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# The Diophantine moment problem and the analytic structure in the activity of the ferromagnetic Ising model

M. Barnsley, D. Bessis, and P. Moussa

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We show that the intensity of magnetization  $I(z, x)$  where  $z = e^{-2\beta H}$  and  $x = e^{-2\beta J}$ , for the ferromagnetic Ising model in arbitrary dimension, reduces, for rational values of  $x$ , to a Diophantine moment problem

$$I(z) = \sum_0^\infty n_k z^k,$$

where

$$n_k = \int_0^\Lambda \sigma(\lambda) \lambda^k d\lambda,$$

$\sigma(\lambda)$  is a positive measure,  $n_0 = 1/2$ , and  $n_k$  is integer for  $k \neq 0$ . The fact that the  $n_k$  are positive integers puts very stringent constraints on the measure  $\sigma(\lambda)$ . One of the simplest results we obtain is that for  $\Lambda < 4$ ,  $\sigma(\lambda)$  is necessarily a finite sum of Dirac  $\delta$  functions whose support is of the form  $4\cos^2(p\pi/m)$ ,  $p = 0, 1, 2, \dots, m-1$ , with  $m$  a finite integer. For  $\Lambda = 4$ , which correspond to the one-dimensional Ising model, we have the result that either  $I(z)$  is a rational fraction belonging to the previous class  $\Lambda < 4$ , or  $I(z) = (1/2)(1-4z)^{-1/2}$  which corresponds precisely to the exact answer for dimension 1. For  $\Lambda > 4$ , which is associated with Ising models in dimension  $d \geq 2$  we show that all cases are reducible to  $\Lambda = 6$ , by a quadratic transformation which transforms integers into integers and positive measures into positive measures. The fixed point of this type of transformation is analyzed in great detail and is shown to provide a devil's staircase measure. Various other results are also discussed as well as conjectures.

## INTRODUCTION

While the ferromagnetic Ising model in two dimensions and zero field is well understood,<sup>1</sup> comparatively little is known in the presence of a magnetic field. The Lee–Yang representation<sup>2</sup> although very interesting has been little exploited. In particular the analytic nature of the singularities in the complex activity plane on the circle  $|z|=1$  is not known, and the possibility that below the critical temperature  $T_c$  when the circle is closed there exists a natural frontier has not been excluded. It thus seems important to know how to characterize the class of analytic functions to which the thermodynamic quantities belong when considered as functions over the complex activity plane.

An associated question is that of the critical indices which appear, at least for those which are exactly computable in two dimensions, to be rational numbers. Is this a general feature of the Ising model, and via the universality principle a characteristic of classes of physical processes? If it is the case, then it would be interesting to be able to classify these processes and to understand how such rational indices are generated.

In this paper we shall only be concerned with the ordinary ferromagnetic Ising model on a lattice of dimension  $d$  with  $c$  nearest neighbors, with interaction limited to the nearest neighbors.

In this case, the perturbative expansion of thermodynamical quantities are of the form

$$\tau(z, x) = \sum_0^\infty \bar{\mu}_l(x) z^l,$$

where  $x$  is the usual temperature variable  $x = \exp(-2\beta J)$  and  $z$  is the activity variable,  $z = \exp(-2\beta H)$ .

The Mayer–Yvon coefficients which arise in the  $z$

expansion of the free energy are *polynomials in  $x$  with integer coefficients*. Furthermore, from the Lee–Yang theorem, we know that these coefficients are moments of a positive measure according to

$$\bar{\mu}_l(x) = \int_0^\pi g(\theta, x) \cos l\theta d\theta, \quad g(\theta, x) > 0.$$

We are thus led to the consideration of a trigonometrical moment problem on the ring of polynomials with integer coefficients.

Such a problem defines a specific class of analytic functions. It is the aim of this paper to set in motion an investigation of the content of such a class. Unexpectedly, the fact that the moments must belong to a ring introduces in many cases stringent constraints on the measure  $g(\theta, x)$ , as will be seen in the sequel.

In Sec. I we show how the Ising model is associated with a moment problem on a ring and how in the case of rational values of  $x$  this reduces to the following ordinary moment problem on the ring of integers: *Find a positive measure  $\sigma(\lambda)$  such that*

$$n_k = \int_0^\Lambda \sigma(\lambda) \lambda^k d\lambda, \quad k=0, 1, 2, \dots,$$

where each  $n_k$  is an integer. The nature of  $\sigma(\lambda)$  depends crucially on  $\Lambda$ . The problem is explicitly solved in Sec. II for the case  $0 \leq \Lambda \leq 4$ , and over this range the measures  $\sigma_\Lambda(\lambda)$  are found to be quantized.

In Sec. III we consider the case  $\Lambda > 4$  and show that it can always be reduced to the situation when  $\Lambda = 6$ . Associated with  $\Lambda = 6$  is an interesting transformation—its fixed point belongs to a measure whose support is contained in and naturally associated with a Cantor set.<sup>3</sup> The analogous transformation in the case  $\Lambda = 4$  is associated with the solution of the one-dimensional Ising model.

In Sec. IV we discuss various examples which cast light on the unsolved part of the problem ( $4 < \Lambda \leq 6$ ), and in conclusion we examine several conjectures.

### 1. THE ISING MODEL VIEWED AS A MOMENT PROBLEM IN A RING

The ferromagnetic Ising model on a lattice of dimension  $d$  with  $c$  nearest neighbors is described by the Hamiltonian

$$H = -J \sum_{i,j} \sigma_i \sigma_j - H \sum_i \sigma_i, \quad J > 0, \quad (1.1)$$

the first sum being performed over nearest neighbor pairs and the second over all sites of the lattice. We introduce the notations

$$x = \exp(-2\beta J), \quad 0 \leq x \leq 1, \quad \text{and } z = \exp(-2\beta H), \quad (1.2)$$

where  $H$  is the magnetic field. Then, following Lee and Yang,<sup>2</sup> the intensity of magnetization per site is found to be

$$I(z, x) = 2(1 - z^2) \int_{\theta_0(x)}^{\pi} \frac{g(\theta, x)}{1 - 2z \cos \theta + z^2} d\theta, \quad (1.3)$$

where  $\theta_0(x)$  is the Lee–Yang angle which vanishes for  $x < x_c$  where  $x_c$  corresponds to the critical temperature, see Fig. 1.  $g(\theta, x)$  is a positive measure, being the density of zeros of the grand partition function on the circle  $|z|=1$  in the complex activity plane in the thermodynamic limit. The representation (1.3) is also valid when the lattice has only finitely many sites, in which case the measure simply consists of a finite sum of delta functions with positive weights.

The measure  $g$  is normalized according to

$$\int_{\theta_0(x)}^{\pi} g(\theta, x) d\theta = \frac{1}{2}, \quad (1.4)$$

and  $I(z, x)$  has the property that

$$I(z, x) = -I(1/z, x) \quad (1.5)$$

which refers to the symmetry of the system under reversal of the magnetic field. In Fig. 2 we represent the domain of analyticity of  $I(z, x)$  in  $z$ .

Developing  $I(z, x)$  around  $z=0$  we obtain the Mayer–Yvon expansion

$$I(z, x) = 1 - 2 \sum_{l \geq 1} LM_l(x) z^l, \quad (1.6)$$

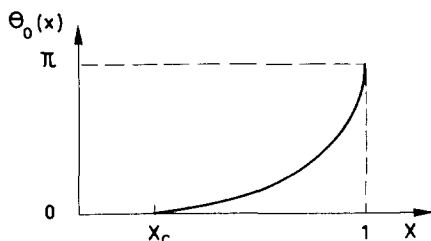


FIG. 1. The Lee–Yang angle  $\theta_0(x)$  as a function of  $x$ . It vanishes when  $x < x_c$ .

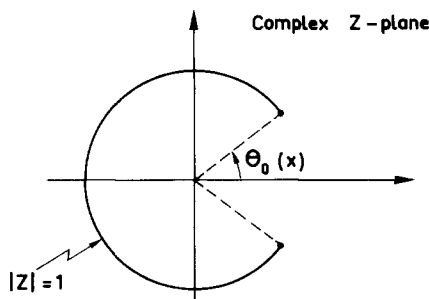


FIG. 2. When  $\theta_0(x) > 0$ ,  $I(z, x)$  is regular both inside and outside the unit circle, and also for  $|z|=1$  when  $|\arg z| < \theta_0(x)$ . When  $\theta_0(x)=0$  its singularities may be dense on the unit circle.

with

$$LM_l(x) = -2 \int_{\theta_0(x)}^{\pi} g(\theta, x) \cos(l\theta) d\theta, \quad l = 1, 2, 3, \dots \quad (1.7)$$

This defines the trigonometrical moment problem, whose moments have the following properties:

(i)  $M_1(x) = x^c$  where  $c$  is the number of nearest neighbors.

(ii)  $LM_l(x)$  is a polynomial of degree  $lc$  in  $x$  whose parity is that of the highest degree term.

(iii) All coefficients of  $LM_l(x)$  are integers.<sup>4</sup>

The proper moment problem associated with (1.7) is

$$\bar{\mu}_l(x) = \int_{\theta_0(x)}^{\pi} g(\theta, x) [\cos^2(\theta/2)]^l d\theta, \quad (1.8)$$

where  $\bar{\mu}_l(x)$  and  $LM_l(x)$  are connected by

$$4^l \bar{\mu}_l(x) = \binom{2l}{l} M_0 - \sum_{p=1}^l \binom{2l}{l-p} p M_p(x), \quad (1.9)$$

$$LM_l(x) = (-1)^{l+1} 2 \bar{\mu}_0 + \sum_{p=1}^l (-1)^{l-p-1} \frac{l}{p} \binom{l+p-1}{l-p} 4^p \bar{\mu}_p(x), \quad (1.10)$$

and, for  $l=0$ ,

$$\bar{\mu}_0 = M_0 = \frac{1}{2}. \quad (1.11)$$

A consequence of (i), (ii), and (iii) is that  $4^l \bar{\mu}_l(x)$  is a polynomial with integer coefficients. A deeper result, proved in Ref. 5, is that in fact this polynomial is exactly divisible by  $(1-x)^l$  and we have

$$4^l \bar{\mu}_l(x) = (1-x)^l P_l(x), \quad (1.12)$$

where  $P_l(x)$  is of degree  $l(c-1)$  and has integer coefficients. When  $c$  is even the  $\bar{\mu}_l(x)$ 's are even polynomials so that setting  $u = x^2$  we have

$$4^l \bar{\mu}_l(x) = (1-u)^l P_l(u), \quad (1.13)$$

where  $P_l(u)$  is of degree  $l(c/2-1)$ . For instance, for both the square and diamond lattices  $c=4$ , the degree of  $P_l$  is exactly  $l$ . Thus from now onwards we will restrict attention to the case where  $c$  is even.

We introduce the variable

$$v = 4z/(1+z)^2 = 1 - \tanh^2 \beta H. \quad (1.14)$$

Then the generating function for the  $\bar{\mu}_l(x)$ 's is given by

the intensity of magnetization<sup>6</sup> in the new variable, namely

$$I(v, x) = 2\sqrt{1-v} \sum_{i=0}^{\infty} v^i \mu_i(x) = 2\sqrt{1-v} \times \int_{\theta_0(x)}^{\pi} \frac{g(\theta, x)}{1-v \cos^2(\theta/2)} d\theta. \quad (1.15)$$

Setting

$$w = (1-u)v/4 = z(1-u)/(1+z)^2 \quad \text{and} \quad \xi = \frac{4}{1-u} \cos^2(\theta/2), \quad (1.16)$$

we obtain

$$\frac{I(w, u)}{2(1-4w/(1-u))^{1/2}} = \sum_{i=0}^{\infty} w^i P_i(u) = \int_0^L \frac{\bar{g}(\xi, u) d\xi}{1-\xi w}, \quad (1.17)$$

where

$$\bar{g}(\xi, u) = \frac{g(2 \text{Arc cos}(\frac{1}{2}\sqrt{\xi(1-u)}), u)}{\sqrt{\xi(4/(1-u)-\xi)}}, \quad (1.18)$$

and

$$L = \frac{4}{1-u} \cos^2(\theta_0/2).$$

This shows that the  $P_i(u)$  are moments of the positive measure  $\bar{g}(\xi, u)$  over its support  $0 \leq \xi \leq L$ , that is

$$P_l(u) = \int_0^L \bar{g}(\xi, u) \xi^l d\xi, \quad l=1, 2, 3, \dots \quad (1.19)$$

When  $u < u_c$  (where  $u_c$  is the value of  $u$  corresponding to the critical temperature),  $\theta_0$  is zero. When  $u \rightarrow 1$ ,  $\theta_0 \rightarrow \pi$ , but we have the asymptotic bound<sup>7</sup>

$$\frac{4}{1-u} \cos^2(\theta_0/2) \Big|_{u=1} \leq c^2, \quad (1.20)$$

so that  $L$  remains finite for all  $0 \leq u \leq 1$ . On the other hand, explicit calculations up to the highest available order have indicated that all of the coefficients of  $P_i(u)$  are positive.<sup>5</sup> If this is true for all orders, then it follows that  $L = \lim [P_i(u)]^{1/i}$  is a monotone increasing function of  $u > 0$ , as is illustrated in Fig. 3 for the case  $c=4$ . Note that for  $c=2$ ,  $P_i(u)$  is simply a positive constant so that  $L$  is independent of  $u$  and equal to 4 as seen from (1.20).

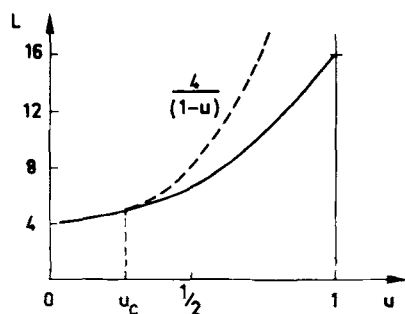


FIG. 3.  $L$  as a function of  $u$  for  $c=4$ . The dotted curve is  $4/(1-u)$  which equals  $L$  just up to  $u_c$ .

We come now to the problem we are interested in. Consider a positive measure  $\bar{g}(\xi, u)$  for which the moments

$$P_l(u) = \int_0^L \bar{g}(\xi, u) \xi^l d\xi, \quad l=0, 1, 2, \dots \quad (1.21)$$

are known to belong to a ring, for example the ring of polynomials with integer coefficients. Then is it possible to characterize, or define in a precise way, the analytic structure of the generating function

$$G(w) = \sum_{i=0}^{\infty} w^i P_i(u). \quad (1.22)$$

One can envisage eventually imposing the additional constraint that the coefficient of the  $P_i(u)$ 's are positive.

For simplicity in this first attempt we shall consider the reduced problem obtained by choosing  $u$  to be a rational number,

$$u = p/r, \quad \text{where } p \text{ and } r \text{ are positive integers, with } p \leq r. \quad (1.23)$$

Then  $P_i(p/r)$  takes the form of an integer  $n_i$  divided by  $r^{i(c/2-1)}$ , so that

$$n_l = r^{l(c/2-1)} \int_0^L \bar{g}(\xi, p/r) \xi^l d\xi = \int_0^L r^{(c/2-1)} r^{-c/2} \bar{g}(\lambda r^{1-c/2}, p/r) \lambda^l d\lambda, \quad (1.24)$$

where we have set  $\lambda = \xi r^{c/2-1}$ . We are thus led to consider the *Diophantine moment problem*

$$n_l = \int_0^\Lambda \sigma(\lambda) \lambda^l d\lambda, \quad l=0, 1, 2, \dots, \quad (1.25)$$

where  $n_0 = \frac{1}{2}$ ,  $n_l$  is a positive integer for  $l \geq 1$ ,  $\sigma(\lambda)$  is a positive measure with support  $0 \leq \lambda \leq \Lambda$ , and where  $\Lambda$  can be chosen to be a positive integer, without loss of generality. Notice that by the trivial modification  $\sigma(\lambda) \rightarrow \sigma(\lambda) + \frac{1}{2}\delta(\lambda)$  we can if we like take  $n_0=1$ . Our new moment problem is still posed over a ring—this time it's the ring of integers.

The generating function associated with (1.25) is

$$G(w) = \sum_{i=0}^{\infty} n_i w^i = \int_0^\Lambda \frac{\sigma(\lambda) d\lambda}{1-w\lambda}. \quad (1.26)$$

This is in fact a Stieltjes function, see Ref. 8, and is holomorphic in the  $w$  plane cut from  $1/\Lambda$  to  $+\infty$ , as shown in Fig. 4. The important point which we will demonstrate and explore in the rest of this paper is that the positivity of the measure in (1.26) combined with the fact that the  $n_i$ 's are integers imposes stringent constraints on the nature of  $\sigma(\lambda)$  and hence on the analytic character of  $G(w)$ .

We mention two practical situations where results along the above lines would be directly applicable.

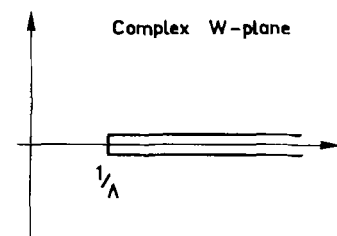


FIG. 4. The location of the cut for the Stieltjes function  $G(w)$ ; elsewhere this function is holomorphic.

First, for the triangular planar lattice which has  $d=2$  and  $c=6$ , the value of  $u_c$  is  $\frac{1}{3}$ .<sup>4</sup> Therefore, on the critical isotherm we have a Diophantine moment problem (D. M. P.) with  $L=6$ , and  $r=3$  so that  $\Lambda=54$ . Second, for  $u=1$  the generating function  $G(w, u)$  reduces to the Monomer-Dimer partition function<sup>9,10</sup> which appears as the solution of a D. M. P. with  $\Lambda=c^2$ .

## 2. THE DIOPHANTINE MOMENT PROBLEM WITH FINITE SUPPORT

### A. The problem

Given that  $\sigma(\lambda)$  is a positive measure defined on  $0 \leq \lambda \leq \Lambda < \infty$ , such that all of the moments

$$n_k = \int_0^\Lambda \lambda^k \sigma(\lambda) d\lambda, \quad k=0, 1, 2, \dots \quad (2.1)$$

are finite integers (with the occasional exception that  $n_0$  is half-integer) we ask what can be said about the generating function

$$G(w) = \int_0^\Lambda \frac{\sigma(\lambda) d\lambda}{1 - \lambda w}, \quad w \in \mathbb{C}. \quad (2.2)$$

In all that follows the support is taken to be the closed interval  $[0, \Lambda]$  so that if  $\sigma(\lambda)$  has delta function contributions at either end point then these must be included in the evaluation of such integrals as (2.1) and (2.2).

The nature of  $G(w)$  depends critically on  $\Lambda$ , as we shall see, and we begin by examining the simplest cases.

### B. The problem when $\Lambda = 1$

The fact that  $\{\lambda^k\}_{k=0}^\infty$  forms a monotone nonincreasing sequence of functions defined on  $[0, 1]$ , together with the positivity of  $\sigma(\lambda)$  implies in this case the inequalities

$$n_0 \geq n_1 \geq n_2 \geq \dots \geq 0. \quad (2.3)$$

It follows that the sequence  $\{n_k\}_{k=0}^\infty$  converges after finitely many steps to a limiting value  $n_{k_0}$  such that

$$n_{k_0} = n_{k_0+1} = n_{k_0+2} = \dots \quad (2.4)$$

Hence

$$n_{k_0} - n_{k_0+1} = \int_0^1 \lambda^{k_0} (1 - \lambda) \sigma(\lambda) d\lambda = 0, \quad (2.5)$$

and since  $\lambda^{k_0}(1 - \lambda)$  is strictly positive when  $\lambda \in (0, 1)$  we discover that

$$\sigma(\lambda) = \sigma_0 \delta(\lambda) + \sigma_1 \delta(\lambda - 1), \quad (2.6)$$

where  $\sigma_0$  and  $\sigma_1$  are both nonnegative. Using (2.1) we now find

$$\sigma_0 = n_0 - n_1 \geq 0, \quad \sigma_1 = n_1 \geq 0, \quad (2.7)$$

and

$$G(w) = (n_0 - n_1) + n_1 / (1 - w). \quad (2.8)$$

We conclude that the Diophantine moment problem on  $[0, 1]$  has a solution if and only if  $n_0 \geq n_1 = n_2 = n_3 = \dots \geq 0$  and in this case  $G(w)$  is a  $[1/1]$  rational fraction which is uniquely specified by the first two terms in its expansion about zero. Note that if  $0 < \Lambda < 1$  then the problem has a solution if and only if  $n_0 \geq n_1 = n_2 = \dots = 0$  and in this case  $G(w)$  is a constant.

### C. The problem when $\Lambda = 2$

Noting that  $\{\lambda^k(2 - \lambda)^k\}_{k=0}^\infty$  forms a monotone nonincreasing sequence of functions defined on  $[0, 2]$ , the sequence of integers

$$m_k = \int_0^2 \lambda^k (2 - \lambda)^k \sigma(\lambda) d\lambda, \quad k=0, 1, 2, \dots \quad (2.9)$$

must be nonincreasing. Thus we again find that there exists an integer  $k_0$  such that

$$m_{k_0} = m_{k_0+1} = m_{k_0+2} = \dots, \quad (2.10)$$

so that

$$m_{k_0} - m_{k_0+1} = \int_0^2 \lambda^{k_0} (2 - \lambda)^{k_0} (1 - \lambda)^2 \sigma(\lambda) d\lambda = 0, \quad (2.11)$$

and hence

$$\sigma(\lambda) = \sigma_0 \delta(\lambda) + \sigma_1 \delta(\lambda - 1) + \sigma_2 \delta(\lambda - 2), \quad (2.12)$$

where the  $c_i$ 's are nonnegative. Identification with the first three moments now provides

$$\sigma_0 = n_0 - \frac{3n_1}{2} + \frac{n_2}{2}, \quad \sigma_1 = 2n_1 - n_2, \quad \sigma_2 = \frac{(n_2 - n_1)}{2} \quad (2.13)$$

and

$$G(w) = \frac{n_0 - (3n_0 - n_1)w + (2n_0 - 3n_1 + n_2)w^2}{1 - 3w + 2w^2}. \quad (2.14)$$

We conclude that the Diophantine moment problem on  $[0, 2]$  has a solution if and only if  $2n_1 \geq n_2 \geq \text{Max}\{n_1, 3n_1 - 2n_0\}$ ,  $n_0 \geq 0$ , and  $n_k = (2n_1 - n_2) + (n_2 - n_1)2^{k-1}$  for  $k \geq 2$ ; and in this case  $G(w)$  is a  $[2/2]$  rational fraction which is uniquely specified by the first three terms in its expansion about zero.

### D. The problem when $\Lambda < 4$

On setting  $\lambda = 4 \cos^2(\theta/2)$  and defining  $\Lambda = 4 \cos^2(\theta_0/2)$  we obtain

$$2n_k/4^k = \int_{\theta_0}^\pi [\cos^2(\theta/2)]^k g(\theta) d\theta, \quad k=0, 1, 2, \dots, \quad (2.15)$$

where

$$g(\theta) = 8 \cos(\theta/2) \sin(\theta/2) \sigma(4 \cos^2(\theta/2)), \quad (2.16)$$

$$\sigma(\lambda) = 2[\lambda(4 - \lambda)]^{-1/2} g(2 \text{Arc cos}(\sqrt{\lambda/2})).$$

The positivity of the measure  $\sigma(\lambda)$  implies that  $g(\theta)$  is also nonnegative over its support. If we now introduce the trigonometrical moments

$$c_k = \int_{\theta_0}^\pi g(\theta) \cos k\theta d\theta, \quad k=0, 1, 2, \dots, \quad (2.17)$$

and use the formulas

$$\cos k\theta = T_{2k}(\cos(\theta/2)) = \bar{T}_k(\cos^2(\theta/2)), \quad (2.18)$$

where  $T_{2k}(x)$  is the (even) Tchebycheff polynomial of order  $2k$  so that

$$\bar{T}_k(\cos^2(\theta/2)) = \sum_{p=0}^k \bar{T}_k^p(\cos^2(\theta/2)^p) \quad (2.19)$$

with

$$\bar{T}_k^0 = (-1)^k \quad \text{and} \quad \bar{T}_k^p = (-1)^{k-p} \binom{k+p-1}{k-p} \frac{k}{p} \frac{4^p}{2} \quad \text{when } p \neq 0, \quad (2.20)$$

then we obtain

$$c_k = \sum_{p=0}^k \bar{T}_k^p \frac{2n_p}{4^p} = (-1)^k 2n_0 + \sum_{p=1}^k (-1)^{k-p} \frac{k}{p} \binom{k+p-1}{k-p} n_p, \quad k=0, 1, 2, \dots \quad (2.21)$$

We now note that

$$\frac{k}{p} \binom{k+p-1}{k-p} = 2 \binom{k+p}{k-p} - \binom{k+p-1}{k-p} \quad (2.22)$$

is an integer for all  $p=1, 2, \dots, k$ , and so all of the  $c_k$ 's are themselves integers even in the physical case which corresponds to  $n_0 = \frac{1}{2}$  and for which  $c_0 = 1$ . Observe that the inverse of the formulas (2.20) is

$$2n_k = \binom{2k}{k} c_0 + 2 \sum_{p=1}^k \binom{2k}{k-p} c_p, \quad k=0, 1, 2, \dots \quad (2.23)$$

The generating function  $G(w)$  for the  $n_k$ 's is

$$G(w) = \sum_{k=0}^{\infty} n_k w^k = \int_0^{\Lambda} \frac{\sigma(\lambda) d\lambda}{(1-\lambda w)} = \int_0^{4 \cos^2(\theta_0/2)} \frac{\frac{1}{2} g(2 \text{Arc cos}(\sqrt{\lambda/2})) d\lambda}{(1-\lambda w) \sqrt{\lambda(4-\lambda)}} \quad (2.24)$$

while the generating function for the  $c_k$ 's is

$$I(z) = \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty} \frac{1}{2} z^k \int_{\theta_0}^{\pi} (\exp(ik\theta) + \exp(-ik\theta)) g(\theta) d\theta = \int_{\theta_0}^{\pi} \frac{(1-z \cos \theta) g(\theta) d\theta}{1-2z \cos \theta + z^2} = \frac{c_0}{2} + \frac{(1-z^2)}{2} \times \int_{\theta_0}^{\pi} \frac{g(\theta) d\theta}{1-2z \cos \theta + z^2} \quad (2.25)$$

The two generating functions  $G(w)$  and  $I(z)$  are connected by *de la Vallée Poussin's* transformation

$$w = z/(1+z)^2 \quad (2.26)$$

according to

$$I(z) = \frac{c_0}{4} + \frac{1-z}{1+z} G\left(\frac{z}{(1+z)^2}\right) \quad (2.27)$$

or

$$G(w) = \frac{1}{\sqrt{1-4w}} \left[ I\left(\frac{1-\sqrt{1-4w}}{1+\sqrt{1-4w}}\right) - \frac{1}{2} I(0) \right]$$

When  $\Lambda < 4$ ,  $I(z)$  is holomorphic in the  $z$  plane less an arc of the circle of radius 1, just as in the case of the function  $I(z, x)$  shown in Fig. 2. The relationship between its values when  $|z| < 1$  and its values when  $|z| > 1$  is

$$I(1/z) + I(z) = c_0, \quad (2.28)$$

which can be verified with the aid of (2.27). In particular, in this situation where  $I(z)$  can be analytically continued from inside the unit circle to outside it, we must have

$$I(\infty) = 0 \quad \text{when } \Lambda < 4. \quad (2.29)$$

We will next make key use of Szegő's Theorem<sup>11</sup>: If

among the coefficients  $c_k$  of a Taylor series  $I(z) = \sum_{k=0}^{\infty} c_k z^k$  there appear only a finite number of different values, then either  $I(z) = P(z)/(1-z^m)$ , where  $P(z)$  is a polynomial of finite degree and  $m$  is a nonnegative integer, or else  $I(z)$  cannot be analytically continued beyond the unit circle.

Now from (2.17) it follows that

$$|c_k| \leq c_0 \quad \text{for all } k, \quad (2.30)$$

and since the  $c_k$ 's are all integers, there appear only finitely many different coefficients in the expansion of  $I(z)$ . Furthermore when  $\Lambda < 4$ ,  $I(z)$  can certainly be continued beyond the unit circle and so Szegő's theorem provides

$$I(z) = \frac{P(z)}{(1-z^m)} = P(z) + z^m P(z) + z^{2m} P(z) + \dots, \quad (2.31)$$

where in view of (2.29) we must have

$$P(z) = c_0 + c_1 z + \dots + c_{m-1} z^{m-1} \quad \text{when } \Lambda < 4. \quad (2.32)$$

Thus we have deduced that when  $\Lambda < 4$ ,  $I(z)$  is a rational fraction whose only singularities are poles located at various roots of unity.

Notice that the polynomial  $P(z)$  in (2.32) necessarily possesses the factor  $(1-z)$  because  $\Lambda < 4$  corresponds to  $\theta_0 > 0$  in (2.25) which means  $I(z)$  cannot have a pole at  $z=1$ . More generally, there can occur many cancellations between the numerator and denominator in (2.31) as can be seen by forming a superposition of two such generating functions—the result is a new generating function corresponding to a D. M. P. for which there will in general be many cancellations when expressed in the form (2.31). This means that there is no straightforward way of expressing  $m$  as a function of  $\theta_0$ .

In general, when  $\Lambda \leq 4$ , we observe that in view of (2.30) the set of numbers

$$\tilde{c}_k = c_k + c_0, \quad k=0, 1, 2, \dots \quad (2.33)$$

are all positive integers bounded above by  $\tilde{c}_0$  and hence the number

$$C = \tilde{c}_0 \cdot \tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \dots \quad (2.34)$$

considered as a representation in the base  $\tilde{c}_0$ , is either rational or irrational, and we have the following proposition.

*Proposition:* The generating function  $I(z)$  is a rational function if and only if the number  $C$  is rational.

*Proof:* Suppose  $C$  is rational. Then the sequence  $\{\tilde{c}_k\}$  must ultimately be periodic so that there exists integers  $m$  and  $n$  with  $\tilde{c}_k = \tilde{c}_{k+m}$  for all  $k$  whenever  $k > n$ . But this implies

$$I(z) = c_0 + c_1 z + \dots + c_n z^n + z^{n+1} \frac{(c_{m-1} + c_{n+2} z + \dots + c_{n+m} z^{m-1})}{(1-z^m)} \quad (2.35)$$

which is rational.

Conversely, suppose  $I(z) = P(z)/Q(z)$  where  $P(z)$  and  $Q(z)$  are polynomials of finite degree with, say,

$$Q(z) = q_0 + q_1 z + \dots + q_m z^m, \quad q_0 \neq 0, \quad q_m \neq 0. \quad (2.36a)$$

Then for all  $k$  greater than some  $k_0$  we must have

$$q_0 c_{k+m} + q_1 c_{k+m-1} + \dots + q_m c_k = 0. \quad (2.36b)$$

Now consider the set of vectors

$$\nu_k = (c_k, c_{k+1}, \dots, c_{k+m-1}), \quad \text{where } k=0, 1, 2, \dots \quad (2.37)$$

Since only finitely many such vectors can be constructed out of the  $c_k$ 's there must exist at least one such vector which reappears infinitely many times as  $k$  varies. It now follows that  $C$  is rational, for if  $\nu_{k_1} = \nu_{k_2}$ , then using (2.36b) we have  $c_{k_1+m} = c_{k_2+m}$  and so on, ..., which completes the proof of the proposition.

In the case where  $\Lambda < 4$  we have already seen that  $I(z)$  is necessarily rational, and in fact takes the special form (2.31) and (2.32). However in the case  $\Lambda = 4$  the number  $C$  may be either rational or irrational. If it is rational, then the proposition provides that  $I(z)$  is itself rational and can be expressed in the form (2.35), but where we no longer have any assurance that  $I(z)$  tends to zero at infinity. That is,  $I(z)$  is the sum of a polynomial and a component of the form (2.31) and (2.32). If, on the other hand  $C$  is irrational, then  $I(z)$  cannot be a rational function and Szegő's theorem provides that it possesses a natural frontier on the unit circle  $|z| = 1$ .

Up to now the only use we have made of the positivity of  $g(\theta)$  is embodied in (2.30). The full positivity constraint is most conveniently expressed in terms of the Toeplitz determinants<sup>12</sup>

$$T_k(c_0, c_1, \dots, c_k) = \begin{vmatrix} c_0 & c_1 & c_2 & \dots & c_k \\ c_1 & c_0 & c_1 & \dots & c_{k-1} \\ c_2 & c_1 & c_0 & \dots & c_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_k & c_{k-1} & \dots & \dots & c_0 \end{vmatrix}. \quad (2.38)$$

A necessary and sufficient condition that  $g(\theta)$  is non-negative for  $0 \leq \theta \leq \pi$  is that

$$T_k(c_0, c_1, \dots, c_k) \geq 0 \quad \text{for all } k=0, 1, 2, \dots, \quad (2.39)$$

while  $g(\theta)$  is nonnegative for  $\theta_0 \leq \theta \leq \pi$  and vanishes on  $0 \leq \theta \leq \theta_0$  if and only if we have both (2.39) and

$$T_k(d_0, d_1, \dots, d_k) \geq 0 \quad \text{for all } k=0, 1, 2, \dots, \quad (2.40)$$

where for each  $k$

$$d_k = 2c_k \cos \theta_0 - c_{|k+1|} - c_{|k-1|}. \quad (2.41)$$

Furthermore, if  $T_{k_0}(c_0, c_1, \dots, c_{k_0}) = 0$ , then  $T_k(c_0, c_1, \dots, c_k) = 0$  for all  $k \geq k_0$  and this is the case if and only if  $I(z)$  takes the form (2.31) and (2.32) for some  $m \leq k_0$ .

We believe, but have so far been unable to prove in general, that the positivity constraints (1.39) are in fact such as to ensure that the number  $C$  is rational and hence that there cannot exist a natural frontier on the unit circle in the case  $\Lambda = 4$ . This belief is supported by the fact that it is certainly true in the physical case mentioned in Sec. 1 where  $n_0 = \frac{1}{2}$  so that  $c_0 = 1$ . In this situation, either

$$0 = c_1 = c_2 = c_3 = \dots \quad (2.42)$$

in which case  $I(z) = c_0 = 1$ , or else there is some first  $c_{k_0}$  such that  $|c_{k_0}| = 1$ . If the latter is true, then  $T_{k_0} = 0$  from which it follows that  $I(z)$  is a rational function of the special form (2.31) and (2.32).

We now return to the  $w$  plane and convert our results concerning  $I(z)$  into statements about the generating function  $G(w)$ . When  $\Lambda < 4$  we see from (2.31) together with (2.27) that  $G(w)$  is a rational fraction with its poles located at various points  $w = w_k$ ,  $k \in \{1, 2, \dots, m-1\}$ , where

$$w_k = \frac{1}{4 \cos^2(k\pi/m)}. \quad (2.43)$$

When  $\Lambda = 4$  and  $n_0 = \frac{1}{2}$ , we either have

$$G(w) = \frac{1}{\sqrt{1-4w}} \left\{ \left[ 1 \pm \left( \frac{1-\sqrt{1-4w}}{1+\sqrt{1-4w}} \right)^m \right]^{-1} - \frac{1}{2} \right\} \quad \text{for some } m, \quad (2.44)$$

which must always reduce to a rational fraction with its poles at all of the points  $w_k$ . For example, when  $m = 7$

$$G(w) = \frac{1-7w+14w^2-7w^3}{2(1-4w)(1-5w+6w^2-w^3)}, \quad (2.45)$$

or else corresponding to (2.42) we have

$$G_4(w) = \frac{1}{2} \cdot \frac{1}{\sqrt{1-4w}}. \quad (2.46)$$

It is interesting to note that  $G_4(w)$  here is precisely the function obtained for the one-dimensional Ising model in the thermodynamic limit. Rational fractions are obtained in the case of a finite Ising chain with periodic boundary conditions.

### 3. THE PROBLEM WHEN $\Lambda > 4$

This situation is the interesting one for physical problems with the exception of the one-dimensional Ising model which corresponds to  $\Lambda = 4$ .

*Proposition:* Any Diophantine moment problem corresponding to  $\Lambda > 4$  can be reduced to one with  $\Lambda = 6$  by repeated application of the transformation

$$G_{2\sqrt{\Lambda}}(w) = \frac{1}{1-w\sqrt{\Lambda}} G_{\Lambda} \left( \frac{w^2}{(1-w\sqrt{\Lambda})^2} \right), \quad (3.1)$$

where  $G_{\Lambda}(w)$  denotes a Diophantine moment generating function associated with the support  $[0, \Lambda]$  and  $\sqrt{\Lambda}$  is integer.

To prove this result we need to use various transformations which take D. M. P.'s into D. M. P.'s. These are of considerable importance and we begin by discussing the simplest ones.

Starting with

$$n_k = \int_a^b \lambda^k \sigma(\lambda) d\lambda, \quad k=0, 1, 2, \dots, \quad (3.2)$$

where  $a$  and  $b$  are real numbers,  $\sigma$  is nonnegative, and the  $n_k$ 's are integers, let us make a translation

$$\lambda = x - q, \quad \text{where } q \text{ is an integer greater than } -a. \quad (3.3)$$

Then we obtain

$$n_k = \int_{a+q}^{b+q} (x-q)^k \sigma(x-q) dx, \quad (3.4)$$

and putting  $\bar{\sigma}(x) = \sigma(x - q)$ , which is also nonnegative, we define a new set of moments by

$$\bar{n}_k = \int_{a+q}^{b+q} x^k \bar{\sigma}(x) dx, \quad k=0, 1, 2, \dots \quad (3.5)$$

These new moments associated with the translated measure  $\bar{\sigma}(x)$  are also integers because

$$\bar{n}_k = \int_a^b (\lambda + q)^k \sigma(\lambda) d\lambda = \sum_{p=0}^k \binom{k}{p} q^{k-p} n_p, \quad (3.6)$$

and the two generating functions

$$G(w) = \sum_{k=0}^{\infty} n_k w^k \quad \text{and} \quad G_{[\square_1]}(w) = \sum_{k=0}^{\infty} \bar{n}_k w^k \quad (3.7)$$

are connected by the relation

$$G_{[\square_1]}(w) = \frac{1}{1-qw} G\left(\frac{w}{(1-qw)}\right). \quad (3.8)$$

The second simple transformation is nonlinear. Setting

$$\lambda = y^2 \quad (3.9)$$

in (3.2) and supposing  $0 < a < b$ , we get

$$n_k = \int_a^b y^{2k+1} \sigma(y^2) dy \quad (3.10)$$

and

$$\begin{aligned} G(w) &= \sum_{k=0}^{\infty} n_k w^k = \int_a^b \frac{\sigma(\lambda) d\lambda}{1-\lambda w} \\ &= \int_a^b \sigma(y^2) y \left( \frac{1}{1-y\sqrt{w}} + \frac{1}{1+y\sqrt{w}} \right) dy \\ &= \int_{-\sqrt{b}}^{+\sqrt{b}} \frac{|y| \sigma(y^2)}{1-y\sqrt{w}} dy, \end{aligned}$$

$$\sigma(y^2) \text{ being zero on } [-\sqrt{a}, +\sqrt{a}]. \quad (3.11)$$

If we now introduce the moments

$$m_k = \int_{-\sqrt{b}}^{+\sqrt{b}} |y| \sigma(y^2) y^k dy, \quad k=0, 1, 2, \dots, \quad (3.12)$$

then we see that the generating function

$$G_{\square}(w) = \sum_{k=0}^{\infty} m_k w^k \quad (3.13)$$

is even and

$$m_{2k} = n_k, \quad m_{2k+1} = 0, \quad k=0, 1, 2, \dots \quad (3.14)$$

The two generating functions are related by

$$G_{\square}(w) = G(w^2). \quad (3.15)$$

Next we combine the two transformations (3.8) and (3.15). Starting with

$$n_k = \int_0^{\Lambda} \lambda^k \sigma(\lambda) d\lambda \quad \text{and} \quad G(w) = \sum_{k=0}^{\infty} n_k w^k, \quad (3.16)$$

and applying the quadratic transformation followed by a translation, we arrive at

$$G_{[\square_1]}(w) = \frac{1}{1-qw} G_{\square}\left(\frac{w}{(1-qw)}\right) = \frac{1}{1-qw} G\left(\frac{w^2}{(1-qw)^2}\right), \quad (3.17)$$

where the support of  $G_{[\square_1]}(w)$  is  $[-\sqrt{\Lambda} + q, \sqrt{\Lambda} + q]$  and its moments are

$$\bar{m}_k = \sum_{p=0}^{[k/2]} \binom{k}{2p} q^{k-2p} n_p, \quad k=0, 1, 2, \dots, \quad (3.18)$$

$[k/2]$  denoting the integer part of  $k/2$ . If we insist that  $\sqrt{\Lambda}$  is an integer, which we can always do by enlarging the support in (3.16), then we arrive at the D. M. P. preserving transformation (3.1), and we see that the original problem with support  $[0, \Lambda]$  is transformed into a problem whose support is  $[0, 2\sqrt{\Lambda}]$ .

The proposition is now proved by repeated application of this transformation as follows. Begin with any value of  $\Lambda > 4$ . Increase  $\Lambda$  until it becomes a perfect integer square. Apply the transformation. Repeat the process. It is readily seen that in this way we can always arrive at a problem whose support is  $[0, 6]$ . For example, for  $\Lambda = 54$ , which corresponds to the critical isotherm of the planar triangular Ising model, we have the sequence

$$\begin{aligned} \Lambda = 54 \text{ enlarged to } 64 &= 8^2 = 64 \text{ trans}^n 16 \\ &= 4^2 \text{ trans}^n 8 \text{ enlarged to } 9^2 = 81 \text{ trans}^n 6. \end{aligned}$$

We have not been able to classify rigorously the classes of solutions that are admitted when  $\Lambda = 6$ , although we have a fair idea of the types of things to be expected as will be seen from the examples given in the next section. The situation to date is summarized in Table I.

## 4. EXAMPLES

### A. A class of algebraic generating functions

We consider the solutions  $G_i(w)$ ,  $i=1, 2, \dots, N$ , of the algebraic equation

$$+p - G(w) + wG(w)^N = 0, \quad (4.1)$$

where  $p$  and  $N$  are positive integers.

*Proposition:* The solution of (4.1) which is regular at  $w=0$  is the generating function for a D. M. P. with

$$\Lambda = \frac{N-1}{p} \left[ \frac{pN}{N-1} \right]^N \quad \text{when } N > 1, \quad \text{and } \Lambda = 1 \text{ when } N = 1.$$

*Proof:* Note that the only possible singular points for any  $G_i(w)$  are  $w = \infty$ ,  $w = 0$ , and

$$w = w_{\text{sing}} = \left( \frac{N-1}{pN} \right)^N \frac{p}{(N-1)} \quad (=1 \text{ when } N=1).$$

The solution with no singularity at zero,  $G_1(z)$ , has the expansion

$$G_1(w) = p + p^N w + \dots, \quad (4.2)$$

wherein all the coefficients are integer. Observing directly from (4.1) that the imaginary part of any  $G_i(w)$  cannot vanish when  $\text{Im}w \neq 0$ , we deduce from (4.2) that

$$\text{Im}G_1(w) < 0 \quad \text{when } \text{Im}w > 0. \quad (4.3)$$

Since the only singularities of  $G_1(w)$  are at  $w_{\text{sing}}$  and infinity, we have by Cauchy's theorem



TABLE I. Summary of what is known about  $G(w)$  as a function of the length of its support.

Support	Nature of $G(w)$	Remarks
$\Lambda = 1$	[1/1] rational fraction	
$\Lambda = 2$	[2/2] rational fraction	
$\Lambda < 4$	rational fraction whose poles have various locations $w = w_k$ $k \in \{1, 2, \dots, m-1\}$ , some $m$ , where $w_k = 1/4 \cos^2(k\pi/m)$ .	Only rational fractions are allowed.
$\Lambda = 4$	If $n_0 = \frac{1}{2}$ while other moments are integer, then either $G(w)$ is a rational fraction with poles at all of the points $w = w_k$ , $k = 1, 2, \dots, m-1$ , where $w_k = 1/4 \cos^2(k\pi/m)$ , some $m$ ; or $G(w) = \frac{1}{2} / \sqrt{1-4w}$ .  In general, $G(w)$ may be a superposition of a rational fraction, an analytic function with branch points of order two at $w = \frac{1}{4}$ and $w = \infty$ , and a Stieltjes function with a natural frontier on $\frac{1}{4} \leq w < \infty$ .	Algebraic functions with branch points of order two are admitted.  Is the possibility of a natural frontier excluded by the positivity constraints?
$4 < \Lambda < 6$	Very little known	No-man's land
$\Lambda = 6$	Examples show that $G(w)$ can have a natural frontier on its second sheet, and that certain hypergeometric functions possessing logarithmic singularities occur. $G(w)$ can also be an algebraic function with high order branch points.	Algebraic functions with high order branch points admitted.  Functions with logarithmic singularities admitted.
$\Lambda > 6$	$G_\Lambda(w)$ is related to $G_6(w)$ by a sequence of purely algebraic transformations.	

$$G_1(w) = \frac{1}{2\pi i} \oint_C \frac{G_1(\xi) d\xi}{(\xi - w)} \quad (4.4)$$

for any  $w$  in the complex plane cut from  $w_c$  to  $\infty$ , and  $C$  is any contour in this cut plane which encloses  $w$ . Choosing  $C$  as in Fig. 5, and letting the circular part tend to infinity it is readily found that the only contribution to (4.4) comes from the integration back and forth along the cut. Since  $G_1(w)$  is real on the real axis for  $w < w_{\text{sing}}$ , the discontinuity along the cut is

$$2 \lim_{\epsilon \rightarrow 0^+} G(x + i\epsilon) = -2\pi i \theta(x), \quad x \geq w_{\text{sing}}, \quad (4.5)$$

where  $\theta(x)$  is positive because of (4.3). Thus we have

$$G_1(w) = \int_{w_{\text{sing}}}^{\infty} \frac{\theta(x)}{x - w} dx. \quad (4.6)$$

Defining  $\sigma(\lambda) = \theta(1/\lambda)/\lambda$  on  $0 \leq \lambda \leq 1/w_{\text{sing}}$  we finally

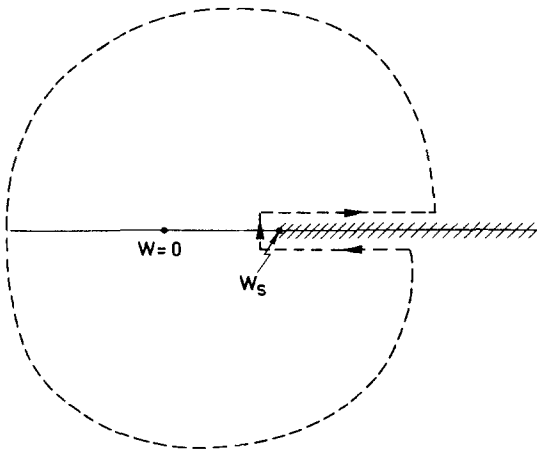


FIG. 5. The contour  $C$  used in (4.4) to reduce  $G_1(w)$  to the form (4.6). The discontinuity on the cut is positive.

have

$$G_1(w) = \int_0^{1/w_{\text{sing}}} \frac{\sigma(\lambda) d\lambda}{1 - \lambda w}, \quad \text{for all } w \text{ in the cut plane,} \quad (4.7)$$

which completes the proof of the proposition.

Notice that for  $p=1$  and  $N$  large,  $\Lambda \sim e^N$ . In this example we have a sequence of algebraic functions which are solutions of D. M. P.'s and whose order increases with increasing  $\Lambda$ .

### B. A generating function with logarithmic singularities

Suppose we have two generating functions

$$G_\Lambda(w) = \sum_{k=0}^{\infty} n_k w^k \quad \text{and} \quad G_{\Lambda'}(w) = \sum_{k=0}^{\infty} n'_k w^k \quad (4.8)$$

belonging to supports  $[0, \Lambda]$  and  $[0, \Lambda']$ , respectively. Then the Hadamard product is defined by

$$G_{\Lambda\Lambda'}(w) = G_\Lambda * G_{\Lambda'}(w) = \sum_{k=0}^{\infty} n_k n'_k w^k. \quad (4.9)$$

$G_{\Lambda\Lambda'}(w)$  is a D. M. P. generating function with support  $[0, \Lambda\Lambda']$  because

$$\begin{aligned} n_k n'_k &= \int_0^\Lambda \int_0^{\Lambda'} (\lambda\lambda')^k \sigma(\lambda) \sigma'(\lambda') d\lambda d\lambda' \\ &= \int_0^{\Lambda\Lambda'} t^k dt \int \sigma'(t/\lambda) \sigma(\lambda) \frac{d\lambda}{\lambda}. \end{aligned} \quad (4.10)$$

Choosing

$$G_4(w) = \sum_{k=0}^{\infty} \binom{2k}{k} w^k = 1/\sqrt{1-4w}, \quad (4.11)$$

we find

$$G_{16}(w) = (G_4 * G_4)(w) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 w^n = F(\frac{1}{2}, \frac{1}{2}, 1; 16w) \quad (4.12)$$

which is the Gauss hypergeometric function. This has a logarithmic singularity at  $w = \frac{1}{16}$ .

Other hypergeometrics involving rational indices and the variable scaled by a suitable integer also give rise to D. M. P. generating functions.<sup>13</sup>

### C. A generating function with a natural boundary on the second sheet<sup>14</sup>

Consider the case  $\Lambda = 8$  so that

$$n_k = \int_0^8 \lambda^k \sigma(\lambda) d\lambda, \quad k=0, 1, 2, \dots \quad (4.13)$$

Then defining  $\tilde{\sigma}(\lambda) = 2\sigma(2\lambda)$  on  $0 \leq \lambda \leq 4$  we find

$$\tilde{n}_k = \int_0^4 \lambda^k d\tilde{\sigma}(\lambda) = n_k/2^k, \quad k=0, 1, 2, \dots \quad (4.14)$$

If we now convert this into a trigonometrical moment problem by applying the transformation (2.26) and (2.27), we find that the trigonometrical moments are of the form

$$\tilde{c}_k = c_k/2^k, \quad k=0, 1, 2, \dots, \quad (4.15)$$

where  $c_k$ 's are integer. Now suppose that the  $\tilde{c}_k$ 's are such that the associated generating function

$$\tilde{I}(z) = \sum_{k=0}^{\infty} \frac{c_k}{2^k} z^k \text{ is holomorphic for } |z| < 2 \quad (4.16)$$

and moreover

$$\operatorname{Re} \tilde{I}(z) \geq c_0/2 \text{ when } |z| < 1, \quad (4.17)$$

so that the associated trigonometrical measure  $\tilde{g}(\theta)$  is nonnegative. Then on inverting the transformation (2.26) and (2.27) we are led to a solution of the D. M. P. (4.13).

Now choose

$$c_0 = 2, \quad c_k = 1 \text{ if } k \text{ is prime, and } c_k = 0, \text{ otherwise.} \quad (4.18)$$

Then (4.16) is satisfied, and moreover

$$\begin{aligned} \operatorname{Re} \tilde{I}(\exp(i\theta)) &= 2 + \sum_{k \text{ prime}} (\cos k\theta)/2^k \\ &\geq 2 - \sum_{k=1}^{\infty} 1/2^k \geq 1 = c_0/2 \end{aligned} \quad (4.19)$$

which is condition (4.17). Thus (4.18) corresponds to a solution of the D. M. P. (4.13), where  $\Lambda = 8$ . Moreover,  $\tilde{I}(z)$  clearly has a natural frontier on  $|z| = 2$  so the solution of the D. M. P. has a natural frontier on its second sheet.

### D. A generating function associated with a Cantor set

We consider the effect of repeated application of the D. M. P. preserving transformation (3.1), starting with  $\Lambda = 6$  and any  $G_6^{(0)}(w)$  whose support is the whole interval  $[0, 6]$ . On increasing  $\Lambda$  to 9 and applying the transformation we obtain  $G_6^{(1)}(w)$  whose support is contained in  $[3 - \sqrt{6}, 3 + \sqrt{6}]$ . Again increasing  $\Lambda = 3 + \sqrt{6}$  to 9, and again applying the transformation, we now obtain a D. M. P. generating function  $G_6^{(2)}(w)$  whose support is contained in  $[3 - (3 + \sqrt{6})^{1/2}, 3 - (3 - \sqrt{6})^{1/2}] \cup [3 + (3 - \sqrt{6})^{1/2}, 3 + (3 + \sqrt{6})^{1/2}]$ . At the  $n$ th iteration we obtain a D. M. P. generating function  $G_6^{(n)}(w)$  whose support is contained in the union of the set of intervals

$$I_j^{(n)} = [a_{2j-1}^{(n)}, a_{2j}^{(n)}], \quad j=1, 2, \dots, 2^{n-1}. \quad (4.20)$$

Here the set  $\{a_k^{(n)}\}_{k=1}^{2^n}$  consists of all the numbers which are expressible in the form

$$3 \pm (3 \pm (3 \pm \dots (3 \pm \sqrt{6})^{1/2} \dots)^{1/2})^{1/2}, \quad (4.21)$$

where the number of square roots involved is  $n$ . These numbers are ordered so that

$$a_1^{(n)} < a_2^{(n)} < \dots < a_{2^n}^{(n)}, \quad (4.22)$$

and the only difference between  $a_{2j-1}^{(n)}$  and  $a_{2j}^{(n)}$  is the sign of the  $\sqrt{6}$  term in their representations in the form (4.21). The following points, which we state here without proof, can now be established. Firstly

$$\bigcup_{j=1}^{2^{n-1}} I_j^{(n)} \supset \bigcup_{j=1}^{2^n} I_j^{(n+1)} \text{ for } n=1, 2, 3, \dots, \quad (4.23)$$

so that each set of intervals forms a covering to all of its successors. Secondly,

$$\lim_{n \rightarrow \infty} \operatorname{meas} \bigcup_{j=1}^{2^{n-1}} I_j^{(n)} = \lim_{n \rightarrow \infty} \sum_{j=1}^{2^n} |a_{2j} - a_{2j-1}| = 0. \quad (4.24)$$

Thus the limiting support, as  $n$  tends to infinity, is the set  $S$  of all points expressible in the form

$$3 \pm \sqrt{(3 \pm \sqrt{(3 \pm \dots \text{ad infimum})})} \quad (4.25)$$

$S$  is uncountable and of measure zero, and is therefore a Cantor set.

The above consideration leads us to seek the answers to the following questions. Does the sequence  $\{G_6^{(n)}(w)\}_{n=1}^{\infty}$  converge to a well defined limiting function  $G_6^{(\infty)}(w)$ ? Is  $G_6^{(\infty)}(w)$  a genuine D. M. P. generating function? What is the nature of the measure of  $G_6^{(\infty)}(w)$ , this being associated with the Cantor set  $S$ ?

*Proposition:* Let  $G_6^{(0)}(w)$  be any given D. M. P. generating function with support on  $[0, 6]$ . Then the sequence  $\{G_6^{(n)}(w)\}_{n=1}^{\infty}$  converges to a function  $G_6^{(\infty)}(w)$ , the convergence being uniform for  $w \in \Omega$ , where  $\Omega$  is any closed bounded region interior to the complex plane cut from  $w = \frac{1}{6}$  to  $w = \infty$ .

Furthermore,  $G_6^{(\infty)}(w)$  is a fixed point of the transformation (3.1) corresponding to  $\Lambda = 9$ . It is a bona fide D. M. P. generating function with support  $[0, 6]$  and is unique up to multiplication by a positive integer  $n_0$ , its Taylor series expansion about the origin being

$$G_6^{(\infty)}(w) = n_0 \sum_{k=0}^{\infty} g_k w^k, \quad (4.26)$$

where the  $g_k$ 's are integers given recursively by the relations

$$\begin{aligned} g_0 &= 1, \\ \sum_{p=0}^{2s+1} g_p (-3)^{-p} \binom{2s+1}{p} &= 0, \end{aligned} \quad (4.27)$$

and

$$\sum_{p=0}^{2s+2} g_p (-3)^{-p} \binom{2s+2}{p} = (-3)^{-2(s+1)} g_{s+1}, \quad s=0, 1, 2, \dots$$

The series (4.26) converges for  $|w| < \frac{1}{6}$ .

*Proof:* First we note that if (3.1) has a fixed point

$G_6^{(\infty)}(w)$  which is regular in some neighborhood of  $w=0$ , then its Taylor series expansion about  $w=0$  would have to be (4.26) where the  $g_n$ 's are given by (4.27). This can be seen by substituting (4.26) into both sides of (3.1) wherein  $\Lambda=9$ , and equating coefficients. Furthermore, it is easy to see that the  $g_k$ 's defined by (4.27) are all integers.

Now let us start with any D.M.P. generating function  $G_6^{(0)}(w)$  with  $G_6^{(0)}(0)=n_0$  and  $\Lambda=6$ , and let us apply (3.1)  $n$  times to obtain a new D.M.P. generating function  $G_6^{(n)}(w)$  belonging to  $\Lambda=6$ . Set

$$G_6^{(n)}(w) = \sum_{k=0}^{\infty} g_k^{(n)} w^k. \quad (4.28)$$

Then a little algebra provides that

$$g_k^{(n)} = n_0 g_k \text{ for } k=0, 1, \dots, (2n-1), \quad (4.29)$$

where the  $g_k$ 's are those defined by (4.27). Moreover, we know from the previous proposition that  $G_6^{(n)}(w)$  is a D.M.P. generating function with support  $[0, 6]$ , and hence

$$g_k^{(n)} = \int_0^6 \lambda^n \sigma^{(n)}(\lambda) d\lambda \text{ for } k=0, 1, 2, \dots, \quad (4.30)$$

where  $\sigma^{(n)}(\lambda)$  is a nonnegative measure on  $[0, 6]$ . In addition, from (4.29) it follows that

$$g_0^{(n)} = n_0 \text{ and } g_1^{(n)} = 3n_0 \text{ for } n \geq 1. \quad (4.31)$$

Applying the theory of upper and lower principal representations<sup>16</sup> to (4.30) and (4.31) we obtain

$$3^k n_0 \leq g_k^{(n)} \leq \frac{1}{2} \cdot 6^k n_0 \text{ for all } k > 0 \text{ and } n \geq 1. \quad (4.32)$$

Combining this with (4.29) we now have

$$3^k \leq g_k \leq \frac{1}{2} \cdot 6^k, \text{ for all } k > 0, \quad (4.33)$$

which says in particular that the series (4.26) is absolutely convergent for  $|w| < \frac{1}{6}$ . Let us denote the function thus defined by  $G_6^{(\infty)}(w)$ .

Finally we prove that

$$G_6^{(\infty)}(w) = \int_0^6 \frac{\sigma(\lambda) d\lambda}{1 - \lambda w} \text{ for some nonnegative } \sigma(\lambda), \quad (4.34)$$

and that  $\{G_6^{(n)}(w)\}_{n=0}^{\infty}$  converges uniformly to  $G_6^{(\infty)}(w)$  for all  $w \in \Omega$ , where  $\Omega$  is any closed bounded region interior to the complex plane cut from  $w = \frac{1}{6}$  to  $\infty$ .

To prove (4.34), we recall that

$$F(w) = \int_0^{\infty} \frac{\theta(\lambda) d\lambda}{(1 - w\lambda)} \text{ for some nonnegative } \theta(\lambda) \quad (4.35)$$

if and only if<sup>17</sup>  $D_{0,k}(F_0, F_1, \dots, F_{2k}) \geq 0$  and  $D_{1,k}(F_1, F_2, \dots, F_{2k+1}) \geq 0$  for all  $k$  where

$$D_{l,k}(F_1, F_{1+1}, \dots, F_{2k+1}) = \text{Det} \begin{vmatrix} F_l & F_{l+1} & \dots & F_{l+k} \\ F_{l+1} & F_{l+2} & \dots & F_{l+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{l+k} & F_{l+k+1} & \dots & F_{l+k+1} \end{vmatrix} \text{ for } l=0 \text{ and } 1, \quad (4.36)$$

and

$$k! F_k = \lim_{w \rightarrow 0} \frac{d^k F(w)}{dw^k} \left( = \int_0^{\infty} \lambda^k \theta(\lambda) d\lambda \right). \quad (4.37)$$

But  $D_{l,k}(n_0 g_l, n_0 g_{l+1}, \dots, n_0 g_{l+k}) = D_{l,k}(g_l^{(2k+1)}, g_{l+1}^{(2k+1)}, \dots, g_{l+k}^{(2k+1)}) \geq 0$  for  $l=0$  and 1 because  $G_6^{(2k+1)}(w)$  is certainly of the form (4.35), and we have (4.29). It now follows that  $G_6^{(\infty)}(w)$  is expressible in the form (4.35). In addition (4.33) implies that the measure is zero for  $w > 6$ , and thus we have (4.34). In particular, the Stieltjes moment problem associated with the set of moments  $\{n_0 g_k\}_{k=0}^{\infty}$  is determinate: Thus<sup>18</sup> the set of moments  $\{n_0 g_k\}_{k=0}^{\infty}$  defines a lense-shaped region  $\omega_N(n_0 g_0, \dots, n_0 g_N; w)$  in the complex plane, for each  $w \in \Omega$ , such that for any function of the form (4.35) whose first  $(N+1)$  moments are precisely  $n_0 g_0, n_0 g_1, \dots, n_0 g_N$ , we have

$$F(w) \in \omega_N(n_0 g_0, \dots, n_0 g_N; w) \text{ for all } w \in \Omega, \quad (4.38)$$

where

$$\text{Size } \omega_N(n_0 g_0, \dots, n_0 g_N; w) \rightarrow 0 \text{ as } N \rightarrow \infty, \text{ uniformly for } w \in \Omega. \quad (4.39)$$

The size of a set  $S$  in the complex plane is simply  $\text{Max}\{|w_1 - w_2| : w_1 \in S, w_2 \in S\}$ . Let  $\epsilon > 0$  be given. Choose  $N$  so large that  $\text{Size } \omega_N(n_0 g_0, \dots, n_0 g_N; w) < \epsilon$  for all  $w \in \Omega$ . Then we clearly have

$$|G_6^{(\infty)}(w) - G_6^{(n)}(w)| < \epsilon \text{ for all } w \in \Omega \quad (4.40)$$

for all  $n > N$ , which completes the proof.

We are now in a position to specify the measure  $\sigma_6^{(\infty)}(\lambda)$  of  $G_6^{(\infty)}(w)$ , and to show how this is defined over the Cantor set  $S$ . Let the measure associated with  $G_6^{(n)}(w)$  be  $\sigma_6^{(n)}(\lambda)$ , and define

$$\theta_6^{(n)}(\lambda) = \int_0^{\lambda} \sigma_6^{(n)}(x) dx \text{ for all } n=0, 1, 2, \dots, \quad (4.41)$$

so that

$$G_6^{(n)}(w) = \int_0^6 \frac{d\theta_6^{(n)}(\lambda)}{(1 - w\lambda)}. \quad (4.42)$$

Then using the uniform convergence of the sequence  $\{G_6^{(n)}(w)\}$ , we have from the theory of moments that

$$\lim_{n \rightarrow \infty} \theta_6^{(n)}(\lambda) = \theta_6^{(\infty)}(\lambda), \text{ uniformly for } \lambda \in [0, 6]. \quad (4.43)$$

Moreover, from the proposition, the result must be independent of the choice of starting function  $G_6^{(0)}(w)$ , subject to the normalization condition

$$G_6^{(0)}(0) = \theta_6^{(0)}(6) = 1. \quad (4.44)$$

Now, the relation between the successive  $\sigma_6^{(n)}(\lambda)$ 's is

$$\sigma_6^{(n+1)}(\lambda) = |\lambda - 3| \sigma_6^{(n)}((\lambda - 3)^2), \quad n=0, 1, 2, \dots. \quad (4.45)$$

Choosing for simplicity

$$\sigma_6^{(0)}(\lambda) = \delta(\lambda - 6) \quad (4.46)$$

we find

$$\sigma_6^{(1)}(\lambda) = \frac{1}{2} [\delta(\lambda - (3 - \sqrt{6})) + \delta(\lambda - (3 + \sqrt{6}))]. \quad (4.47)$$

Iterating, we thus have

$$\sigma_6^{(n)}(\lambda) = \frac{1}{2^n} \sum_{k=1}^{2^n} \delta(\lambda - \alpha_k^{(n)}), \quad (4.48)$$

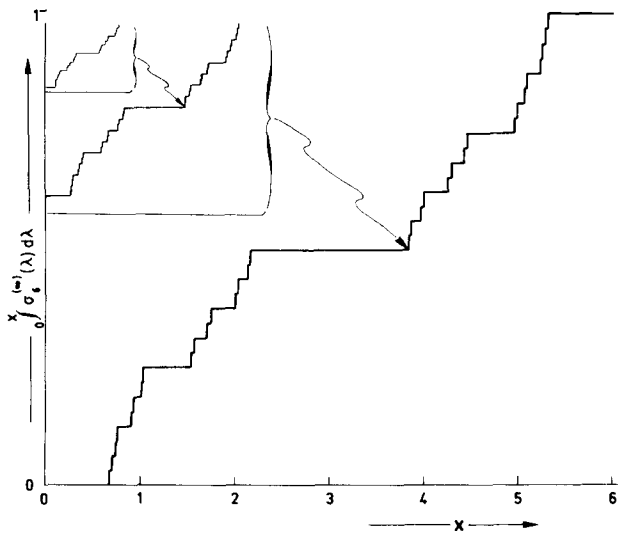


FIG. 6. The devil's staircase. This represents the integrated measure  $\int_0^x \sigma_6^{(\infty)}(\lambda) d\lambda$  associated with the D.M.P. generating function  $G_6^{(\infty)}(w)$ . Each "vertical" segment has the detailed structure shown in the inset.

where the  $a_k^{(n)}$ 's are defined in (4.21). Hence

$$\theta_6^{(n)}(\lambda) = \frac{1}{2^n} \times \{\text{The number of } a_k^{(n)}\text{'s less than or equal to } \lambda\}. \quad (4.49)$$

Taking the limit as  $n \rightarrow \infty$ , we obtain  $\theta_6^{(\infty)}(\lambda)$  which is an example of the infamous devil's staircase, illustrated in Fig. 6. It is a continuous bounded monotone nondecreasing function whose derivative exists and is zero at every point except on the Cantor set  $S$ .

More generally we can consider the fixed point,  $G_{2q}^{(\infty)}(w)$ , of the transformation (3.1) corresponding to an initial D.M.P. generating function  $G_{2q}^{(0)}(w)$  with support on  $[0, 2q]$ , where  $q$  is an integer  $\geq 2$ . Starting with  $\Lambda = 2q$ , one increases  $\Lambda$  to  $q^2$  and applies (3.1) to obtain a new D.M.P. generating function  $G_{2q}^{(1)}(w)$  with  $\Lambda = 2q, \dots$ , and so on. The analysis proceeds just as before except that the role of 3 is played by  $q$  throughout.  $G_{2q}^{(\infty)}(w)$  is a D.M.P. generating function which has for support the set of all numbers of the form

$$q \pm \sqrt{q} \pm \sqrt{q \pm \sqrt{q} \pm \dots \text{ad infimum} \dots}). \quad (4.50)$$

This ensemble appears to be a Cantor set of measure zero for all  $q > 2$ . When  $q = 2$  the ensemble is dense in  $[0, 4]$  and we find

$$G_4^{(\infty)}(w) = \frac{n_0}{\sqrt{1-4w}}, \quad (4.51)$$

where  $n_0$  is a positive integer, and the corresponding measure is

$$\sigma_4^{(\infty)}(\lambda) = \frac{(n_0/2\pi)}{\sqrt{\lambda(4-\lambda)}}. \quad (4.52)$$

What happens in this limiting case is that all of the gaps in the support of the measure become filled up and the resulting measure is very smooth.

We note that the support of  $G_{2q}^{(\infty)}(w)$  is in fact contained in  $[q - \sqrt{\Lambda_0}, \Lambda_0]$ , where

$$\Lambda_0 = [2q + \sqrt{4q + 1}]/2. \quad (4.53)$$

In particular, on supposing that near  $\Lambda_0^{-1}$

$$G_{2q}^{(\infty)}(\Lambda_0^{-1} - \epsilon) \sim \epsilon^\alpha + \text{higher order in } \epsilon, \quad (4.54)$$

and substituting into the fixed point equation

$$G_{2q}^{(\infty)}(w) = \frac{1}{1-qw} G_{2q} \left( \frac{w^2}{(1-qw)^2} \right) \quad (4.55)$$

we obtain after some calculation the consistent conclusion that

$$\alpha = \ln \Lambda_0^{1/2} / [\ln \Lambda_0 - \ln 2(\Lambda_0 + q\Lambda_0^{1/2})]. \quad (4.56)$$

Choosing  $n$ , not necessarily an integer, so that  $\Lambda_0 = 4^n$ , and the integer  $q = 4^n - 2^n$ , we find the index

$$\alpha = -n/(n+1), \quad (4.57)$$

which agrees with (4.51) where  $n = 1$ .

These functions have many interesting properties—analyticity in a cut plane, positive discontinuity, expansion with integer coefficients, and behavior like  $(w_s - w)^\alpha$  near the first singular point  $w_s = \Lambda_0^{-1}$ . The occurrence of such functions in our problem was unexpected and a physical interpretation of them can only be for the moment hypothetical. However, functions of this type occur elsewhere in a physical context—in particular, reference are made to such functions in recent studies of crystallographic structures.<sup>19,20</sup>

## 5. CONCLUSION AND OUTLOOK

An interesting result of our analysis is the quantification of the D.M.P. when the length of the support is less than four. The fact that all supports greater than four can be reduced to a length six is a mathematical equivalence which may correspond to some physical property such as universality. We have not been able to classify the families of solutions which appear when  $4 < \Lambda \leq 6$ , and suspect that such a classification must depend not only on  $\Lambda$  but on other parameters as well, in distinction from the case  $\Lambda \leq 4$ .

The nature of families of solutions admitted when  $\Lambda > 4$  might be classifiable in terms of the smoothness of the measure  $\sigma(\lambda)$ . Various solutions of the D.M.P. with continuous  $\sigma(\lambda)$  are provided by the hypergeometric functions  $F(a, b, c; Tz)$  with  $a$  and  $b$  rational but not both integers,  $c$  integer, and  $T$  an integer which depends upon  $a, b$ , and  $c$ . Similarly, solutions are also provided by functions which are algebraic over hypergeometrics,  $F(a_1, a_2, \dots, a_r; zT)$ .<sup>13</sup>

An insight into the arithmetical nature of the critical indices and an arithmetical interpretation of their universality will only be revealed when the D.M.P. has been completely classified. However, we anticipate that the class of transformations introduced in Sec. 3 may have surprising consequences for the physical Ising model considered at the outset. These transformations admit fixed points, the simplest of which corresponds exactly to the known solution for the one-dimensional Ising chain. For the moment we can only guess that this transformation may be interpreted as a scale

transformation in both the magnetic field and temperature variables. In general we ask if the strange devil's staircase type functions play a role related to the behavior of the thermodynamical functions in a purely imaginary field. The latter has received attention only very recently,<sup>10</sup> being treated from the renormalization point of view. For us, this provides a motivation, over and above mathematical interest, to extend the investigation to the case of polynomial moments and to see if similar transformations prevail.

More generally, the study of the moment problem when the moments belong to special classes, such as integers, rings, discrete sets, etc., may provide a complete new insight into statistical mechanics models as well as quantum systems such as field theory, bearing in mind that most of the physical problems of this sort can be formulated as moment problems on abstract fields.

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<sup>1</sup>L. Onsager, Phys. Rev. **65**, 117 (1944); B. M. McCoy and T. T. Wu, *The Two-Dimension Ising Model* (Harvard U. P., Cambridge, Mass., 1973).

<sup>2</sup>T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952); D. Ruelle, Commun. Math. Phys. **31**, 265 (1973).

<sup>3</sup>F. Hausdorff, *Set Theory* (Chelsea, New York, 1962).

<sup>4</sup>C. Domb, *Phase Transition and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1974), Vol. 3, p. 357.

<sup>5</sup>J. D. Bessis, J. M. Drouffe, and P. Moussa, J. Phys. A **9**, 2105 (1976).

<sup>6</sup>Analogous Stieltjes representation for the intensity of magnetization was introduced in G. A. Baker, Jr., Phys. Rev. Lett. **20**, 990 (1968).

<sup>7</sup>See Eq. (3.18) in Ref. 5, which contains a misprint and must be read:  $\pi - \Theta_0 \leq (c/2)(\pi - \psi)$ , where  $x = \sin(\psi/2)$ .

<sup>8</sup>See G. A. Baker, Jr., Adv. Theor. Phys. **1**, 1 (1965).

<sup>9</sup>D. S. Gaunt, Phys. Rev. **179**, 174 (1969); G. A. Baker, Jr., and P. Moussa, J. Appl. Phys. **43**, 1360 (1978).

<sup>10</sup>M. E. Fisher, Phys. Rev. Lett. **40**, 1610 (1978).

<sup>11</sup>P. Dienes, *The Taylor series* (Dover, New York, 1957), p. 324.

<sup>12</sup>U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications* (University of California Press, Berkeley and Los Angeles, 1958).

<sup>13</sup>D. and G. Choodnovsky (private communication).

<sup>14</sup>P. Deligne (private communication).

<sup>15</sup>Some of the results in this paragraph have been obtained in collaboration with B. Derrida.

<sup>16</sup>S. Karlin and W. J. Studden, *Chebycheff Systems* (Interscience, New York, London, Sydney, 1966).

<sup>17</sup>J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, Providence, Rhode Island, 1950).

<sup>18</sup>G. A. Baker, Jr., J. Math. Phys. **10**, 814 (1969).

<sup>19</sup>S. Aubry, "The new concept of transitions by breaking of analyticity in a crystallographic model," communication at the Symposium on Non-Linear Soliton Structure and the Dynamic of Condensed Matter, Oxford, England, June 1978.

<sup>20</sup>R. J. Rubin, J. Math. Phys. **11**, 1857 (1970).