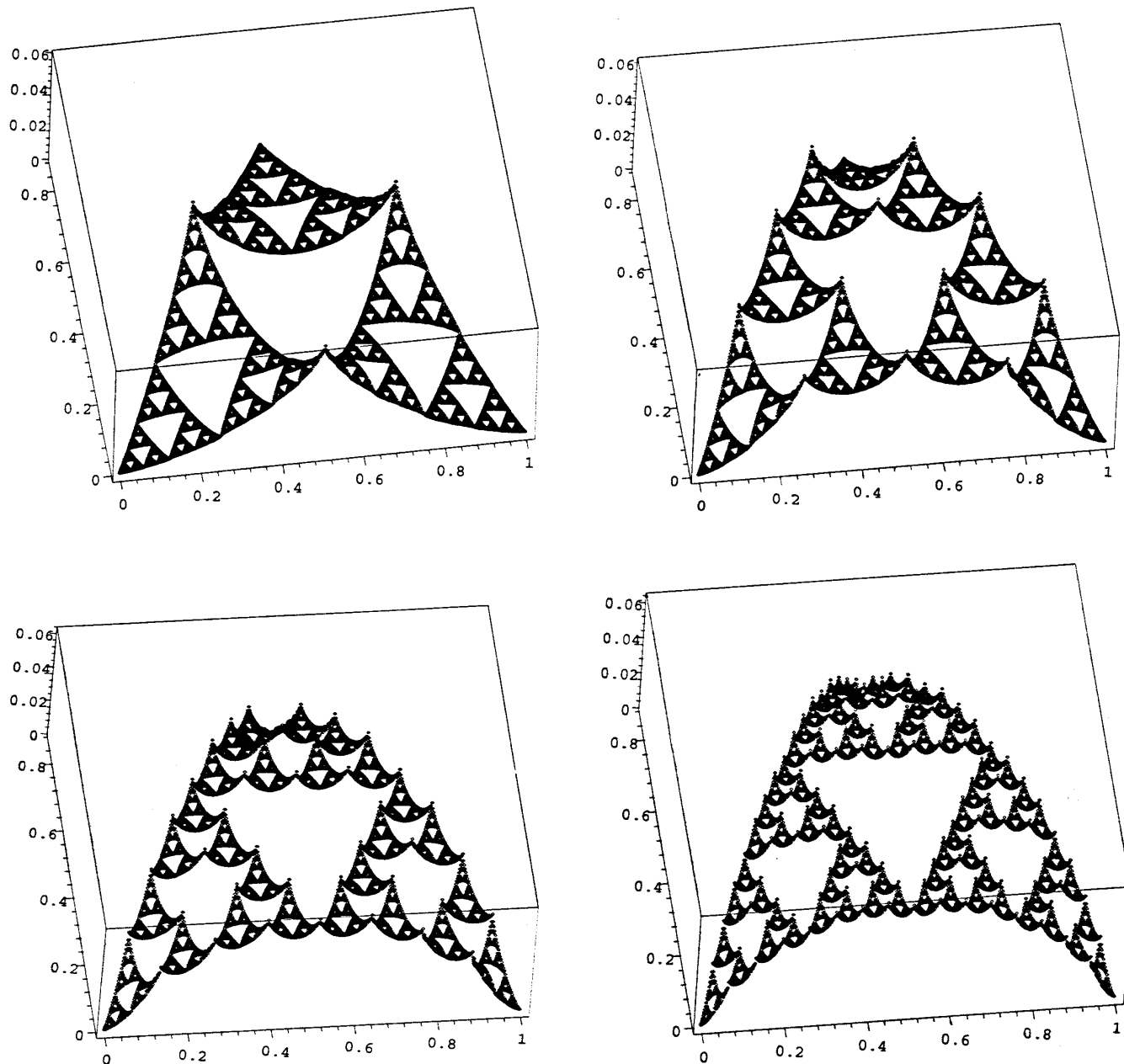


Fig. 3.3 The first four partial sums of the series (1.7) for the Sierpinski gasket.

$x = F_3 F_2 x_0$, where x_0 is the fixed point of the map $F_1 F_2 F_3$, at points symmetric to x , and at no other points.

In Table 3.1, we present the maximum values M_m of $G(x, x)$ over V_m , along with the differences $\Delta M_m = M_m - M_{m-1}$ and the ratio $\Delta M_m / \Delta M_{m-1}$, suggesting the possibility that this ratio approaches $3/5$ as $m \rightarrow \infty$.

We are able to compute exactly the value $G(x, x)$ at the point x described in Conjecture 3.1. Because of the eventual periodicity of the points $F_w^{-1}x$,

expression (1.7) is simply

$$\begin{aligned}
 G(x, x) &= \Psi(F_3 F_2 x_0, F_3 F_2 x_0) + \frac{3}{5} \Psi(F_2 x_0, F_2 x_0) \\
 &+ \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^{3n} \left(\left(\frac{3}{5}\right)^2 \Psi(x_0, x_0)\right) \\
 &+ \left(\frac{3}{5}\right)^3 \Psi(F_2 F_3 x_0, F_2 F_3 x_0) \\
 &+ \left(\frac{3}{5}\right)^4 \Psi(F_3 x_0, F_3 x_0) \tag{3.1}
 \end{aligned}$$

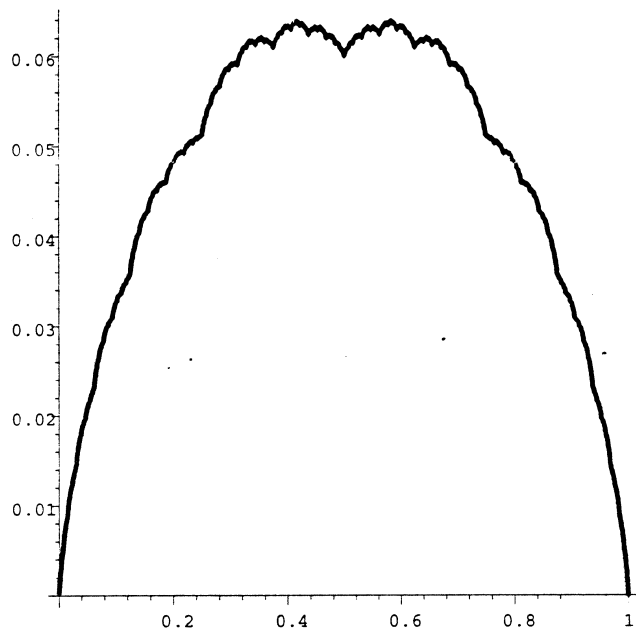


Fig. 3.4 The restriction of $G(x, x)$ to the line segment joining two boundary points of the Sierpinski gasket.

Table 3.1

m	M_m	ΔM_m	$\Delta M_m / \Delta M_{m-1}$
1	0.18	—	—
2	0.1848	0.0048	—
3	0.188448	0.003648	0.76
4	0.1927053	0.0042573	1.167023
5	0.1948998	0.002945	0.6917529
6	0.1916376	0.0012378	0.4203056
7	0.1969737	0.0008361	0.6754726
8	0.197445	0.0004713	0.5636885
9	0.1977288	0.0002838	0.6021642
10	0.1979022	0.0001734	0.6109936

and $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^{3n} = 125/98$. Also, (1.6) on the diagonal is

$$\begin{aligned} \Psi(y, y) &= \sum_{j=1}^3 \left(\frac{9}{50}\right) \psi_j(y)^2 \\ &\quad + \sum_{j=1}^3 \left(\frac{3}{25}\right) \psi_j(y) \psi_{j+1}(y) \end{aligned} \tag{3.2}$$

(interpreting $\psi_4 = \psi_1$). Thus we need to evaluate $\psi_j(y)$ for each of the five values of y that occur in (3.1). It turns out that we can do this exactly. For the point x_0 we note that $x_0 = \lim_{n \rightarrow \infty} (F_1 F_2 F_3)^n v_j$ for any j , so by repeated use of (1.9) $\lim_{n \rightarrow \infty} (A_3 A_2 A_1)^n h|_{V_0}$ will be the constant

vector with entries equal to $h(x_0)$, for any harmonic function h . But

$$A_3 A_2 A_1 = \begin{pmatrix} 68/125 & 32/125 & 1/5 \\ 62/125 & 37/125 & 26/125 \\ 13/25 & 6/25 & 6/25 \end{pmatrix}$$

which has right eigenvector $(1, 1, 1)$ and left eigenvector $(10/19, 5/19, 4/19)$ with eigenvalue 1, so by the Perron-Frobenius theorem we have

$$\lim_{n \rightarrow \infty} (A_3 A_2 A_1)^n = \frac{1}{19} \begin{pmatrix} 10 & 5 & 4 \\ 10 & 5 & 4 \\ 10 & 5 & 4 \end{pmatrix}.$$

This means $h(x_0) = (10h(v_1) + 5h(v_2) + 4h(v_3))/19$ for any harmonic function. A simple computation then shows

$$\begin{aligned} \Psi(x_0, x_0) &= \Psi(F_3 x_0, F_3 x_0) \\ &= \Psi(F_2 F_3 x_0, F_2 F_3 x_0) = 3/38 \end{aligned}$$

$$\Psi(F_2 x_0, F_2 x_0) = 642/9025$$

and

$$\Psi(F_3 F_2 x_0, F_3 F_2 x_0) = 38097/451250.$$

From (3.1,) we find

$$G(x, x) = \frac{178839}{902500} \approx 0.1981595\dots$$

The other example we consider is the pentagasket (Fig. 3.5), generated by five similarities with contraction ratio $\rho = \tau^{-2}$ (τ is the golden ratio $\frac{\sqrt{5}+1}{2}$), and fixed points the five vertices of a regular pentagon. The value of ρ is chosen so that the images $F_j K$ intersect at vertices of the corresponding pentagons. To get a Dirichlet form with the full dihedral-5 symmetry group, we must take

$$\begin{aligned} \mathcal{E}_0(u, u) &= \sum_{j=1}^5 p_1 (u(v_j) - u(v_{j+1}))^2 \\ &\quad + \sum_{j=1}^5 p_2 (u(v_j) - u(v_{j+2}))^2 \end{aligned}$$

(using mod 5 notation for indices), and all r_j are equal. In fact, it is known¹⁸ that the values

$$\begin{aligned} p_1 &= \frac{\sqrt{161} - 7}{16}, & p_2 &= \frac{15 - \sqrt{161}}{16}, \\ r &= \frac{\sqrt{161} - 9}{8} \end{aligned}$$

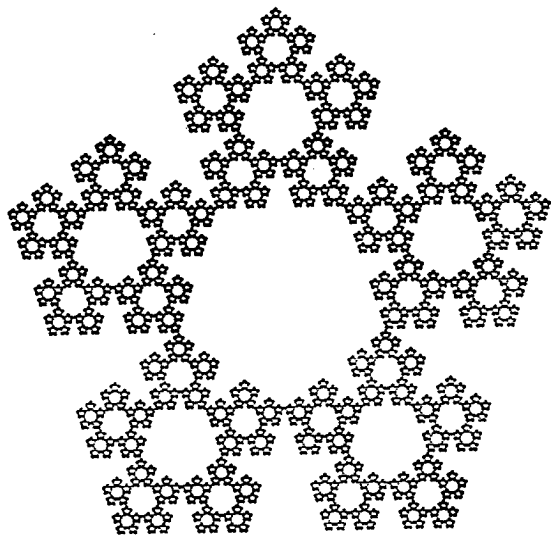


Fig. 3.5 The pentagasket.

Conjecture 3.2. For the pentagasket, the maximum value of $G(x, x)$ occurs at $x = F_{j+1}v_j$ or $F_{j-1}v_j$ and no other points.

To test this conjecture we did a spot check of points in V_m for $m \leq 18$, looking at all points within F_wV_0 for $|w| = m-1$ where F_wV_0 contains the maximum for V_{m-1} .

4. LOCAL GREEN'S FUNCTIONS

On the unit interval we can solve the equation $-u'' = f$ subject to the initial conditions $u(z) = u'(z) = 0$ at a given point z via the integral

$$u(x) = \int_z^x (y - x)f(y)dy.$$

Thus the function

$$G_z(x, y) = \begin{cases} y - x & \text{if } z < y < x \\ x - y & \text{if } x < y < z \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

serves as a local Green's function for this problem. We would like to have an analogous local Green's function for all harmonic structures. We can avoid defining the precise local vanishing conditions by considering the problem $-\Delta_\mu u = f$ where f vanishes in a neighborhood of z , and u is also required to vanish in a neighborhood of z . Of course we already have a solution to $-\Delta_\mu v = f$, namely

$$v(x) = \int G(x, y)f(y)d\mu(y)$$

so $u = v - h$ will solve our problem if h is a harmonic function and $h = v$ on a neighborhood of z . But v is already harmonic on the neighborhood of z where f vanishes, so the question is one of extendability of harmonic functions from neighborhoods of z . In general, this problem may only have a local solution, so the local Green's function is only locally defined.

Here we will examine the simplest situation, in which the local Green's function is in fact globally defined. Assume z is a boundary point that is a fixed point of a mapping F_i , and assume that the matrix A_i is invertible. This gives us a system of neighborhoods $F_i^m K$ of z with the property that harmonic functions on $F_i^m K$ are uniquely extendable to harmonic functions on K . Then the unique

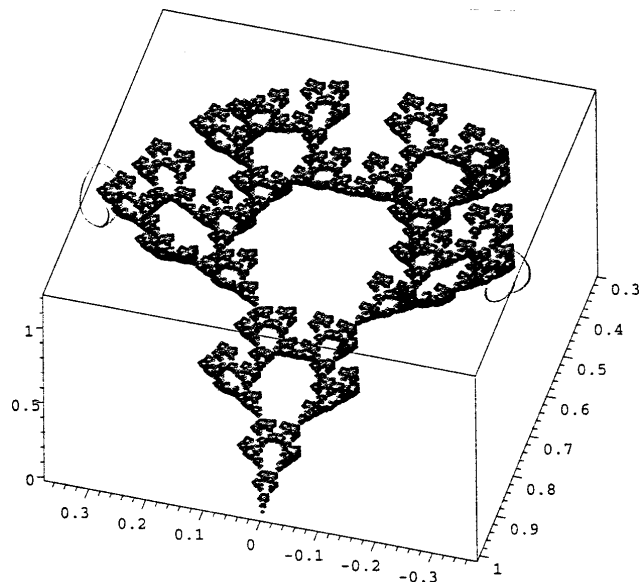


Fig. 3.6 A portion of the graph of $G(x, x)$ on one-fifth of the pentagasket (computed on V_6).

yield a unique harmonic structure with these symmetries.

We computed $G(x, x)$ using (1.7). Figure 3.6 shows a portion of the graph on V_6 restricted to F_1K . By symmetry the graph looks like the same on each F_jK . A search for the maximum value on V_m for $m \leq 6$ shows that it occurs at ten points in V_1 of the form $F_{j+1}v_j$ and $F_{j-1}v_j$ that lie along the outer edges of the pentagon.

solution to our problem can be represented as an integral

$$u(x) = \int G_z(x, y)f(y)d\mu(y) \tag{4.2}$$

against a local Green's function $G_z(x, y)$ that can be written

$$G_z(x, y) = G(x, y) + H_z(x, y) \tag{4.3}$$

where $H_z(x, y)$ is characterized by the condition that for $y \neq z$, $H_z(\cdot, y)$ is the harmonic function satisfying

$$H_z(x, y) = -G(x, y) \tag{4.4}$$

for x in a neighborhood of z (specifically for $x \in F_i^m K$ when m is chosen large enough that $y \notin F_i^m K$). In particular, $G_z(\cdot, y)$ vanishes in a neighborhood of z . Note that this neighborhood varies with y . $H_z(x, y)$ is not defined for $y = z$, but in the integral (4.2) it is assumed that f vanishes near z . Also H_z and G_z depend only on the harmonic structure, and not on the choice of measure.

In contrast to the tame behavior of G_z in (4.1), we will see that $G_z(x, y)$ can be unbounded as $y \rightarrow z$. We do this for the standard harmonic structure on the Sierpinski gasket, but it seems likely that this behavior is typical. For simplicity of notation we take $z = v_1$. The space of harmonic functions on $F_i^m K$ is 3-D, as these are just the restrictions to $F_1^m K$ of global harmonic functions. However, on the closure of complement of $F_1^m K$ there is a 4-D space of harmonic functions. The boundary of this set consists of the four points $v_2, v_3, F_1^m v_2, F_1^m v_3$, and a harmonic function is uniquely determined by specifying its values at these boundary points. It is easy to say exactly what this space of harmonic functions is: in addition to restrictions of the 3-D space of global harmonic functions, it is spanned by a harmonic function on $K \setminus \{v_1\}$ that has a pole at v_1 . This function, which was described in Ref. 15, is illustrated in Fig. 4.1. We denote it by h_4 .

In terms of these functions, we can give the explicit solution to (4.4). Take $m = 1$ for simplicity. For any y in the complement of $F_1 K$, we know that $H_z(\cdot, y)$ is harmonic in $F_1 K$, hence can be expressed as a linear combination of the standard basis $\{h_j(x)\}_{j=1,2,3}$ of global harmonic functions characterized by $h_j(v_k) = \delta_{jk}$. The coefficients

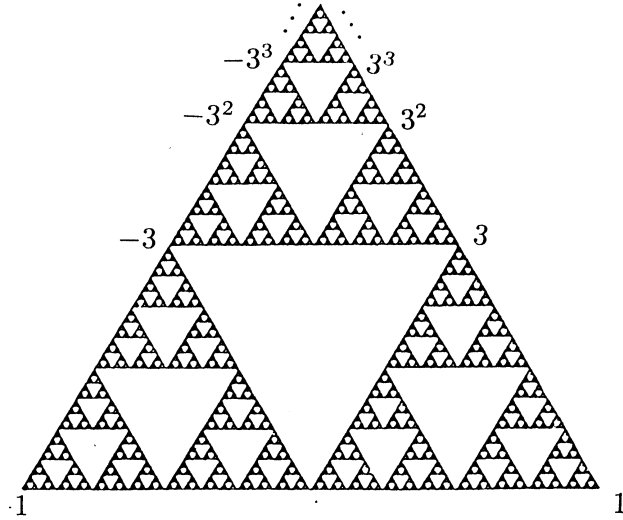


Fig. 4.1 The values of the harmonic function h_4 with a pole at v_1 .

depend on y , so

$$H_z(x, y) = \sum_{j=1}^3 a_j(y)h_j(x). \tag{4.5}$$

But from (4.4), we know $H_z(x, \cdot)$ is harmonic on the complement of $F_1 K$ for each fixed $x \in F_1 K$. This implies $a_j(y)$ is harmonic in the complement of $F_1 K$. Thus we must have

$$H_z(x, y) = \sum_{j=1}^3 \sum_{k=1}^4 a_{jk}h_j(x)h_k(y) \tag{4.6}$$

for certain coefficients a_{jk} . However, if we impose vanishing conditions and symmetry, we can reduce the number of coefficients from 12 down to 2. Indeed we know that $G(x, y)$ vanishes if $x = v_1$ or $y = v_2, v_3$, so the same is true for $H_z(x, y)$. Also $G(x, y)$ is preserved if x and y are both reflected in the line through v_1 and the midpoint between v_2 and v_3 . The vanishing requires that we use only $h_2(x)$ and $h_3(x)$, and $h_1(y)$ and $h_4(y) - h_2(y) + h_3(y)$. The reflection symmetry requires that

$$\begin{aligned} H_z(x, y) = & A(h_2(x) + h_3(x))h_1(y) \\ & + B(h_2(x) - h_3(x))(h_4(y) \\ & - h_2(y) + h_3(y)) \end{aligned} \tag{4.7}$$

for some constants A and B . The determination of the constants requires a computation. The unboundedness of G_z hinges on the constant B not vanishing.

Theorem 4.1. (4.7) holds for all $y \neq z$ for $A = -\frac{1}{2}$ and $B = -3/28$. In particular, H_z and G_z are unbounded.

Proof. If $y \neq z$ then y is in the complement of $F_1^m K$ for some m . The same argument as above shows that we must have (4.7) holding for $x \in F_1^m K$. But $H_z(\cdot, y)$ is a global harmonic function, so (4.7) holds for all x . It is clear that by consistency the constants must be independent of m . So we take $m = 1$ and use the values

$$G(q_j, q_k) = \begin{cases} \frac{9}{50} & j = k \\ \frac{3}{50} & j \neq k \end{cases} \quad (4.8)$$

where $q_1 = F_2 v_3 = F_3 v_2$, $q_2 = F_1 v_3 = F_3 v_1$, $q_3 = F_1 v_2 = F_2 v_1$ are the vertices in $V_1 \setminus V_0$ (in our notation q_j is opposite v_j). Note that we may take $x = q_2$ or q_3 but not q_1 since q_1 is not in $F_1 K$. Combining (4.4), (4.7) and (4.8) yields six equations, but by symmetry there are three identical pairs. Since

$$h_1(q_1) = h_2(q_2) = h_3(q_3) = 1/5$$

$$h_1(q_2) = h_1(q_3) = h_2(q_3) = h_3(q_2) = 2/5$$

$$h_4(q_1) = 0, h_4(q_2) = -3, h_4(q_3) = 3$$

we end up with the three equations

$$\frac{3}{25}A = -\frac{3}{50}, \quad \frac{6}{25}A + \frac{14}{25}B = -\frac{9}{50},$$

$$\frac{6}{25}A - \frac{14}{25}B = -\frac{3}{50}.$$

These equations are consistent and yield the desired values. \square

We have much more precise information about the singularity of H_z . We see that it is contained entirely in the odd part in y (with respect to the reflection symmetry). If we consider the points $y = F_1^m q_1$ along the symmetry line, then $G_z(x, y)$ will remain uniformly bounded. Taking $y = F_1^m q_2$, $G_z(x, y)$ will grow at the rate 3^m for any x not on the symmetry line. This is illustrated in Figs. 4.2 and 4.3.

It is easy to see that

$$\int_{F_1^{m+1}K \setminus F_1^m K} |h_4(y)| d\mu(y)$$

is constant, since h_4 is multiplied by three and μ is divided by three as we change m . Thus $G_z(x, \cdot)$

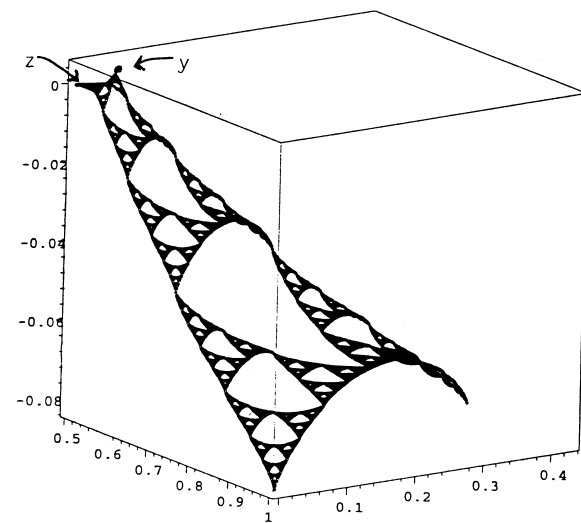
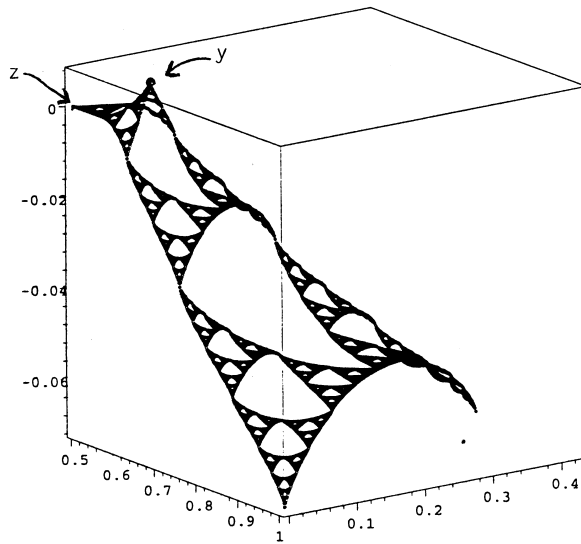
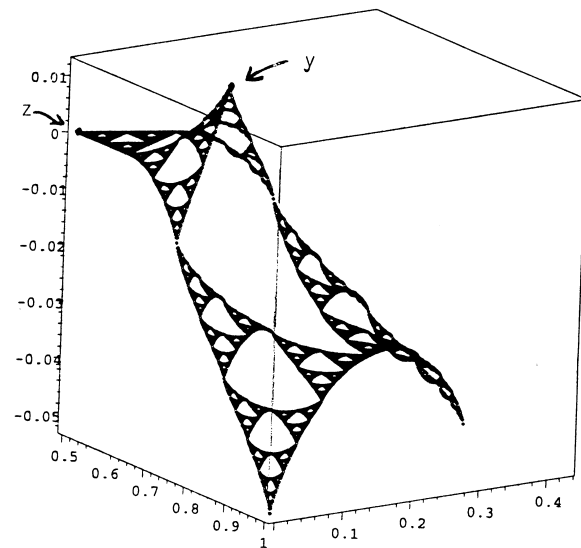


Fig. 4.2 The graphs of $G_z(\cdot, y)$ for $z = v_1$ and $y = F_1^m q_1$ for $m = 1, 2, 3$.

is not integrable. However, if we assume that f is Hölder continuous of any order, then the integral (4.2) exists as a singular integral

$$u(x) = \lim_{k \rightarrow \infty} \int_{K \setminus F_1^k K} G_z(x, y) f(y) d\mu(y) \quad (4.9)$$

without requiring f to vanish in a neighborhood of z . We can then interpret the behavior of u in a neighborhood of z as vanishing to first order as defined in Ref. 16. This means $u(z) = 0$, the normal derivative vanishes at z ,

$$\lim_{k \rightarrow \infty} \left(\frac{5}{3}\right)^k (2u(v_1) - u(F_1^k v_2) - u(F_1^k v_3)) = 0 \quad (4.10)$$

and the transverse derivative vanishes at z

$$\lim_{k \rightarrow \infty} 5^k (u(F_1^k v_2) - u(F_1^k v_3)) = 0. \quad (4.11)$$

Theorem 4.2. *Let f satisfy a Hölder condition of any order on the Sierpinski gasket with the standard harmonic structure and normalized Hausdorff measure μ . Then (4.9) defines a continuous solution of $-\Delta_\mu u = f$ with first order vanishing at z .*

Proof. Write

$$h_m(x) = \int_{K \setminus F_1^m K} H_z(x, y) f(y) d\mu(y). \quad (4.12)$$

Then (4.9) is just

$$u = v + \lim_{m \rightarrow \infty} h_m. \quad (4.13)$$

We claim h_m converges uniformly. To see this we may assume that f is odd with respect to the reflection symmetry, since the even part of $H_z(x, \cdot)$ is uniformly bounded. Then

$$\begin{aligned} h_{m+1}(x) - h_m(x) &= \frac{1}{2} \int_{F_1^m K \setminus F_1^{m+1} K} H_z(x, y) (f(y) \\ &\quad - f(\tilde{y})) d\mu(y) \end{aligned}$$

where \tilde{y} is the reflection of y . Using the Hölder estimate

$$|f(y) - f(\tilde{y})| \leq c\beta^m \quad \text{for } y \in F_1^m K \quad (4.14)$$

for some $\beta < 1$, we obtain

$$\|h_{m+1} - h_m\|_\infty = O(\beta^m)$$

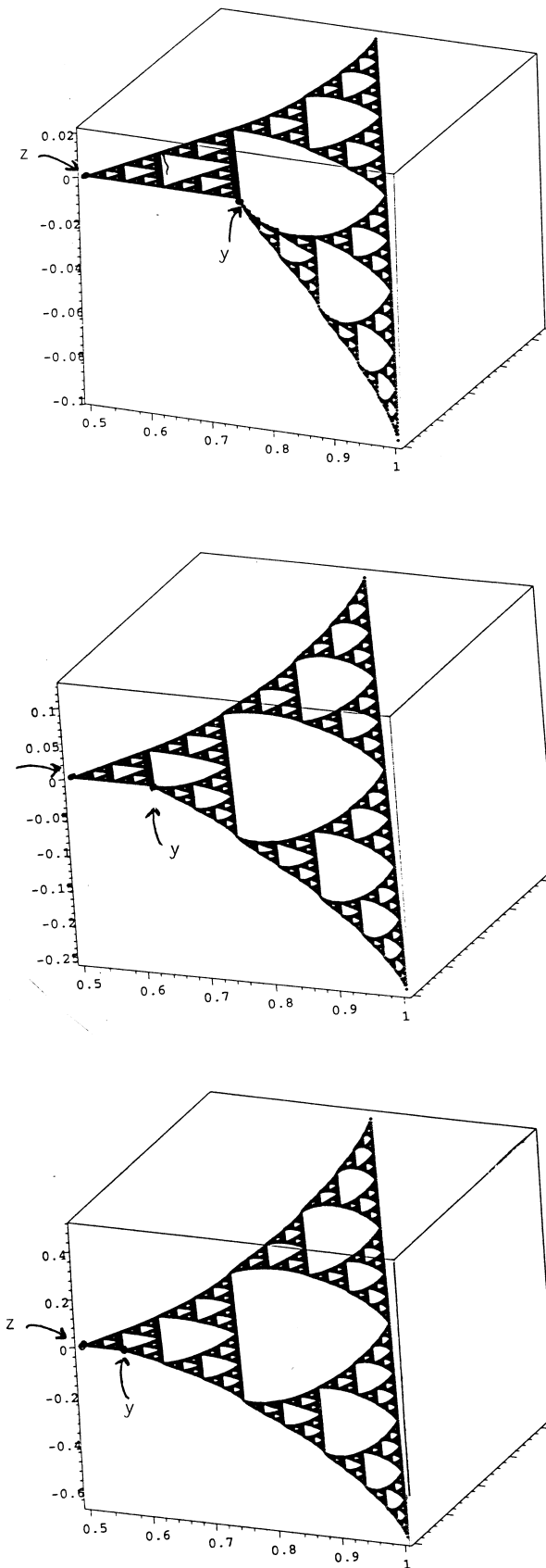


Fig. 4.3 The graphs of $G_z(\cdot, y)$ for $z = v_1$ and $y = F_1^m q_2$ for $m = 0, 1, 2$.

since the measure of $F_1^m K$ contributes a factor of $(1/3)^m$ to cancel the 3^m estimate for $H_z(x, \cdot)$ on $F_1^m K \setminus F_1^{m+1} K$. Thus u is continuous, and moreover $-\Delta_\mu u = f$ since $-\Delta_\mu v = f$ and the uniform limit of harmonic functions is harmonic.

We now prove the vanishing to first order conditions. We get $u(z) = 0$ easily since $v(z) = 0$ and $h_m(z) = 0$ for each m . It remains to show the more precise statements (4.10) and (4.11) (note that $u(v_1) = 0$ in (4.10), so that together (4.10) and (4.11) describe a rate of vanishing as we approach z). In fact, we will show

$$u(x) = o((3/5)^m) \quad \text{on } F_1^m K \tag{4.15}$$

and

$$u(x) - u(\tilde{x}) = o(5^{-m}) \quad \text{on } F_1^m K \tag{4.16}$$

which imply (4.10) and (4.11).

For $x \in F_1^m K$, the integral in (4.9) can be restricted to $F_1^m K$ by (4.4). We write

$$u(x) = \sum_{k=0}^{\infty} u_k(x) \tag{4.17}$$

with

$$u_k(x) = \int_{F_1^{m+k} K \setminus F_1^{m+k+1} K} G_z(x, y) f(y) d\mu(y). \tag{4.18}$$

We study the contribution to (4.17) from the three terms that make up $G_z(x, y)$. First, for $G(x, y)$, we use the estimate $O(3/5)^m$ from Theorem 2.2 and the fact that $\mu(F_1^m K) = 1/3^m$ to estimate

$$\left| \int_{F_1^m K} G(x, y) f(y) d\mu(y) \right| \leq c5^{-m}.$$

Since G is symmetric, $G(\tilde{x}, y) = G(x, \tilde{y})$ so

$$\begin{aligned} & \int_{F_1^m K} (G(x, y) - G(\tilde{x}, y)) f(y) d\mu(y) \\ &= \int_{F_1^m K} G(x, y) (f(y) - f(\tilde{y})) d\mu(y) \end{aligned}$$

and so we can improve the estimate to $o(5^{-m})$.

The estimates for the contribution due to $(h_2(x) + h_3(x))h_1(y)$ is similar, except that this function is even in x alone and so contributes zero to the odd part estimate.

It remains to estimate

$$\begin{aligned} & \int_{F_1^{m+k} F \setminus F_1^{m+k+1} K} (h_2(x) - h_3(x))(h_4(y) \\ & - h_2(y) + h_3(y)) f(y) d\mu(y). \end{aligned}$$

Here we may replace f by its odd part and use the Hölder estimate to pick up a factor of β^{m+k} , because $h_4(y) - h_2(y) + h_3(y)$ is odd. We have an estimate of 3^{m+k} for $|h_4(y) - h_2(y) + h_3(y)|$, an estimate of 3^{-m-k} for the measure, and an estimate of 5^{-m-k} for $|h_2(x) - h_3(x)|$. Combining all these estimates yields $O((\beta/5)^{m+k})$ and summing over k in (4.17) yields $o(5^{-m})$. \square

Remark. The proof actually yields slightly stronger estimates, with $O(5^{-m})$ in (4.15) and $O((\beta/5)^m)$ in (4.16), where β is the Hölder constant in (4.14). Also, we only needed the Hölder continuity in the form (4.14).

We can also study local Green's functions G_z when z is any vertex point. However, these functions will only be locally defined (the vanishing to order one at z includes conditions analogous to (4.10) and (4.11) on both sides of z). See Ref. 16 for details.

We indicate briefly the analogous results for two other examples. Both are examples with $\#V_0 = 3$ and dihedral-3 symmetry for the harmonic structure and the self-similar measure. Under these assumptions the matrix A_1 must have the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - a - b & a & b \\ 1 - a - b & b & a \end{pmatrix}$$

for some constants satisfying $a > 0, b > 0, a + b < 1$ (the matrices A_2 and A_3 are just permutations of A_1), with eigenvalues $1, a + b = r$, and $a - b$. As long as $a \neq b$, these matrices are invertible. In that case, just as for the Sierpinski gasket, we have the basis h_1, h_2, h_3 for global harmonic functions, and their restrictions give a basis for harmonic functions on $F_1^m K$. The space of harmonic functions on the closure of the complement of $F_1^m K$ is 4-D, and the problem is to find explicitly the fourth basis element h_4 , that will extend to $K \setminus v_1$ with a pole at v_1 . It is easy to see that this function must be odd, since 1 and $h_2 + h_3$ give a basis for the even functions, and $h_2 - h_3$ gives a single odd function. The same reasoning as before gives (4.7), and assuming that the coefficient B is nonzero, the growth rate of h_4 governs the unboundedness of G_z .

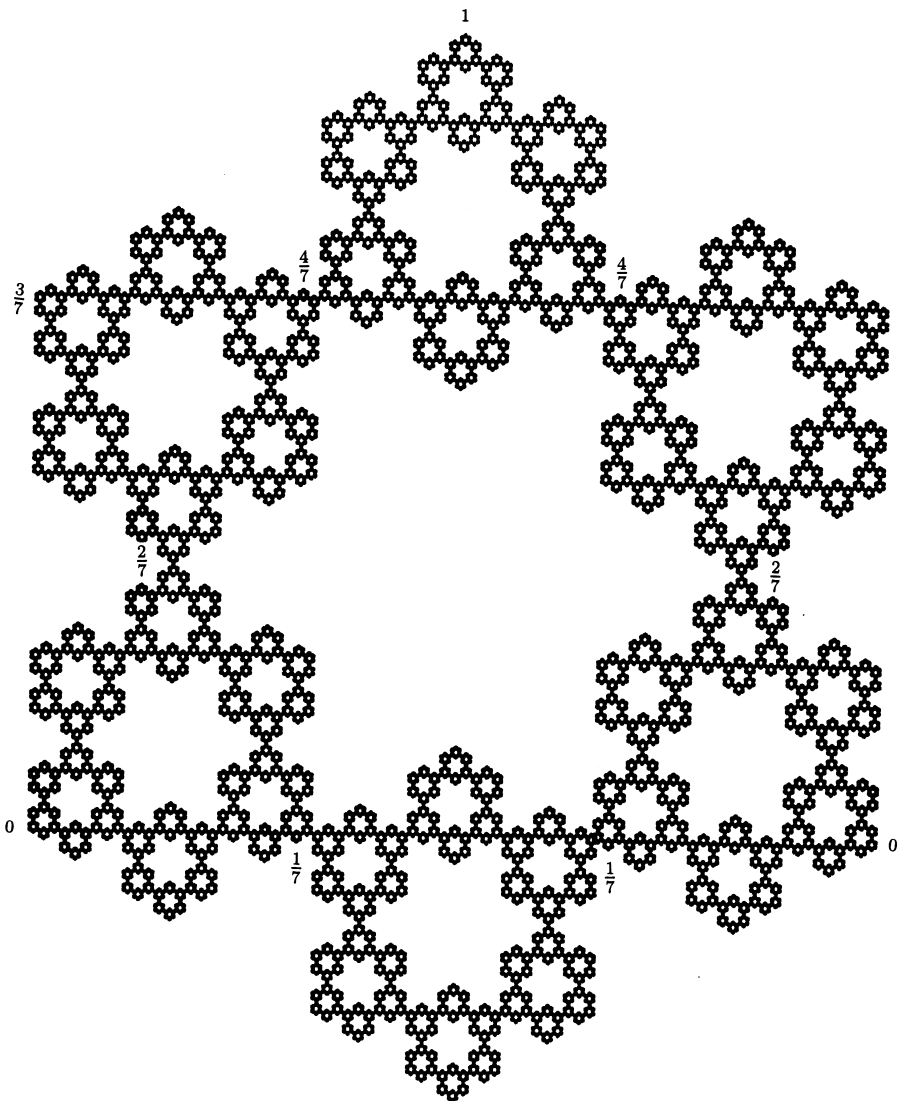


Fig. 4.4 The values of the harmonic function h_1 on the hexagasket.

Example 4.3. The hexagasket, or fractal Star of David, is shown in Fig. 4.4 with the values of h_1 . Here $a = 2/7$, $b = 1/7$ and so $r = 3/7$. Since there are six contractions, the measure is reduced by a factor of $1/6$ in each contraction. In Fig. 4.5, we show the values of h_4 . The expansion factor is again three. In this case h_4 is integrable, so that (4.2) makes sense for any continuous function f . The analogs of (4.15) and (4.16) that we need are $u(x) = o((3/7)^m)$ on $F_1^m K$, and $u(x) - u(\tilde{x}) = o((1/7)^m)$ on $F_1^m K$, and these follow easily, actually improving to $O((1/14)^m)$.

Example 4.4. The level three Sierpinski gasket is shown in Fig. 4.6 with the values of h_1 . Here

$a = 4/15$, and $b = 3/15$ so $r = 7/15$. There are again six contractions so the measure is reduced by $1/6$ in each contraction. In Fig. 4.7, we show the values of h_4 . The expansion factor is seven, so in this case h_4 is not integrable, and in fact we need the Hölder estimate (4.14) to hold for $\beta < 6/7$ in order to make (4.9) converge. The analogs of (4.15) and (4.16) that we need are $u(x) = o((7/15)^m)$ on $F_1^m K$ and $u(x) - u(\tilde{x}) = o((1/15)^m)$ on $F_1^m K$. These again follow by the same reasoning provided $\beta < 6/7$, and we in fact get the improvements $O((7/90)^m)$ for the first and $O((7\beta/90)^m)$ for the second. The requirement that f satisfy such a Hölder estimate is fairly reasonable: it is shown in

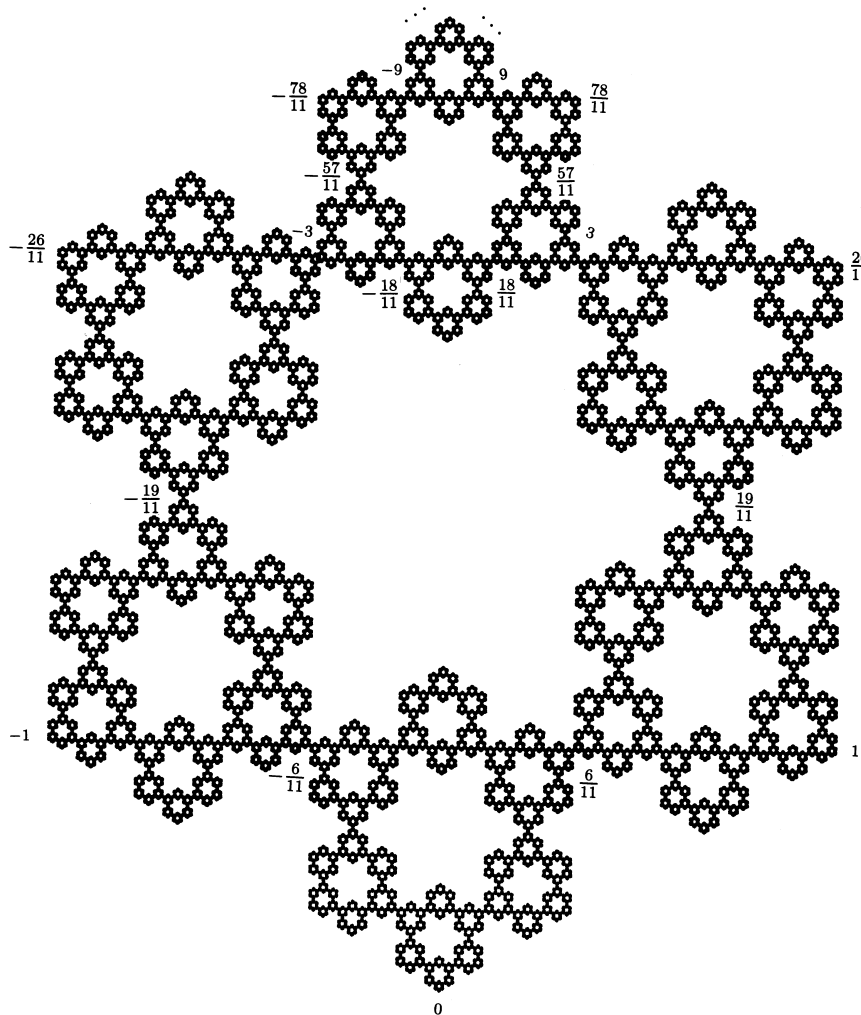


Fig. 4.5 The values of the harmonic function h_4 on the hexagasket.

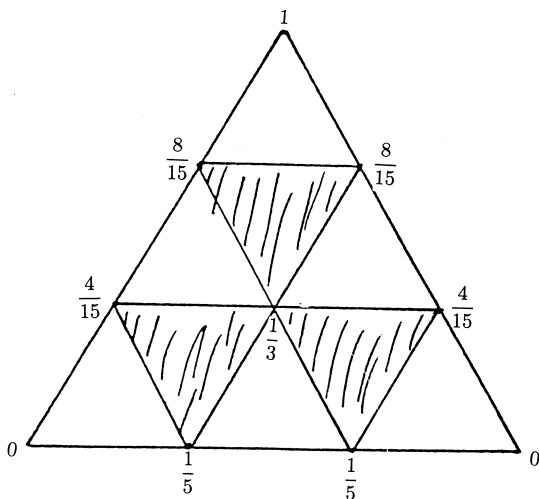


Fig. 4.6 The values of the harmonic function h_1 on the level three Sierpinski gasket.

Ref. 16 that if $f \in \text{dom}(\Delta_\mu)$, then it will satisfy such an estimate for $\beta = 7/15$.

In all three cases we have looked at, the growth factor for h_4 was equal to $(a + b)/(a - b)$. In each case this was the result of a lengthy and unenlightening computation (we did not present the details), but it seems unlikely that this is just a coincidence.

Conjecture 4.5. For a harmonic structure with dihedral-3 symmetry and $\#V_0 = 3$, there exists a nontrivial odd harmonic function h_4 on $K \setminus v_1$ with a pole at v_1 satisfying

$$h_4(F_1x) = \left(\frac{a+b}{a-b}\right)h_4(x).$$

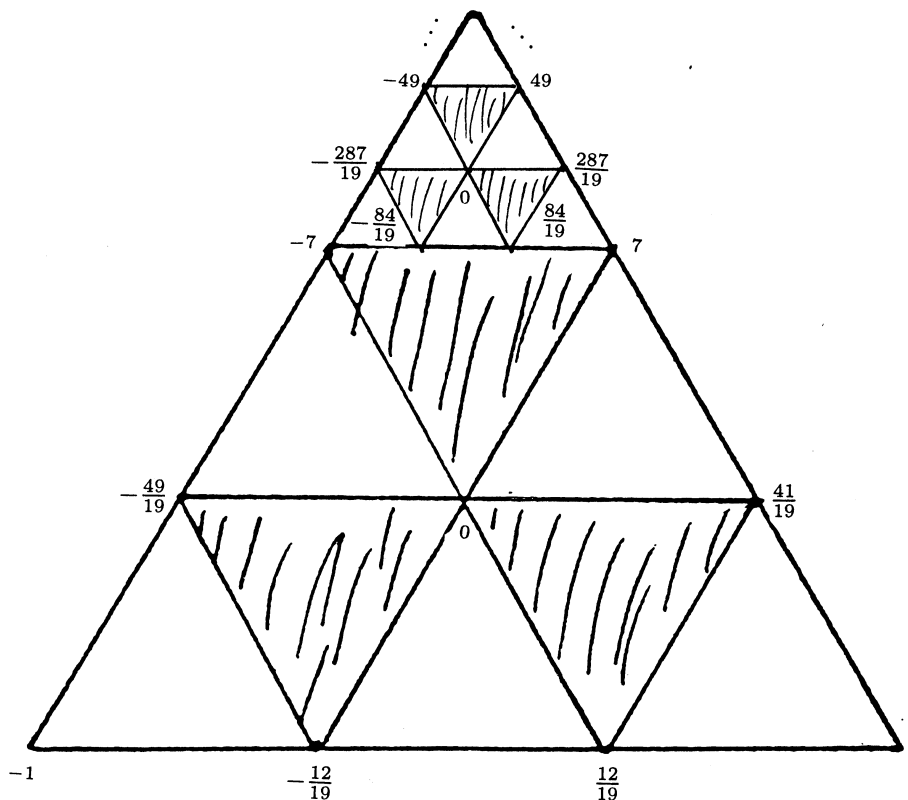


Fig. 4.7 The values of the harmonic function h_4 on the level three Sierpinski gasket.

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