

ORTHOGONAL POLYNOMIALS ON A FAMILY OF CANTOR SETS AND THE PROBLEM OF ITERATIONS OF QUADRATIC MAPPINGS

D. BESSIS, M.L. MEHTA and P. MOUSSA

Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

ABSTRACT. We first study a family of invariant transformations for the integer moment problem. The fixed point of these transformations generates a positive measure with support on a Cantor set depending on a parameter q . We analyze the structure and properties of the set of orthogonal polynomials with respect to this measure. Among these polynomials, we find the iterates of the canonical quadratic mapping: $F(x) = (x - q)^2$, $q \geq 2$. It appears that the measure is invariant with respect to this mapping. Algebraic relations among these polynomials are shown to be analytically continuable below $q = 2$, where bifurcation doubling among stable cycles occurs. As the simplest possible consequence we analyze the neighborhood of $q = 2$ (transition region) for $q < 2$.

1. INTRODUCTION

In a previous paper [1] devoted to the Ferromagnetic Ising Model, we introduced the integer moment problem characterized by positive measure $d\mu(x)$ with support $[0, \Lambda]$ and moments

$$n_k = \int_0^\Lambda x^k d\mu(x), \quad k = 0, 1, 2, \dots, \quad (1.1)$$

where the n_k are positive integers. Equation (1.1) is also characterized by the generating functions of the n_k 's

$$G_\Lambda(z) = \sum_{k=0}^{\infty} n_k z^k = \int_0^\Lambda \frac{d\mu(x)}{1-xz}. \quad (1.2)$$

For Λ increasing from 0 to 4, there exist bifurcations for each value of Λ of the form

$$\Lambda_m = 4 \cos^2 \frac{\pi}{m}, \quad m = 2, 3, 4, \dots \quad (1.3)$$

and the corresponding measures are built up with a finite number of Dirac measures, whose support are points of the form [1]

$$x_m^{(p)} = 4 \cos^2 \frac{p\pi}{m}, \quad p = 1, 2, \dots (m-1). \quad (1.4)$$

For $\Lambda = 4$, and $n_0 = 1$, $n_k = \text{even integer}$, the general solution of the problem is either given by the generating function

$$G_4(z) = [1 - 4z]^{-1/2} \quad (1.5)$$

or by its truncated expressions in the continued fraction expansion of (1.5).

This result is interesting, because it allows one to reconstruct the exact solution of the one-dimensional Ising model.

The case $\Lambda > 4$, which contains the physically interesting Ising models of higher dimensions, has not yet been classified as the previous case $\Lambda \leq 4$, and displays new unexpected features.

However, the support $[0, \Lambda]$ can be shortened for $\Lambda > 4$, by an algebraic transformation which transforms positive measures into positive measures and integers into integers. When iterating this transformation which physically looks like a decimation, the support of the measure breaks down into disjoint intervals of smaller and smaller size.

We look at the fixed point of these transformations when the support ultimately reduces to a Cantor set C_q included in the interval $[q - \xi, q + \xi]$ with $\xi = \frac{1}{2} + \sqrt{q + \frac{1}{4}}$. The positive measure associated to it is a purely singular measure which generates, in the usual way [2], a family of orthogonal polynomials associated to it.

The structure of these polynomials, and the structure of the Cantor set (which depend on a real parameter q) are analyzed. These polynomials satisfy a three-term recursion relation [2]

$$P_{n+1}(x) = (x - q)P_n(x) - R_n(q)P_{n-1}(x), \quad (1.6)$$

where the $R_n(q)$ are rational fractions in q , generated by

$$\begin{aligned} R_{2n}(q) + R_{2n+1}(q) &= q, & R_1(q) &= q \\ R_{2n-1}(q)R_{2n}(q) &= R_n(q), & R_2(q) &= 1. \end{aligned} \quad (1.7)$$

We study in some detail for $q > 2$ the set $\{R_n\}$ which is shown to generate a new perfect set \tilde{C}_q in correspondence with C_q . The measure with respect to which these polynomials are orthogonal is shown to satisfy

$$\begin{aligned} \mu(x) &= \mu(q + \sqrt{x}) - \mu(q - \sqrt{x}) \\ &= \mu[F_{(1)}^{-1}(x)] - \mu[F_{(2)}^{-1}(x)] \end{aligned} \quad (1.8)$$

where $F_i^{-1}(x)$ are the two inverse transformations of the quadratic mapping

$$F(x) = (x - q)^2. \quad (1.9)$$

We then consider the iterates of order n of the canonical quadratic transformation [3] (1.9). Its n th iterate is a polynomial of degree 2^n , which satisfies

$$F_n(x) = [F_{n-1}(x) - q]^2. \quad (1.10)$$

We show that

$$F_n(x) \equiv P_{2^n}(x) + q. \quad (1.11)$$

For $q > 2$, up to the additive constant q , the $F_n(x)$ form a subset of orthogonal polynomials with respect to the measure $d\mu(x)$ defined on the Cantor set C_q . Because $\mu(x)$ satisfies (1.8), which is the Perron–Frobenius [4] relation, it is an invariant measure defined on C_q which appears to be the complementary set of the domain of attraction of the point at infinity. For any q , (1.6) defines polynomials, the degree of which interpolates the degree of the $F_n(x)$. We observe that the $R_n(q)$ are rational fractions in q , which allows analytic continuation toward the region $0 < q < 2$ where bifurcation doubling among stable cycles occur. As an example, we study the distribution of the superstable values $q_s < 2$ in the vicinity of $q = 2$.

2. INVARIANT TRANSFORMATIONS

We consider a family of orthogonal polynomials $P_n(x)$, with respect to a positive measure $d\mu(x)$ of support $[a, b]$.

$$\int_a^b P_m(x)P_n(x) d\mu(x) = h_n \delta_{nm} \quad (2.1)$$

with the normalization

$$P_m(x) = x^m + \dots. \quad (2.2)$$

To the measure $d\mu(x)$ we associate the moments

$$\mu_k = \int_a^b x^k d\mu(x) \quad (2.3)$$

and the generating function of the moments

$$G(z) = \sum_{k=0}^{\infty} \mu_k z^k = \int_a^b \frac{d\mu(x)}{1-xz}. \quad (2.4)$$

Performing a translation q on the measure, we define

$$d\mu_q(x) = d\mu(x - q) \quad \text{with supporting } [a + q, b + q]. \quad (2.5)$$

Then one finds that

$$G_q(z) = \frac{1}{1 - qz} G\left(\frac{z}{1 - qz}\right) \quad (2.6)$$

and the orthogonal polynomials $P_m^q(x)$ associated to $d\mu_q(x)$ are simply given by

$$P_m^q(x) = P_m(x - q); \quad h_m^q = h_m. \quad (2.7)$$

We consider now a slightly more intricate transformation, defined on the generating function $G(z)$ for convenience (we assume $0 \leq a < b$)

$$G_\square(z) = G(z^2) = \int_a^b \frac{d\mu(x)}{1 - xz^2} = \sum_{n=0}^{\infty} \mu_n z^{2n}, \quad (2.8)$$

to which corresponds a positive measure $d\mu_\square(x)$

$$G_\square(z) = \int_E \frac{d\mu_\square(x)}{1 - xz}, \quad (2.9)$$

where $E = [-\sqrt{b}, \sqrt{a}] \cup [\sqrt{a}, \sqrt{b}]$

$$\mu_\square(x) = \frac{1}{2} \epsilon(x) \mu(x^2), \quad \mu(0) = 0 \quad (2.10)$$

and

$$\epsilon(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0. \end{cases} \quad (2.11)$$

The support of $d\mu_\square(x)$ being the set $E = [-\sqrt{b}, -\sqrt{a}] \cup [\sqrt{a}, \sqrt{b}]$ when the support of $d\mu(x)$ was $[a, b]$. The corresponding orthogonal polynomials are defined by

$$P_{2m}^\square(x) = P_m(x^2), \quad h_{2m}^\square = h_m \quad (2.12)$$

for the even polynomials, and for the odd ones

$$P_{2m+1}^\square(x) = x P_m^{(1)}(x^2), \quad h_{2m+1}^\square = h_m^{(1)} \quad (2.13)$$

where we have introduced the orthogonal polynomials $P_m^{(1)}(x)$ associated with the measure $x d\mu(x)$, to which corresponds the generating function

$$G^{(1)}(z) = \int_a^b \frac{x d\mu(x)}{1-xz} = \int_a^b \frac{d\mu^{(1)}(x)}{1-xz} = \frac{G(z) - G(0)}{z}. \quad (2.14)$$

Clearly due to the symmetry of $\mu_{\square}(x)$ with respect to $x = 0$, the $P_{2m}^{\square}(x)$ are even, as shown by (2.12), while the $P_{2n+1}^{\square}(x)$ are odd. We shall now combine the two operations (q) and \square , to build the operation $(q) * \square$.

In this way we get

$$G_{(q)*\square}(z) = \frac{1}{1-qz} G\left(\left(\frac{z}{1-qz}\right)^2\right), \quad (2.15)$$

$$\mu_{(q)*\square}(x) = \frac{1}{2} \epsilon(x-q) \mu[(x-q)^2], \quad (2.16)$$

$$P_{2m}^{(q)*\square}(x) = P_m[(x-q)^2], \quad (2.17)$$

$$P_{2m+1}^{(q)*\square}(x) = (x-q)P_m^{(1)}[(x-q)^2]. \quad (2.18)$$

The interest of the transformation $(q) * \square$ lies in the fact that the integer moment problem (now $a = 0, b = q^2$)

$$n_k = \int_0^{q^2} x^k d\mu(x) \quad (2.19)$$

with support of $[0, q^2]$ ($q > 2$) is transformed, as can be checked, into a new integer moment problem

$$n_k^{(q)*\square} = \int_0^{2q} x^k d\mu_{(q)*\square}(x) \quad (2.20)$$

where the $n_k^{(q)*\square}$ are again positive integers, provided q is an integer, which can always be satisfied by enlarging the support of the initial measure. However, for $q > 2$, $2q < q^2$ and the support is reduced, the smallest q being 3 to which any integer moment problem can be reduced. Here we want to analyze the fixed point of such transformations, not requiring for the time being that q be an integer.

3. SUPPORT PROPERTIES OF THE MEASURE AND INVARIANCE

The fixed point of the transformation (2.15) satisfies

$$G(z) = \frac{1}{1 - qz} G\left(\left(\frac{z}{1 - qz}\right)^2\right). \quad (3.1)$$

It is easy to see that Equation (3.1) uniquely determines (up to a multiplicative constant) the formal series expansion of $G(z)$. Therefore if the solution is analytic, it is also unique (up to a multiplicative constant). For $q = 2$, the solution analytic near the origin is

$$G_{q=2}(z) = \frac{1}{\sqrt{1 - 4z}} \quad (3.2)$$

apart from a multiplicative constant, and

$$G_{q=2}(z) = \frac{1}{\pi} \int_0^4 \frac{dx}{(1 - xz)\sqrt{x(4-x)}}$$

the support is the full interval $[0, 4]$ and the measure is absolutely continuous. For $q > 2$, the measure will become purely singular with no absolutely continuous part, and no Dirac measure in it, and is defined on a Cantor set, as we shall see.

Starting with a measure which fills up the interval $[0, q^2]$, after performing $(q) * \square$, we get a measure which fills up $[0, 2q]$, therefore we see that

$$\mu(x) = \mu(2q), \quad x > 2q. \quad (3.3)$$

Performing again $(q) * \square$ we get a measure which fills up the interval $[q - \sqrt{2q}, q + \sqrt{2q}]$, performing again $(q) * \square$, we get a measure which is non-zero on $[q - \sqrt{q + \sqrt{2q}}, q - \sqrt{q - \sqrt{2q}}] \cup [q + \sqrt{q - \sqrt{2q}}, q + \sqrt{q + \sqrt{2q}}]$, which shows that in particular $\mu(x)$ is continuous at $x = q$. After n iterations, the support consist of 2^{n-1} disjoint intervals $[a(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, \epsilon_n), a(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, -\epsilon_n)]$ where we have set

$$a(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = q + \epsilon_1 \sqrt{q + \epsilon_2 \sqrt{q + \dots + \epsilon_{n-1} \sqrt{q + \epsilon_n \sqrt{2q}}} \quad (3.4)$$

and $\epsilon_j = \pm 1$. For $q > 2$, all these numbers are real and positive. It has been shown [5] that the limiting support of the measure is a closed perfect compact set of measure zero: a Cantor set C_q , which for $q > 2$, is the set

$$\{C_q\} = \{q + \epsilon_1 \sqrt{q + \epsilon_2 \sqrt{q + \dots}}\}. \quad (3.5)$$

We now sketch the proof of the existence of the limiting measure corresponding to the fixed point

and its construction.

Considering the transformation

$$G_1(z) = \frac{1}{1 - qz} G_0 \left(\frac{z^2}{(1 - qz)^2} \right). \quad (3.6)$$

We define the polynomials

$$T_1(z) = 1 - qz \quad (3.7)$$

$$T_{n+1}(z) = (1 - qz)^{2^n} T_n \left(\frac{z^2}{(1 - qz)^2} \right), \quad n \geq 1$$

where $T_n(z)$ are polynomials of degree 2^{n-1} in z . Then the n th iterate of the transformation (3.6) is

$$G_n(z) = \frac{T_1(z)T_2(z) \dots T_{n-1}(z)}{T_n(z)} G_0 \left\{ \frac{z^{2^n}}{T_n^2(z)} \right\}. \quad (3.8)$$

Noting that the degree 2^{n-1} of the polynomial $T_n(z)$ is just one more than that of the product $\prod_{j=0}^{n-1} T_j(z)$, and that the power series expansion of $G_0(z^{2^n}/T_n^2(z))$ starts with z^{2^n} , we see that

$$\mathfrak{R}_n(z) = \prod_{j=0}^{n-1} T_j(z)/T_n(z) = [2^{n-1} - 1/2^{n-1}](z) \quad (3.9)$$

which means that this rational fraction is the Padé Approximation [10] of $G_n(z)$ of order $[N - 1/N]$ with $N = 2^{n-1}$. It is also, of course, the Padé Approximation of the fixed point $G(z)$.

Introducing the associated polynomials

$$\mathcal{F}_n(x) = x^{2^{n-1}} T_n \left(\frac{1}{x} \right) \quad (3.10)$$

which fulfil

$$\mathcal{F}_1(x) = x - q \quad (3.11)$$

$$\mathcal{F}_{n+1}(x) = \mathcal{F}_n((x - q)^2) \quad n \geq 1$$

and noting that

$$\frac{d\mathcal{F}_n(x)}{dx} = 2^{n-1} \mathcal{F}_1(x) \mathcal{F}_2(x) \dots \mathcal{F}_{n-1}(x) \quad (3.12)$$

we get

$$\frac{1}{z} \mathcal{R}_n\left(\frac{1}{z}\right) = \frac{1}{2^{n-1}} \frac{\mathcal{F}'_n(z)}{\mathcal{F}_n(z)} = \frac{1}{2^{n-1}} \sum_{i=1}^{2^{n-1}} \frac{1}{z - x_i^{(n)}} \quad (3.13)$$

where the $x_i^{(n)}$ are the roots of $\mathcal{F}_n(x)$. We rewrite (3.9) as

$$\mathcal{R}_n(z) = \frac{1}{2^{n-1}} \sum_{i=1}^{2^{n-1}} \frac{1}{1 - x_i^{(n)}z} = \int \frac{d\mu_n(x)}{1 - xz} \quad (3.14)$$

where the positive measure $d\mu_n(x)$ reads

$$\mu_n(x) = \frac{1}{2^{n-1}} \{\text{number of } x_i^{(n)} \text{ less than or equal to } x\}. \quad (3.15)$$

The $x_i^{(n)}$, roots of $\mathcal{F}_n(x)$ are the numbers previously introduced

$$a(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, 0) = q + \epsilon_1\sqrt{q} + \epsilon_2\sqrt{q} + \dots + \epsilon_{n-1}\sqrt{q}. \quad (3.16)$$

When $n \rightarrow \infty$, the set $\mu_n(x)$ has a limit $\mu(x)$ which is defined on the Cantor set C_q . The convergence property is best seen by noticing that

$$\frac{1}{z} \mathcal{R}_n\left(\frac{1}{z}\right) = \frac{1}{2^{n-1}} \sum_{i=1}^{2^{n-1}} \frac{1}{z - x_i^{(n)}} \quad (3.17)$$

form a set of analytic functions in the complex plane holomorphic outside a neighborhood of C_q , C_q^c , and that this set of functions is uniformly bounded by $1/d$, where d is the distance of z to C_q^c . By Vitali's theorem $(1/z)\mathcal{R}_n(1/z)$ tends uniformly to an analytic Stieltjes function in the complex plane deprived of C_q .

However, due to a classical result on Stieltjes functions [6], $d\mu_n(x)$ tends towards a positive measure $d\mu(x)$. Using the fixed point relation (2.16) for $\mu(x)$ we get

$$\mu(x) = -\frac{1}{2}\mu[(x - q)^2] + \frac{1}{2}\mu(q^2) \quad \text{for } x < q \quad (3.18)$$

if we normalize $\mu(x)$ at zero by

$$\mu(0) = 0 \quad (3.19)$$

then setting $x = q$ in (3.18) we get

$$\mu(q) = \frac{1}{2}\mu(q^2). \quad (3.20)$$

Therefore

$$\mu(x) = -\frac{1}{2}\mu[(x - q)^2] + \mu(q), \quad x < q. \quad (3.21)$$

In the same way for $x < q$, we get

$$\mu(2q) - \mu(2q - x) = \frac{1}{2} [\mu(q^2) - \mu[(q - x)^2]]. \quad (3.22)$$

Setting $x = q$ we get

$$\mu(2q) - \mu(q) = \frac{1}{2} \mu(q^2). \quad (3.23)$$

A consequence of (3.18) to (3.23) is

$$\mu[(x - q)]^2 = 2[\mu(q) - \mu(x)], \quad \mu(2q - x) = 2\mu(q) - \mu(x), \quad x < q \quad (3.24)$$

or whatever x is positive

$$\begin{aligned} \mu(q + \sqrt{x}) + \mu(q - \sqrt{x}) &= 2\mu(q), \\ \mu(q + \sqrt{x}) - \mu(q - \sqrt{x}) &= \mu(x). \end{aligned} \quad (3.25)$$

Introducing the transformation

$$F(x) = (x - q)^2 \quad (3.26)$$

and its two inverses

$$F_{(1)}^{-1}(x) = q + \sqrt{x}, \quad F_{(2)}^{-1}(x) = q - \sqrt{x}, \quad (3.27)$$

the last relation (3.25) can be rewritten as

$$\mu(x) = \mu(F_{(1)}^{-1}(x)) - \mu(F_{(2)}^{-1}(x)). \quad (3.28)$$

This relation is the Perron–Frotenius identity which expresses the invariant character of the positive measure $d\mu(x)$ under $F(x)$.

4. RECURSION RELATIONS

A family of orthogonal polynomials with respect to a positive measure fulfils a three-term recursion relation [2]

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - R_n P_{n-1}(x) \quad (4.1)$$

where α_n and R_n are real numbers and R_n is positive. The measure $d\mu(x)$ being symmetrical with respect to $x = q$ we have

$$P_m(-x+q) = (-)^m P_m(x+q), \quad (4.2)$$

from which we deduce

$$\alpha_n = q. \quad (4.3)$$

To find the $R_n(q)$, we rewrite the recursion relation (4.1) as

$$xP_n(x+q) = P_{n+1}(x+q) + R_n P_{n-1}(x+q). \quad (4.4)$$

Multiplying by x Equation (4.4) and iterating once more, we get

$$x^2 P_m(x+q) = P_{m+2}(x+q) + (R_m + R_{m+1})P_m(x+q) + R_m R_{m-1} P_{m-2}(x+q) \quad (4.5)$$

or changing m into $2m$, and using

$$P_{2m}(x+q) = P_m(x^2) \quad (4.6)$$

which follows from (2.12), we have

$$[x^2 - (R_{2m}R_{2m+1})]P_m(x^2) = P_{m+1}(x^2) + R_{2m}R_{2m-1}P_{m-1}(x^2). \quad (4.7)$$

A comparison with Equation (4.1) gives

$$\alpha_n = q = R_{2m} + R_{2m+1} \quad (4.8)$$

$$R_m = R_{2m}R_{2m-1}$$

with

$$R_1 = q, \quad R_0 = 0. \quad (4.9)$$

The last Equation (4.8) is equivalent to

$$h_{2m} = h_m. \quad (4.10)$$

From (4.8), it results that $R_m(q)$ are rational fractions in q , ratio of polynomials with integer coefficients. The polynomials $P_m(x)$ are therefore themselves rational fractions in the parameter q . We note that the polynomials $P_m^{(1)}(x)$ introduced in (2.13) are simply connected to $P_m(x)$ by

$$P_{2m+1}(x) = (x-q)P_m^{(1)}[(x-q)^2]. \quad (4.11)$$

Some polynomials $P_m(x)$ are exceptionally also polynomials in q , more precisely, polynomials

with an index

$$m = p2^n \quad (4.12)$$

p, n non-negative integers, fulfil the relation

$$P_{p2^n}(x) = P_p [P_{2^n}(x) + q] \quad (4.13)$$

which is straightforward to derive by iteration of (4.6). Therefore the two subclasses with $p = 1$ and $p = 3$ are polynomials in q .

Setting

$$Q_p(x) = P_p(x) + q \quad (4.14)$$

we have

$$Q_{p2^n}(x) = Q_p \{Q_{2^n}(x)\}. \quad (4.15)$$

Choosing $p = 2^k$, we get a more symmetrical relation

$$Q_{2^{k+n}}(x) = Q_{2^k} \{Q_{2^n}(x)\} = Q_{2^n} \{Q_{2^k}(x)\}. \quad (4.16)$$

It is easy to see that $Q_{2^k}(x)$ is just the k th iterate of

$$Q_2(x) \equiv (x - q)^2. \quad (4.17)$$

Before analyzing the consequence of this remark, we shall study

5. THE ANALYTIC STRUCTURE OF THE CAUCHY TRANSFORM OF THE INVARIANT MEASURE AND THE STRUCTURE OF THE ENSEMBLE $R_n(q)$ FOR $q > 2$.

The Padé approximation $\mathfrak{R}_n(z)$ (defined by (3.14)) have a limit, which is the fixed point of (2.15)

$$G(z) = \int \frac{d\mu(x)}{1 - xz}. \quad (5.1)$$

This function is clearly analytic everywhere except in the Cantor set C_q .

However, relation (3.8) tells us that

$$G(z) = \frac{T_1(z)T_2(z) \dots T_{n-1}(z)}{T_n(z)}, \quad G \left\{ \frac{z^{2^n}}{T_n^2(z)} \right\} \quad (5.2)$$

where $T_n(z)$ are simply

$$T_n(z) = z^{2^{n-1}} P_{2^{n-1}}\left(\frac{1}{z}\right). \quad (5.3)$$

The zeros of $T_n(z)$ are the inverse of the zeros of $P_{2^{n-1}}(z)$, which in turn are real numbers of the form

$$a(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-2}) = q + \epsilon_1 \sqrt{q + \epsilon_2 \sqrt{q + \dots + \epsilon_{n-2} \sqrt{q}}}. \quad (5.4)$$

These zeros are *not* in the Cantor set C_q but all their limiting points belong to it. Because the zeros of T_n and T_{n-1} can never coincide, being related to orthogonal polynomials, it results that any zero of $T_{n-1}(z)$ is a zero of $G(z)$. It then results that $G(z)$ has the following analytic structure: it is holomorphic in all the complex plane, including the point at ∞ , it has only real zeros of the form (5.4) accumulating to each point of the Cantor set C_q which therefore appear as essential singularities. While the measure $d\mu(x)$ is made of a non-denumerable set of steps of infinitely small height (purely singular measure), its Cauchy transform $G(z)$ has a simpler structure with families of zeros accumulating at each point of the Cantor set C_q .

We come now to the structure of the ensemble $R_n(q)$ for fixed $q > 2$. The R_n , being positive, are numbers between 0 and q .

We have for them the following results, valid for $q > 2$

PROPOSITION 1.

$$0 < R_{2n} < R_n \quad \text{and} \quad 0 < R_{2n} \leq 1. \quad (5.5)$$

The proof is by induction. Let $0 < R_{2p} \leq 1$ and $R_{2p} < R_p$ for $p = 1, 2, \dots, n-1$. Then

$$R_{2n} = \frac{R_n}{R_{2n-1}} = \frac{R_n}{q - R_{2n-2}} \quad (5.6)$$

using (4.8).

So that $0 < R_{2n} < R_n$ since $R_{2n-2} \leq 1$ and $q > 2$. If n is even, then by the induction hypothesis $R_n \leq 1$ and $0 < R_{2n} < R_n \leq 1$. If n is odd, then $R_{2n-2} < R_{n-1} \leq 1$ and $q - R_{n-1} < q - R_{2n-2}$ or

$$R_{2n} = \frac{R_n}{q - R_{2n-2}} = \frac{q - R_{n-1}}{q - R_{2n-2}} < 1. \quad (5.7)$$

PROPOSITION 2.

$$\lim_{k \rightarrow +\infty} R_{p^{2^k}} = 0. \quad (5.8)$$

From

$$R_{2n} = \frac{R_n}{q - R_{2n-2}} \quad \text{and} \quad 0 < R_{2n-2} \leq 1$$

we deduce that R_{2n} lies between R_n/q and $R_n/(q-1)$. Therefore $R_{p^{2^k}}$ lies between R_p/q^k and $R_p/(q-1)^k$. Taking the limit $k \rightarrow \infty$, one sees the truth of the proposition.

PROPOSITION 3.

$$\lim_{k \rightarrow \infty} R_{p^{2^k+s}} = R_s. \quad (5.9)$$

The proof is again by induction [7]. The proposition is true for $s = 0$ as shown above. Let it be true for $s = 0, 1, \dots, S-1$, then

$$\begin{aligned} R_{p^{2^k+s}} &= q - R_{p^{2^k+s-1}} && \text{if } S \text{ is odd} \\ &= (R_{p^{2^{k-1}+S/2}})/(R_{p^{2^k+s-1}}) && \text{if } S \text{ is even.} \end{aligned} \quad (5.10)$$

Taking the limit $k \rightarrow \infty$ and using the induction hypothesis, one sees that $R_{p^{2^k+s}} \rightarrow R_s$. The structure of the set $\tilde{C}_q = \{R_{2^n}(q), n = 0, 1, 2, \dots\}$ is made more transparent with the help of these propositions: each point of \tilde{C}_q is a limit point. More precisely, the numbers $\{R_n(q)\}$ for $q > 2$ fixed, lie in two disjoint intervals: $0 \leq R_{2n} \leq 1$; $(q-1) \leq R_{2n+1} \leq q$. Numerical studies show that these sets separate again into disjoint subsets

$$0 \leq R_{4n} \leq R_4 < R_6 \leq R_{4n+2} \leq R_2 = 1$$

and

$$R_3 \leq R_{4n+3} \leq R_7 < R_5 \leq R_{4n+1} \leq R_1 = q.$$

We conjecture that this splitting at each step continues, so that $R_{k^{2^r+n}}$ lies between R_n and R_{2^r+n} . If it is so, the closure of the set \tilde{C}_q is nowhere dense, and \tilde{C}_q is perfect.

6. ITERATION OF QUADRATIC POLYNOMIALS

The iteration of quadratic polynomials can, by a linear change of variable, be reduced to the iterations of the canonical transformation

$$F(x) = (x - q)^2, \quad q \text{ is real.} \quad (6.1)$$

By iterating n times, we get $F_n(x)$ which is a polynomial of degree 2^n in x . We have shown that

for $q > 2$

$$F_n(x) \equiv P_{2n}(x) + q \quad (6.2)$$

where $P_k(x)$ is a set of orthogonal polynomials with respect to the measure $d\mu(x)$.

It is known [5] that for $q > 2$, the complementary set of the domain of attraction of the point at infinity is the Cantor set C_q . The Perron–Frobenius identity tells us that $d\mu(x)$ is an invariant measure on C_q . However, it is in the region $q < 2$ that stable cycles and bifurcations among stable cycles occur. When we go from $q > 2$ to $q < 2$, the $R_n(q)$ are no more positive numbers and the $F_n(x)$ lose their character of being orthogonal polynomials with respect to a positive measure. But all algebraic relations which have been established for $q > 2$ remain valid for $q < 2$ by analytic continuation in q , provided we have analyticity in q . This is certainly the case, for all our relations involve only rational fractions or algebraic functions, in q , such as recursion relation identities, and expressions for the roots of polynomials. We shall apply this remark to analyze the distribution of the superstable values of q .

The superstable values of q , that we shall index q_s , correspond to cycles with a special value of the multiplier $S = 0$. The fixed points which form a cycle of order n are solutions of $F_n(x) = x$ and the corresponding multiplier (which has the same value for all the points of the cycle) is

$$S = F'_n(x)|_{x=F_n(x)}.$$

From

$$F_n(x) = [F_{n-1}(x) - q]^2, \quad F'_n(x) = 2[F_{n-1}(x) - q]F'_{n-1}(x)$$

we see that $S = 0$ implies in particular

$$F_{n-1}(x) - q = 0, \text{ that is } F_n(x) = 0,$$

for at least one x . Therefore, a superstable cycle always contains $x = 0$ and also, of course, the maps by F_n of $x = 0$.

As a consequence, the superstable value $q_{s,n}$ corresponding to a cycle of order n , is the solution of

$$F_n(x, q)|_{x=0} = 0$$

or equivalently

$$F_{n-1}(0, q) = q$$

but we recall that

$$F_{n-1}(x, q) = P_{2n-1}(x, q) + q$$

and the superstable values $q_{s, n}$ are to be found among the real zeros of

$$P_{2n-1}(0, q) = 0. \quad (6.3)$$

However the $P_k(x, q)$ are, for $q \geq 2$, orthogonal polynomials, the value $x = 0$ can never be a root in this case, because it would then belong to the support of the measure, which never includes zero. It then follows that the superstable values can belong only to the interval $q < 2$.

More generally we want to study the equation $P_k(0, q) = 0$ not necessarily for k of the form 2^{n-1} .

Starting from the three-term recursion relation

$$P_{m+1}(x, q) = (x - q)P_m(x, q) - R_m(q)P_{m-1}(x, q)$$

we get setting $m = 2p - 1$

$$P_{2p}(q, q) = -R_{2p-1}(q)P_{2p-2}(q, q).$$

Making use of

$$P_{2p}(x + q, q) = P_p(x^2, q)$$

we get

$$P_{2p}(q, q) = P_p(0, q).$$

Therefore, $P_k(0, q) = -R_{2k-1}(q)P_{k-1}(0, q)$ and the superstable values are to be found among the zeros of

$$\prod_{h=1}^{2^{(n-1)}} R_{2h-1}(q) = 0. \quad (6.4)$$

Let us consider the vicinity of $q = 2$. Because the exact solution of (4.8) for $q = 2$ is

$$R_0(2) = 0, \quad R_1(2) = 2, \quad R_n(2) = 1, \quad n \geq 2$$

we can linearize the problem and write

$$R_n(q) = R_n(2) + (q - 2)A_n + 0[(q - 2)^2].$$

Then

$$\begin{aligned} A_{2n} + A_{2n+1} &= 1, & A_0 &= A_2 = 0 \\ A_{2n} + A_{2n-1} &= A_n, & A_1 &= A_3 = 1. \end{aligned} \quad (6.5)$$

We find

$$\begin{aligned} A_{2n} &= -n + p + 1 \\ A_{2n+1} &= n - p, \end{aligned} \tag{6.6}$$

where p is a non-negative integer defined by

$$n = (2p + 1)2^k, \quad k \geq 0, \quad k \text{ integer.}$$

We see that the dependence of the $R_n(q)$ on their index n is of an arithmetical nature.

If we look at the zeros of

$$\prod_{h=1}^{2^{(n-1)}} R_{2h-1}(q) = 0$$

near $q = 2$, we have

$$\prod_{h=1}^{2^{(n-1)}} R_{2h-1}(q) = 2 + \frac{1}{8}(q-2)(2+4^n) + O((q-2)^2)$$

and the superstable values are given by

$$q_{s,n} \approx 2 - 3.4^{1-n}, \tag{6.7}$$

these superstable values accumulate at $q = 2$. By Fatou's theorem [8] we see that there exist stable cycles of an arbitrarily large order near $q = 2^-$. The analogue of the Feigenbaum [9] number would be

$$\lim_{n \rightarrow \infty} \frac{q_{s,\infty} - q_{s,n}}{q_{s,\infty} - q_{s,n+1}} = 4, \tag{6.8}$$

this result is in agreement with reference [11].

7. CONCLUSION

It is remarkable that a connection between the physical problem of ferromagnetism modeled by the Ising model and the problem of the iterations of the quadratic map, is made through the integer moment problem. All these problems present by themselves critical phenomena or bifurcations. It may be that a further investigation will show a precise mathematical connection among the various invariants such as the critical indices for the ferromagnetic Ising model, and the Feigenbaum [9] numbers for the mappings of the interval.

Here we want to point out that a deeper analysis of the content of the recursion relations; for $q < 2$

$$R_{2n}(q) + R_{2n+1}(q) = q, \quad R_{2n}(q)R_{2n-1}(q) = R_n(q) \quad (7.1)$$

may lead to an understanding of the arithmetical nature of Feigenbaum numbers, which are given for δ_2 , for instance, by

$$\delta_2 = \lim_{n \rightarrow \infty} \frac{\sum_{h=1}^{2^{(n+1)}-1} \frac{\partial}{\partial q} \ln R_{(2h-1)}}{\sum_{h=1}^{2^{(n)}-1} \frac{\partial}{\partial q} \ln R_{(2h-1)}} \Bigg|_{q=q_{2^\infty}}$$

where q_{2^∞} is the first Myrberg [3] point accumulation of the bifurcations of cycles of order 2^n . For the simple case $q = 2$, we have analyzed the content of this relation in Section 6.

Besides the result that the iterations of quadratic maps are simply orthogonal polynomials with respect to an invariant measure defined on a Cantor set C_q , we would like to point out that a three-term recursion relation of the form

$$P_{n+1}(q) = (x - q)P_n(q) - R_n(q)P_{n-1}(q)$$

can be reinterpreted (being associated with a Jacobi matrix) in terms of a physical crystallographic problem – in which q is the energy level and the $P_n(q)$ the wave function. For such a system, the spectrum will be purely singular, produced by the term $R_n(q)$ which represents the interaction. This was also a motivation for studying this ensemble more carefully.

ACKNOWLEDGEMENTS

We want to thank Dr B. Derrida for fruitful discussions and Drs A. Pandey and B. Duplantier for their contributions to the study of the ensemble $\{R_n(q)\}$ for $q > 2$. We also thank Drs M.F. Barnsley and J.S. Geronimo for helpful discussions on their own work prior to publication.

REFERENCES

1. Barnsley, M., Bessis, D., and Moussa, P., *J. Math. Phys.* **20**, 535 (1979).
2. Szegő, G., *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, 1939.
3. Julia, G., *J. Math. Pur. Appl.* **4**, 47 (1918).
Myrberg, P.J., *J. Math. Pur. Appl.* **41**, 339 (1962).
4. Collet, P. and Eckmann, J.P., 'Iterated Maps on the Interval as Dynamical System', *Progress in Physics*, Vol. 1, Birkhäuser, 1980.

5. Misiurewicz, M., to appear in *Publ. Math. IHES* 53, see also lecture note at 1980 Les Houches Summer School. After this work was finished, we received a preprint by M.F. Barnsley, J.S. Geronimo, and A.N. Harrington, Georgia Institute of Technology, Atlanta (1981), in which a direct proof is given.
6. Akhiezer, N.I., *The Classical Moment Problem*, Oliver and Boyd, Edinburgh and London, 1965, p. 93.
7. Pandey, A., private communication.
8. Fatou, P., *Bull. Soc. Math. France* 47, 161 (1919); 48, 33 (1920); 48, 208 (1920).
9. Feigenbaum, M., *J. Stat. Phys.* 19, 25 (1978).
10. Baker, G.A., Jr, *The Essentials of Padé Approximants*, Academic Press, New York, 1978.
11. Dias de Deus, J. and Taborda Duarte, J., CFMC Preprint, E 10-81, Instituto Nacional de Investigação Científica, Lisbon, Portugal.

(Received September 25, 1981)