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## Wave propagation in almost-periodic structures

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### Abstract

The unusual phenomena occurring in the wave propagation in almost-periodic structures – both classical and quantum – are studied in this paper. Focusing our attention on structures with spectral measures in the family of disconnected iterated function systems, we describe and characterize the concept of “quantum intermittency”. This theory shows that the non-trivial renormalization properties of the set of orthogonal polynomials associated with these systems are the origin of such “intermittency”, and leads to a new determination of the exponents of the asymptotic growth of the moments of the position operator.

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### 1. Introduction

Imagine an infinite, linear chain of particles of mass  $m_j$ , linked by springs of rigidity  $k_j$ , and bounded at one end to a fixed wall, in its state of equilibrium. Energy can be input in this system by suddenly displacing the zeroth particle from its equilibrium position, while holding all other particles at rest. Letting then the system free to follow its time evolution, one observes an excitation of the neighbouring masses, and an energy flow towards the infinite end of the chain. Is this flow permanent, so that all the energy will eventually escape to infinity, or is the motion recurrent in some sense? As is well known, either case may occur, depending on the nature (continuous or discrete, respectively) of the frequency spectrum of the system. In the former case, can one exactly (and quantitatively) characterize the energy flow?

Quite interestingly, this problem has a long history, nicely reviewed in the volume *Wave Propagation in Periodic Structures* by Leon Brillouin [1]. Indeed, Newton himself studied a doubly infinite linear chain with equal masses,  $m$ , and spring constants,  $k$ , in the second book of the *Principia*, in 1686. His aim was to derive a relation for the

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speed of sound; letting  $d$  be distance between neighbouring “air” particles in the state of equilibrium, he obtained for the velocity  $v$  of propagation of an elastic wave the formula

$$v = d\sqrt{\frac{k}{m}}, \quad (1)$$

which he then related to the density  $\rho$  and to the isothermal bulk modulus  $kd$  of air:  $v = \sqrt{kd/\rho}$ . Since Newton’s result is akin to the theory we are about to expose, it is convenient to pause for a moment and rederive it in modern notation.

Newton’s equations of motion for the system are quite simply given by

$$m \frac{d^2 x_i}{dt^2} = k(x_{i+1} - x_i) - k(x_i - x_{i-1}), \quad (2)$$

where  $x_i$  is the displacement of the  $i$ th particle from its equilibrium position.

Normal modes for this system were discovered by John and Daniel Bernoulli around 1727. Simply putting  $x_i = a_i e^{i\omega t}$ , one finds the following equation for the *mode frequency*,  $\omega$ , and *amplitudes*,  $a_i$ :

$$k(a_{i+1} - 2a_i + a_{i-1}) = -m\omega^2 a_i. \quad (3)$$

Eq. (3) can also be seen as the eigen-problem of the infinite, symmetric, tridiagonal, *Jacobi* matrix  $J$  with constant diagonal and outer diagonal elements:  $J_{i,i} = -2k$ ,  $J_{i,i+1} = J_{i,i-1} = k$ . Its spectrum,  $\sigma(J)$ , is the interval  $[-4k, 0]$  and hence the proper frequencies of vibration,  $\omega$ , must belong to  $[-2\sqrt{k/m}, 2\sqrt{k/m}]$ .

In 1753, Daniel Bernoulli formulated the *Principle of Superposition*, according to which the general motion of the system above is given by a suitable superposition of “proper oscillations”. Indeed, introducing the momenta  $p_i = m dx_i/dt$  in the set of equations (2), we obtain

$$\frac{d}{dt} \begin{pmatrix} \underline{x} \\ \underline{p} \end{pmatrix} = H \begin{pmatrix} \underline{x} \\ \underline{p} \end{pmatrix}, \quad (4)$$

where  $H$  is an infinite, square block matrix having  $m^{-1}$  times the identity in the top right quarter, the Jacobi matrix  $J$  in the bottom left quarter, and zeros elsewhere. The spectrum of  $H$  is trivially related to that of  $J$ :  $\sigma(H) = i[-2\sqrt{k/m}, 2\sqrt{k/m}]$ , and the formal solution of the equations of motion,

$$\begin{pmatrix} \underline{x}(t) \\ \underline{p}(t) \end{pmatrix} = e^{Ht} \begin{pmatrix} \underline{x}(0) \\ \underline{p}(0) \end{pmatrix}, \quad (5)$$

can be “spectralized” in an eigen-function (“proper oscillations”) expansion; of course, this has to be continuous. Choosing as initial condition at  $t = 0$  the vector  $x_n = \delta_{n,0}$ ,  $p_n = 0$ , which stands for the physical situation of sudden displacement of the zeroth mass from its equilibrium position, we find that the displacement of the generic,  $n$ th mass, at time  $t$ , is given by the real part of

$$x_n(t) = \int T_n(y) e^{i\sqrt{2k/mt}\sqrt{1-y}} d\mu(y), \quad (6)$$

where  $T_n$  is the (properly normalized)  $n$ th Chebychev polynomial, extended by symmetry to negative values of the index  $n$ , and  $\mu$  is the continuous measure with density  $d\mu(y) = dy/\pi\sqrt{1-y^2}$ . The motion of the  $n$ th element of the chain is therefore given by the Fourier transform of the  $n$ th *orthogonal polynomial* associated with the spectral measure  $\mu$ , with respect to the spectral measure itself. This latter,  $\mu$ , is associated with the initial state of the evolution: in quantum mechanical language, it is called the *local density of states*. These characteristics are quite general, and will be found in all systems treated in this paper.

The vector of the instantaneous values of the squares of the displacements,  $x_n^2(t)$ , gives a snapshot of the elastic energy propagation; this can be gauged by the *moments*  $v_\alpha$

$$v_\alpha(t) := \sum_n n^\alpha x_n^2(t). \quad (7)$$

When the moment problem is *determined* [2] (as in all cases we shall discuss in this paper) the full distribution  $\{x_n^2\}$  is in one-to-one relation with the set of its moments with integer index  $\alpha \geq 0$ .

In the problem at hand, the integration in (6) can be performed in terms of Bessel functions:  $x_n(t) = J_{2n}(2t\sqrt{k/m})$ . As a consequence, at fixed time  $t$ , the distribution of elastic energy is localized in the region  $|n| \leq t\sqrt{k/m}$ , with a negligible tail outside. One finds that  $v_\alpha(t) \sim (t\sqrt{k/m})^\alpha$ , and therefore, at least in a suitable average sense, the distribution of elastic energy scales as a function of the argument  $n/t$ , i.e. space over time: wave propagation is *ballistic*, and it is correct to associate a *velocity* to it, which is exactly that appearing in Newton's formula (1).

One may wonder what are the characteristics of the physical system (2) which give rise to this simple and regular behaviour – Certainly, the equality of masses and spring constants, which leads to a constant Jacobi matrix  $J$ . The presence of an absolutely continuous spectral measure is another distinct feature. Indeed, since Lord Kelvin in 1881, and Born in 1912, it is known that richer spectral properties can be obtained by modulating the sequence of masses. Nevertheless, when the Jacobi matrix associated with the problem is still periodic, the same qualitative picture shown above is to be found. It was then a remarkable fact when, in 1953, Dyson allowed masses in the chain to take random values, and found that with probability one the spectrum of the system is pure point, with exponentially localized eigen-functions [3]. In such systems, typically, energy stays confined, and cannot propagate to infinity! This is the content of *quantum localization*. These dynamical behaviours are extremal, and a full spectrum of possibilities exists in between them.

This has become evident more recently, in the study of *quasi-* and *almost-*periodic systems [4–7]. Partly because of the experimental discovery of quasi-crystals, and partly because of their mathematical appeal, these structures have been intensely studied, unveiling the deep relations which link, in sequence:

- The “character” of the sequence of coefficients of the Jacobi matrix (i.e. periodicity, quasi/almost periodicity, or lack thereof).
- The nature of the spectral measure of the system (i.e. pure point, singular continuous, absolutely continuous).
- The nature of the time evolution of the system (recurrencies, intermittent energy propagation along the chain, etc.).

In our work [8,9], putting the spectral measure, rather than the Jacobi matrix, at the first place, we have observed that singular continuous properties of the spectral measure imply almost periodicity of the Jacobi matrix, and a non-trivial evolution of the dynamical system. This phenomenon (non-trivial dynamical evolution) can be defined as the lack of a simple scaling relation for the energy transport over the linear chain: moments  $v_\alpha$  do not behave asymptotically as  $t^{\gamma\alpha}$ , not even allowing the presence of a constant  $\gamma$  different from one at the exponent, and, as a consequence, the distribution  $x_n^2(t)$  does not scale as a function of  $nt^{-\gamma}$ .

For reasons to be apparent in the following, we have termed the coexistence of these phenomena *quantum intermittency*. We believe that this paradigm plays a fundamental role in the physics of these systems, and this paper is devoted to the discussion of some of its multi-faceted aspects.

Its content can be summarized as follows: a class of spectral measures to be considered throughout the paper is introduced in Section 2. A Jacobi matrix can be associated with each of these spectral measures. The dynamics generated via Schrödinger equation by these Jacobi Hamiltonians is introduced in Section 3, where we also briefly review known results, and anticipate new developments, for the asymptotic behaviour of this dynamics, at *fixed* site. While some aspects of intermittency are already perceivable at this level, its full content is appreciated when looking comparatively at the evolution of the moments  $v_\alpha$  defined above, suitably generalized to the quantum case.

This leads to the definition of the intermittency function  $\beta$ , for which, in Section 4, we develop a novel theory which shows its origin in the renormalization properties of the orthogonal polynomials associated with the system. Finally, the conclusions try to put these results into a general perspective.

## 2. Almost-periodic lattice systems derived from their spectral properties

Many different systems have been studied in the research on the dynamical properties of almost-periodic systems; these include notably Fibonacci and Harper Jacobi matrices [10–15]. Nevertheless, we have preferred to build our research on the basis of an *inverse* approach: a Jacobi (i.e. tridiagonal, semi-infinite, real symmetric) matrix is uniquely determined by, and uniquely defines, its spectral measure  $\mu$  – with respect to the first basis vector – in all cases when the moment problem is determined [2]. Therefore, we have assumed *at the start* a particular spectral measure as given, and derived from it the associated Jacobi matrix, i.e. the physical system it describes, and its time evolution. Because we have chosen  $\mu$  in a class for which many important results are known, our approach has allowed us to make very precise statements, not only from the theoretical viewpoint, but also in numerical experimentations.

The class just mentioned consists of the invariant measures of iterated function systems (IFS) [16–18], among which we have considered in detail the sub-classes of disconnected linear IFS, and of real julia set measures. The techniques to derive the associated Jacobi matrices are presented in [19–21]. IFS measures can be defined by the *balance* equation

$$\int f(x) d\mu(x) = \sum_{j=1}^M \pi_j \int f(\phi_j(x)) d\mu(x), \quad (8)$$

which holds for any continuous function  $f$ ;  $\phi_j$  are a set of  $M$  contractive real maps, with associated weights  $\pi_j$ ,  $\pi_j > 0$ ,  $\sum_j \pi_j = 1$ . When  $\phi_i(x) = \delta_i x + \beta_i$ , we are in the presence of a linear IFS measure. When an IFS is generated by two non-linear maps,  $\phi_{\pm}(x) = \pm\sqrt{x + \lambda}$ ,  $\pi_+ = \pi_- = \frac{1}{2}$ , with  $\lambda \geq 2$  a real parameter, we are facing a real Julia set measure: indeed, this is also the invariant measure of the dynamical system  $z \rightarrow z^2 - \lambda$ , which lives on a real Julia set.

The simplicity and at the same time, the richness of these measures can be appreciated in the so-called *thermodynamical formalism*, which studies the limiting behaviour of sums of the kind  $\mathcal{S}(\{I_k\}, \mu) = \sum_k \mu(I_k)^q l(I_k)^{-\tau}$ . Here,  $\{I_k\}$  is a cover of the support of the measure  $\mu$  by intervals, and  $l$  denotes their length. For  $q$  fixed, as the size of the cover (i.e. the largest value of  $l(I_k)$ ) tends to zero, the sum  $\mathcal{S}$  may tend to zero or to infinity according to the value of  $\tau$ . The border-line value between the two behaviours,  $\tau(q)$ , defines the generalized dimension,  $D_q$ , via  $D_q = \tau(q)/q - 1$ . With a certain generality (sufficient for the cases considered here) we can say that  $D_0$  is the fractal dimension of the support of the measure,  $D_1$  the information, and  $D_2$  the correlation dimension of  $\mu$ . Usually, the generalized dimensions of a singular spectral measure are hard to compute. Since they are strictly related to the dynamics, their *precise* determination is nevertheless crucial. For disconnected, linear IFS, the dimensions  $D_q$  follow simply from the equation

$$\sum_{j=1}^M \pi_j^q \delta_j^{-\tau} = 1, \quad (9)$$

which contains only a finite number of terms, and can be easily solved to the precision wanted. The case of real Julia set measures is just a bit harder, since they can be approximated by linear IFS: a thorough study is presented in [22].

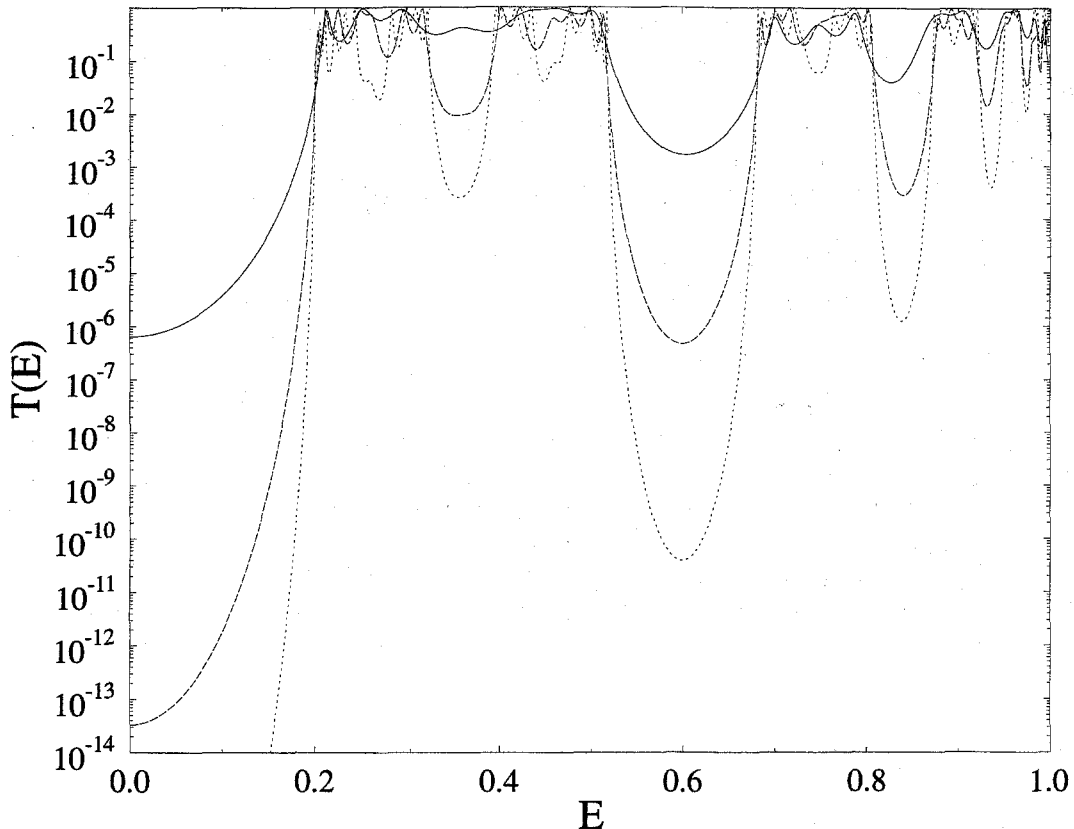


Fig. 1. Transmission coefficient,  $T(E)$ , as a function of energy,  $E$ , for the scattering of a wave through a (symmetrized) lattice system of length  $N = 40$  (continuous curve), 80 (dashes), 120 (dots), described by the Jacobi matrix of the measure  $\mathcal{C}$ . The energy dips correspond to gaps in the spectrum of  $\mu$ . Because of symmetry, only half of the picture is shown.

A few particular cases of these measures may serve to highlight the generality of IFS theory:

- The spectral measure  $\mu$  of Newton's example in Section 1: this is a Julia set measure with  $\lambda = 2$ .
- The uniform Lebesgue measure on  $[-1, 1]$ : this comes from a linear IFS with equal weights and  $\phi_{\pm}(x) = \frac{1}{2}x \pm \frac{1}{2}$ .
- Devil's staircase measures on  $[-1, 1]$ : we shall consider repeatedly the one given by  $\phi_{\pm}(x) = \frac{2}{5}x \pm \frac{4}{5}$ ,  $\pi_{+} = \pi_{-} = \frac{1}{2}$ . For short, it will be indicated as  $\mathcal{C}$ , because its support is a Cantor set.

The first two examples above are somehow exceptional: for typical values of the parameters, IFS measures are singular continuous, and one may wonder to what sort of Jacobi matrices they give rise. It turns out that these are typically almost periodic [9] (limit periodic in the Julia case [23]). Therefore, in this approach, two of the three defining criteria of quantum intermittency are met since the very beginning.

Of a certain interest, worth of being mentioned at this point, is the fact that the techniques leading to the construction of such Jacobi matrices [19] can be also translated into the design of electrical or mechanical filters, like that originally constructed by Vincent in 1898 [1]. In our examples, these filters are *fractal-pass*, in the sense that they suppress all frequencies, except those belonging to a fractal set, which can be chosen as wished by tuning the IFS parameters! Fig. 1 shows a typical transmission profile, at different lengths of the filter, which renders evident the hierarchical construction of the measure  $\mathcal{C}$  effected by the corresponding IFS.

### 3. Quantum dynamics of almost-periodic Jacobi matrices

Schrödinger's equation,  $i d\psi/dt = J\psi$ , together with the initial condition  $\psi(0) = e_0 := (1, 0, \dots)$ , describes a physical situation quite similar to that we have met in Section 1. Here, the Jacobi matrix  $J$  describes a lattice system with site energies  $J_{i,i}$  and hopping terms  $J_{i,i+1}$ . The initial site of the evolution is  $e_0$ , the zeroth lattice state, and, as time grows, the wave-packet  $\psi$  spreads on the lattice. The analogy is most clearly seen when exhibiting the solution of the equation of motion, as in [8,29]

$$c_n(t) := (e_n, e^{-itJ} e_0) = \int e^{-itE} p_n(E) d\mu(E). \quad (10)$$

Here,  $p_n$  is the  $n$ th orthogonal polynomial associated with the spectral measure  $\mu$ , and, as indicated,  $c_n(t)$  is the projection of  $\psi(t)$  on the  $n$ th lattice state,  $e_n$ .

Already at this level we can see one of the distinct features of quantum intermittency: Let us consider the projection of the motion over a finite sub-lattice, which for convenience can be taken to be the set of sites from 0 to  $N$ :  $\mathcal{E}_N(t) = \sum_{n=0}^N |c_n(t)|^2$ . According to the celebrated RAGE theorems [4], when the spectral measure  $\mu$  is absolutely continuous,  $\mathcal{E}_N(t)$  decays to zero: as in Newton's example, energy flows to infinity. When  $\mu$  is pure point,  $\mathcal{E}_N(t)$  is almost-periodic, as in Dyson's case. But when  $\mu$  is singular continuous,  $\mathcal{E}_N(t)$  follows neither of these behaviours [24]: all it can be said is that it goes to zero in Cesàro time average. Indeed, thanks to a Mellin

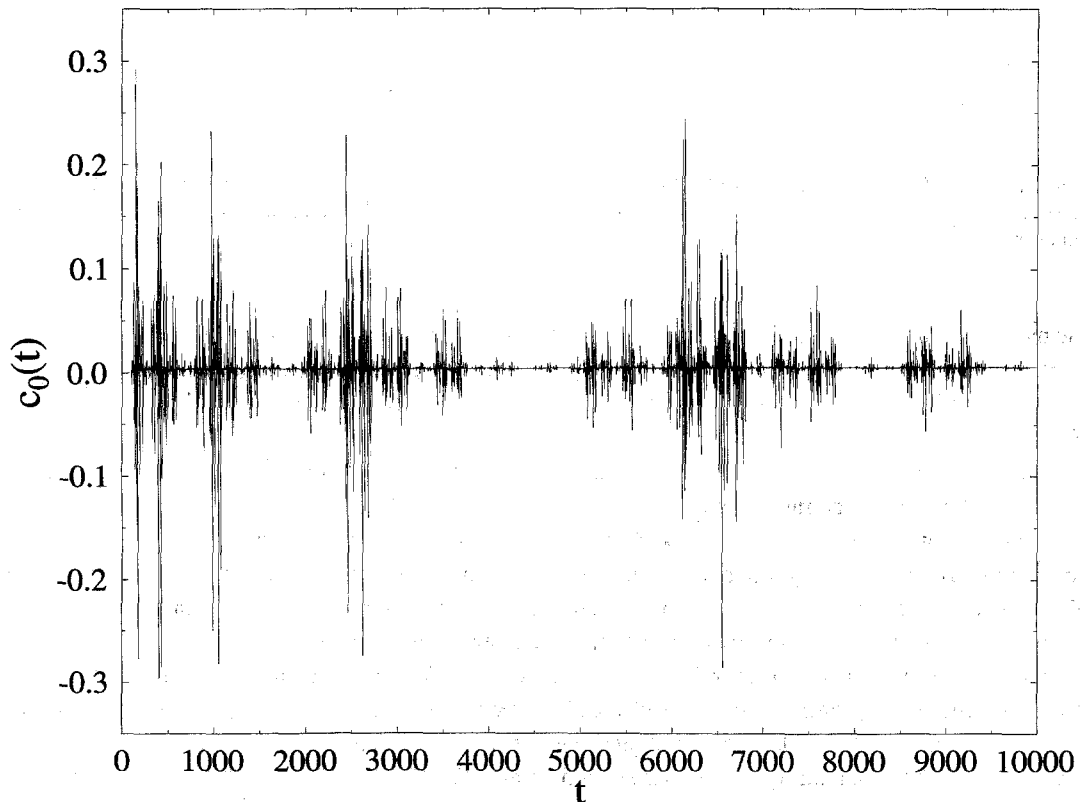


Fig. 2. Projection of the motion on site zero,  $c_0$ , versus time,  $t$ , for the quantum dynamics of the IFS  $\mathcal{C}$ .

type analysis, and generalizing previous results of Bessis et al. [25,26], Ketzmerick et al. [27], Makarov [28], and ourselves [8,9,29], we have shown [30] that, in the IFS class described above

$$\frac{1}{2T} \int_{-T}^T \Xi_N(s) ds = \sum_p R_p e^{-z_p \log T} + O(T^{-1}), \quad (11)$$

where  $z_p$  are poles in the Mellin transform of  $\Xi_N$ , of the form  $z_p = a_p + ib_p$ , with  $z_0 = D_2$ , and  $a_p \geq D_2$  for all  $p$ ;  $R_p$  are the corresponding residues. The presence of poles with  $a_p$  arbitrarily close, or equal to,  $D_2$ , gives rise to a log-oscillatory behaviour in  $\Xi_N(T)$ , superimposed to the leading decay. Without Cesàro time-averaging,  $\Xi_N(t)$  shows typically an “intermittent” behaviour (Fig. 2, see also [31]) with zones of “high activity” (which means here return of probability density close to the origin), and quiescence.

#### 4. The intermittency function $\beta$

Perhaps the most important manifestation of intermittency in wave propagation in almost-periodic structures is the fact that the distribution of excitations ( $x_n^2(t)$  in the classical case,  $|c_n(t)|^2$  in the quantum, see Fig. 3) does *not*

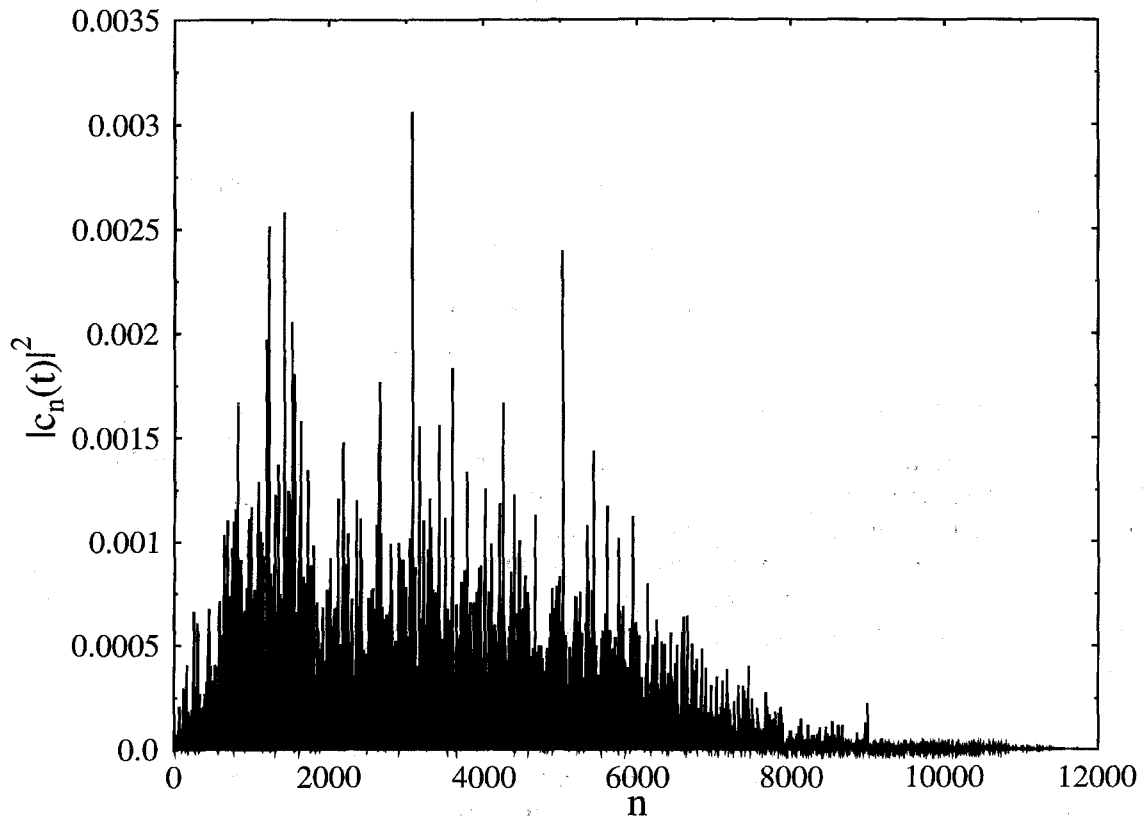


Fig. 3. Snapshot of the probability distribution,  $|c_n(t)|^2$ , versus site,  $n$ , for the quantum dynamics of the IFS  $C$ .

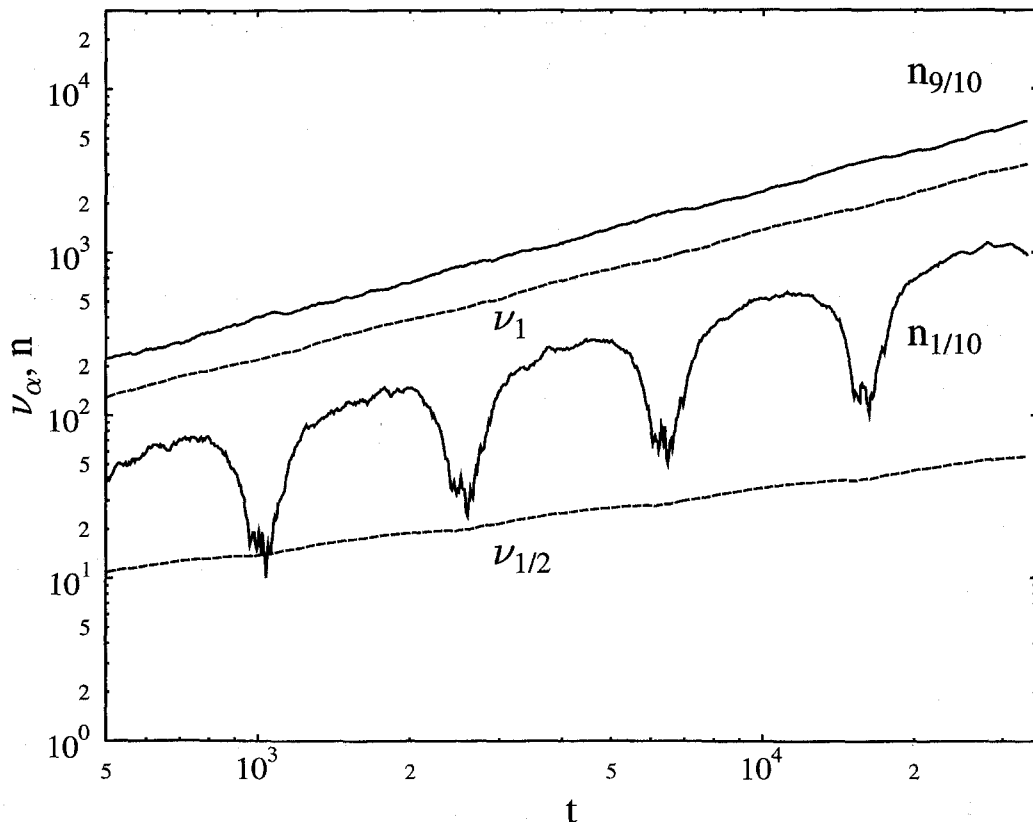


Fig. 4. Moments,  $v_\alpha$  ( $\alpha = \frac{1}{2}, 1$ ), versus time,  $t$ , for the quantum dynamics of the IFS  $C$ . Also plotted are the percentiles,  $n_p$  ( $p = \frac{1}{10}, \frac{9}{10}$ ), defined by the minimal value  $m$  such that  $\sum_{n=0}^m |c_n(t)|^2 \geq p$ : at time  $t$ , the fraction  $p$  of probability density is located between 0 and  $n_p(t)$ . Notice that the oscillations of the leftmost 10% of the distributions are associated with the recurrences seen in Fig. 2.

scale asymptotically as a function of the product  $nt^\gamma$ . This is manifest in the non-uniform scaling of the moments  $v_\alpha$  (Eq. (7) and Fig. 4), which can be encoded in an *intermittency function*  $\beta$  of the moment index  $\alpha$ , via

$$\log(v_\alpha(t)) \simeq \alpha\beta(\alpha) \log t + g(\log t). \quad (12)$$

Here, the leading correction  $g(s)$  is a bounded function. The origin of the function  $g$  can be easily explained by analysing in detail the distribution of the complex poles of the Mellin transform of  $\mathcal{E}_N$ , and therefore also of  $v_\alpha$ , as done in [30]. For particular values of the parameters,  $g$  turns out to be asymptotically log-periodic.

To the contrary, the nature of the convex, non-decreasing function  $\beta(\alpha)$  [8], which is limited from below by  $D_1$  (Guarneri's inequality [32,33]), is much more subtle, and elusive. In a previous paper [9], we have shown that *in the case of real Julia sets* the function  $\beta$  is *uniquely determined by the thermodynamical properties of the spectral measure*  $\mu$ , via the relation

$$\beta(\alpha) = D_{1-\alpha}. \quad (13)$$

To our knowledge, this is the sole non-trivial case for which an exact result has been derived. In view of what is noticed above, this also permits to obtain Newton's result in a synthetic form, since all the generalized dimensions relevant to the case take the value one.



In the general case of which IFS are an excellent example, thanks to their approximation properties [34–37], this estimate is not exact [9]. It is then plausible that understanding why IFS measures do not follow relation (13) will give us a clue on the general behaviour.<sup>2</sup> To do this, we shall now present a novel, complete theory of the intermittency function  $\beta$  of IFS measures, which contains Eq. (13) as a particular case.

According to Eq. (10), the probability density on site  $n$  is

$$|c_n(t)|^2 = \iint d\mu(x) d\mu(y) p_n(x) p_n(y) \Phi(t, x - y), \tag{14}$$

where  $\Phi = e^{-it(x-y)}$  for instantaneous values, or  $\Phi = \sin t(x - y)/t(x - y)$  for Cesàro averages. Applying the balance property (8), we get

$$|c_n(t)|^2 = \sum_{i,j=1}^M \pi_i \pi_j \iint d\mu(x) d\mu(y) p_n(\phi_i(x)) p_n(\phi_j(y)) \Phi(t, \phi_i(x) - \phi_j(y)), \tag{15}$$

in which we can disregard terms with  $i \neq j$ , for they decrease in Cesàro average like  $t^{-1}$ . To the contrary, diagonal terms decay like  $t^{-D_2}$  [9,30], and will therefore be retained.

Since the composite functions  $p_n(\phi_i(x))$  are still polynomials of degree  $n$ , they can be expanded on the basis of the first  $n$  orthogonal polynomials of  $\mu$ :

$$p_n(\phi_i(x)) = \sum_{k=0}^n \Gamma_{i,k}^n p_k(x). \tag{16}$$

The coefficients  $\Gamma$  have a profound meaning, as they are the *Lanczos vectors* associated with a generalization of the Jacobi matrix  $J$  [19].

For convenience, we shall assume that  $\Gamma_{i,k}^n$  are also defined, and null, for  $k > n$ . This relation can be used within Eq. (15), to get

$$|c_n(t)|^2 \sim \sum_{i=1}^M \pi_i^2 \iint d\mu(x) d\mu(y) \sum_{k,l=0}^n \Gamma_{i,k}^n \Gamma_{i,l}^n p_k(x) p_l(y) \Phi(\delta_i t, x - y), \tag{17}$$

where the sign  $\sim$  makes it evident that we are looking for asymptotic behaviours, and where linearity of the maps has allowed us to shift the multiplicative action of  $\delta_i$  from energy to time, in the argument of  $\Phi$ .

Multiplying both sides of (17) by  $n^\alpha$ , and summing over  $n$ , produces the moment  $\nu_\alpha(t)$  at LHS; exchanging the order of summation at RHS (which is certainly allowed, since all terms in the summation decay sufficiently fast in the relative indices) gives

$$\nu_\alpha(t) \sim \sum_{i=1}^M \pi_i^2 \sum_{k,l=0}^{\infty} \iint d\mu(x) d\mu(y) p_k(x) p_l(y) \Phi(\delta_i t, x - y) \sum_n n^\alpha \Gamma_{i,k}^n \Gamma_{i,l}^n. \tag{18}$$

We must now analyse in detail the terms appearing in this equation. The Stieltjes integrals

$$c_{k,l}(t) := \iint d\mu(x) d\mu(y) p_k(x) p_l(y) \Phi(t, x - y) \tag{19}$$

<sup>2</sup> In a recent paper by Piéchon [38], a relation formally equivalent to Eq. (13), but radically different in meaning, has been conjectured to hold exactly, and generally. Without commenting here on the validity of this conjecture, I remark that in such proposed relation  $\beta$  is the growth exponent of moments *averaged over initial site* (which mathematically amounts to different Jacobi Hamiltonians) and  $D$  are the thermodynamical dimensions of the *global* density of states – which is not the spectral measure  $\mu$ , and can indeed have rather different characteristics from the latter.

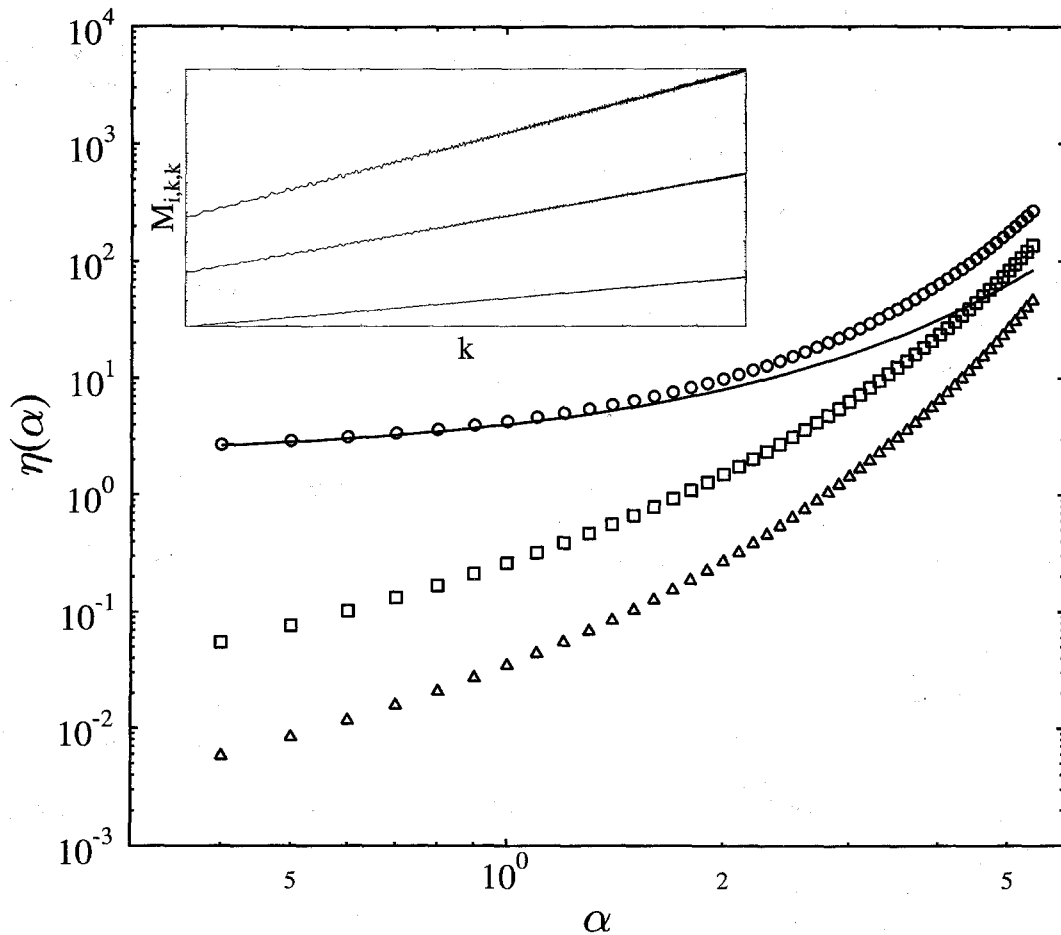


Fig. 5. Curves  $\eta_{i,p}$ , versus  $\alpha$  (see text), with  $p = 0$  (circles),  $p = 2$  (squares), and  $p = 4$  (triangles), for the IFS  $C$ . The continuous line marks the approximation  $\pi_1^{1-\alpha}$ . In the inset,  $M_{i,k,k}(\alpha)$  versus  $k$ , for three values of  $\alpha$ , in doubly logarithmic scale, showing an almost perfect power-law behaviour.

are the instantaneous (or Cesàro averaged) values of the product  $c_k c_l^*$ . They, too, decay in time average like  $t^{-D_2}$ . The discrete Mellin transforms of the numerical coefficients  $\Gamma$ ,

$$M_{i,k,l}(\alpha) := \sum_n n^\alpha \Gamma_{i,k}^n \Gamma_{i,l}^n, \quad (20)$$

exist for all values of  $\alpha$ . Relabelling indices, to introduce the site distance  $p = l - k$ , it can be shown that  $M_{i,k,k+p}(\alpha)$  behaves like  $\eta_{i,p}(\alpha) k^\alpha$  as  $k$  tends to infinity. The coefficients  $\eta_{i,p}(\alpha)$  can be found numerically, as we shall discuss later. We observe that  $\eta_{i,p}(\alpha)$ , for  $\alpha$  fixed, is a fastly decreasing function of  $p$  (see Fig. 5).

Making use of this analysis in Eq. (18) leads to

$$v_\alpha(t) \sim \sum_{i=1}^M \pi_i^2 \left( \eta_{i,0}(\alpha) v_\alpha(\delta_i t) + \sum_{p>0} \eta_{i,p}(\alpha) v_{\alpha,p}(\delta_i t) \right), \quad (21)$$

where the generalized moment  $\nu_{\alpha,p}(t)$  is defined by  $2 \sum_k k^\alpha \Re(c_{k,k+p}(t))$ ,  $\Re$  denoting the real part. A weighted relation between different moments  $\nu_{\alpha,p}$ , at different times  $t$ , and  $\delta_i t$ , is established. This remarkable fact, together with the asymptotic behaviour of these moments,  $\nu_{\alpha,p} \sim \gamma_p(\alpha) t^{\alpha\beta}$  (we can normalize to  $\gamma_0 = 1$ ), leads us to an implicit equation for the function  $\beta(\alpha)$ :

$$1 = \sum_{i=1}^M \pi_i^2 \delta_i^{\alpha\beta} \left( \eta_{i,0}(\alpha) + \sum_{p>0} \gamma_p(\alpha) \eta_{i,p}(\alpha) \right). \tag{22}$$

This equation fully describes quantum intermittency in IFS systems.

In fact, on the one hand, Eq. (22) can be turned into a computational scheme. The constants  $\eta_{i,p}(\alpha)$  can be easily determined from the coefficients  $\Gamma$  (whose calculation requires only the Jacobi matrix  $J$ : one can employ the techniques in [19] or directly the definition). The constants  $\gamma_p(\alpha)$  are the ratios between the asymptotic behaviours of  $\nu_{\alpha,0}$  and  $\nu_{\alpha,p}$ , and can be computed independently of  $\beta$ . In a moment, we shall present the results of these computations.

At the same time, Eq. (22) is invaluable to understand quantum intermittency from the theoretical point of view. To begin with, the result (13) can now be rederived in this general setting: In the Julia set case, one has the crucial renormalization equation  $p_{2n} \circ \phi_{\pm} = p_n$ . Approximating the non-linear maps  $\phi_{\pm}$  by an IFS with  $M = 2^q$  linear maps (the larger the integer  $q$ , the better the approximation), leads to  $|T_{i,k}^n| = 1$  for  $k = n/M$  and zero otherwise. Then, we have that

$$\pi_i = 2^{-q} = \frac{1}{M}, \quad \eta_{i,0}(\alpha) = \pi_i^{-(\alpha+1)}, \quad \eta_{i,p} = 0 \text{ for } p > 0. \tag{23}$$

With these values, it is easy to see that Eq. (22) coincides with the definition of generalized dimensions, Eq. (9).

Let us now turn our attention to the devil’s staircase measure  $\mathcal{C}$  introduced in Section 2. Here, Eq. (13) predicts a flat function  $\beta(\alpha) = D_0$ , while a non-constant function is observed in numerical experiments: something else than thermodynamics is determining this enigmatic *intermittency without multi-scaling* [9]. Because of the symmetries of this IFS, the values of  $\eta_{i,p}(\alpha)$  do not depend on the map index  $i$ , and we can solve for  $\beta$  in Eq. (22), obtaining

$$\beta(\alpha) = \frac{1}{\alpha \log \delta^{-1}} \log \left( \frac{1}{2} \left[ \eta_0(\alpha) + \sum_{p>0} \gamma_p \eta_p(\alpha) \right] \right). \tag{24}$$

Now, the renormalization form (23) of the  $\eta$ -functions leads to  $\beta = D_0$ , which is an extreme case of a convex, non-decreasing function. It is therefore clear that any deviations from (23) must produce a convex, increasing function, which means intermittency. Such deviations are apparent in Fig. 5.

A close analysis of these data reveals that  $\eta_0$  differs very little – at least for small values of  $\alpha$  – from the estimate (23): at  $\alpha = 0.4$  one has  $\eta_0 = 2.68$  and  $\pi_1^{-1-\alpha} = 2.64$ . Nonetheless, inserting these values in Eq. (24), and neglecting terms with  $p > 0$ , leads to the  $\beta$  values:  $\beta = 0.806$  and  $0.756$ , respectively, which – on the scale of precise numerical computations – are appreciably different from themselves, and from the numerically computed value  $\beta(0.4) = 0.777$ . Fig. 6 shows the  $\beta$ -functions obtained by truncating the summation in Eq. (24) to  $p = 0$ ,  $p = 0, 2$ , and  $p = 0, 2, 4$  (the coefficients  $\gamma_p$  being null for odd  $p$ ). Convergence of the successive approximations to the experimental data (stars) is observed. At fixed values of the maximal  $p$ , the precision of the approximation gets worse with increasing  $\alpha$ , because of the increase of the curves  $\eta_p(\alpha)$ .

As seen, small variations in the values of the  $\eta$ -functions can change appreciably the value of the dynamical exponent  $\beta$ ; moreover, they can be responsible for intermittency, when this does not originate from the thermodynamical properties of the spectral measure. What is then the origin of these variations? It stems from the complicate

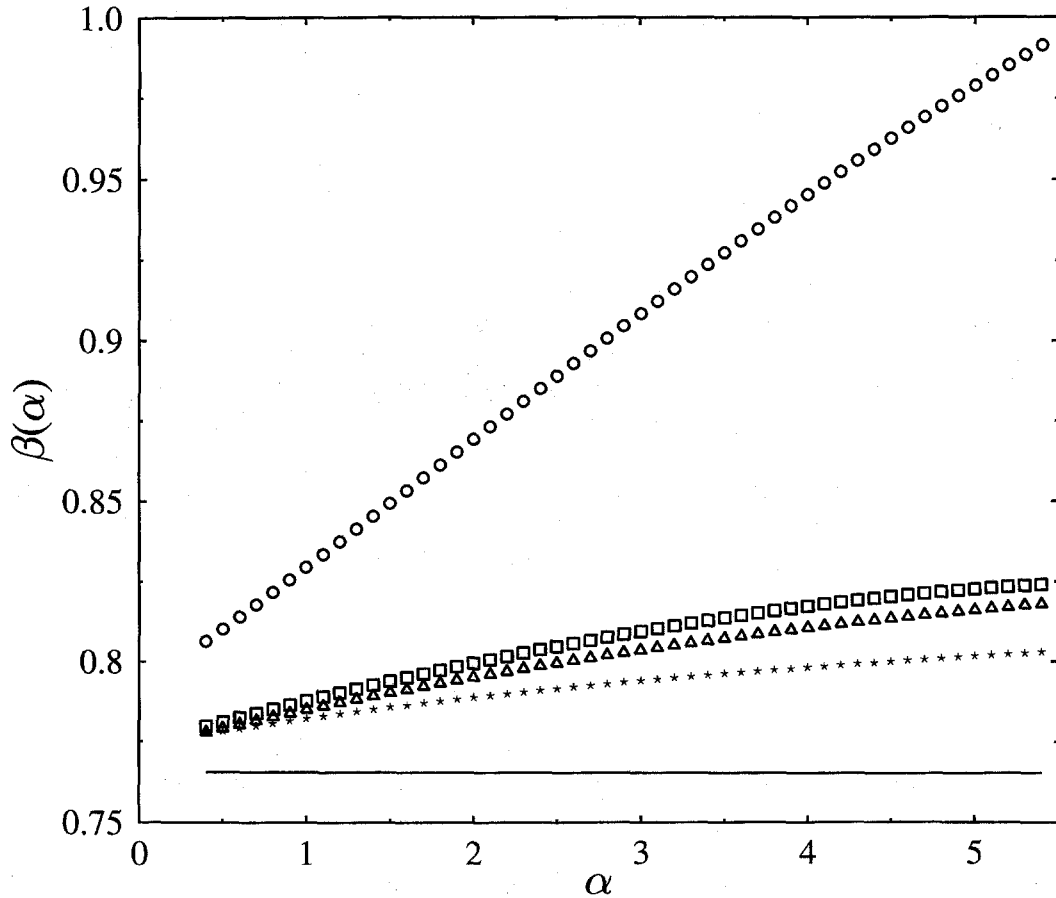


Fig. 6. Exact (stars), and approximate intermittency functions  $\beta$ , versus  $\alpha$ , for the IFS  $C$ . See also Fig. 5 and text. Different approximations are  $p = 0$  (circles),  $p = 0, 2$  (squares),  $p = 0, 2, 4$  (triangles). The continuous line marks the curve  $\beta = D_0$ .

structure of the coefficients  $\Gamma$ , pictured in Fig. 7, quite different from the simple renormalization ansatz  $|\Gamma_{i,k}^n| = 1$  for  $k = n/M$  and zero otherwise. Eq. (16) shows that these coefficients encode the renormalization properties of the orthogonal polynomials of the spectral measure. Since these latter are the “eigen-functions” to be properly associated with a continuous spectrum, it is then natural that they play the significant role which has been unveiled by our analysis.

## 5. Conclusions

The dynamical analysis of quantum motion has shown that except for a few non-physical examples this cannot be chaotic in a proper sense [39,40]. This fact poses a serious challenge to the pretension of quantum mechanics to be a complete theory of nature [41]. Nevertheless, the quest for the Holy Grail of quantum chaos, which has animated the physicists’ community for more than a decade, has widely enlarged our understanding of the motion induced by particular classes of Schrödinger’s Hamiltonians. Among these, the case of almost periodic systems has interested physicists and mathematicians alike.

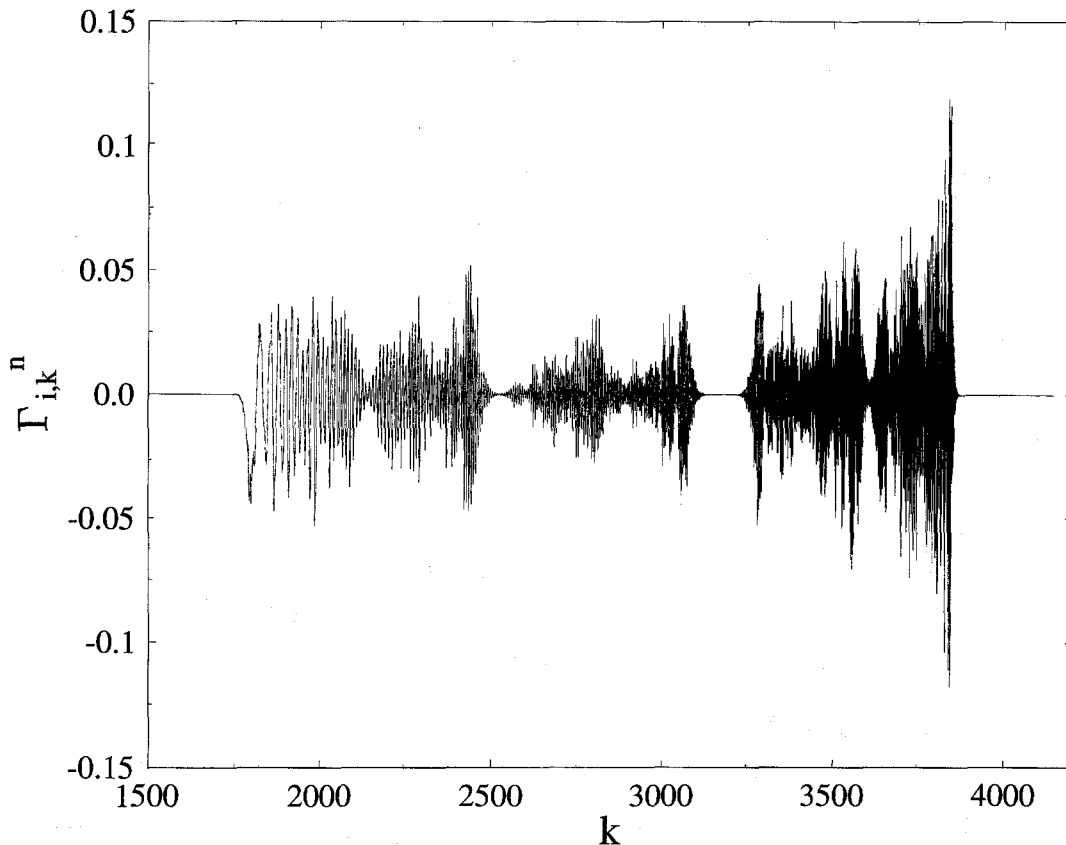


Fig. 7. Coefficients  $\Gamma_{1,k}^n$  with  $n = 6000$ , versus  $k$ , for the IFS  $\mathcal{C}$ .

Somehow boldly, we have proposed the term *quantum intermittency* for what appears to be the typical coexistence of three phenomena: almost periodicity of lattice Hamiltonians, singular spectral measures, and non-trivial dynamical phenomena. Most of these phenomena are of course well-known: RAGE-like results have a rich and long history, as well as the “odd” kind of recurrences associated with singular continuous spectra, and the study of *diffusion* in relation with properties of the spectral measure. Nonetheless, the *intermittency function*  $\beta$ , on which we focus mostly our attention, has a fresh look: it permits a quantitative description of the spreading of the motion over the lattice basis, and has so far eluded a complete theoretical analysis.

The particular value of  $\beta$  at  $\alpha = 2$  (second moment) is certainly endowed with an important physical meaning [42]. Yet, only studying the full curve  $\beta$  as a function of  $\alpha$  permits to unveil the richness of the dynamics: in this sense quantum intermittency is a much stronger concept than *anomalous diffusion*, i.e., the inequality  $\beta(2) \neq \frac{1}{2}$ . It is indeed much harder to describe theoretically (and sometimes to compute numerically) the *variations* of  $\beta$  with  $\alpha$ , than to obtain approximate estimates – more or less precise – of  $\beta$  in a range of small values of  $\alpha$ : see also the papers by Artuso and Geisel in this volume, and Ref. [12]. The results presented in this paper are an important step in this direction, since they solve the problem in the vast class of IFS measures.

Relations between dynamics and thermodynamical properties of the spectral measure emerge when we do not use the geometry of the Hilbert basis, and consider non-basis dependent quantities to describe the motion, as done by Guarneri [33,43,44]. The intermittency function  $\beta$  itself can be put in relation with the thermodynamical properties of

the spectral measure; this relation is exact and transparent in the case of Julia set Jacobi matrices. This paper explains why this is so, and why this fails in general: mathematically speaking, only under strong algebraic properties, like those of Julia sets, the orthogonal polynomials of a spectral measure are linked by a simple renormalization relation.

Indeed, the fact that the motion in almost periodic systems should be described by such renormalization properties is known, and has been already employed, e.g., in the study of Fibonacci systems [12,13,45,46]. What is added by our results is the discovery that, typically, the renormalization procedure cannot be given a simple form, and must itself present a “singular” structure, which is the origin of the non-trivial intermittency function  $\beta$ . The linear disconnected IFS studied in this paper are an evident illustration of this phenomenon. Since this class of systems is an excellent approximation tool for singular measures [34–37], we expect similar techniques and results to apply to a more general setting.

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