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The effects of intermittency on statistical characteristics of turbulence and scale similarity of breakdown coefficients

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The influence of intermittency on turbulent diffusion is expressed in terms of the statistics of the dissipation field. The high-order moments of relative diffusion are obtained by using the concept of scale similarity of the breakdown coefficients (bdc). The method of bdc is useful for obtaining new models and general results, which then can be expressed in terms of multifractals. In particular, the concavity and other properties of spectral codimension are proved. Special attention is paid to the logarithmically periodic modulations. The parametrization of small-scale intermittent turbulence, which can be used for large-eddy simulation, is presented. The effect of molecular viscosity is taken into account in the spirit of the renorm group, but without spectral series, ϵ expansion, and fictitious random forces.

I. INTRODUCTION

The imbedding of the theory of turbulence into the theory of Markov processes has been proposed in a previous paper by the author.¹ It was shown in Ref. 1 that the hypothesis of independent increments of velocity, which is widely used by many authors in the Lagrangian description of turbulence, is inconsistent with the Navier–Stokes equations in a fundamental way. This hypothesis violates the main feature of turbulent flow—the energy transport between motions of various scales. A more general Lagrangian description of turbulent velocity such as the Markov process with dependent increments, which recognizes the condition of incompressibility and the important phenomenon of intermittency, was presented in Ref. 1.

In order to obtain some insight into the behavior of high-order moments and transition probability for the Markov process, the model of intermittent relative motion of fluid particles was proposed.¹ The idea of this model is that strong pressure gradients, which cause the jumps of velocity, are accompanied by strong energy dissipation [most probably in the areas of vortex reconnections, which have been described by the method of three-dimensional solenoidal vortex singularities (vortons)^{2–7}]. The relative velocity of fluid particles [$u(t)$] and the distance between them [$r(t)$] have been connected with the rate of energy dissipation ϵ_l , averaged over the sphere of the radius $l = \langle r^2(t) \rangle_L^{1/2}$, where the symbol $\langle \rangle_L$ means statistical averaging over the Lagrangian ensemble of trajectories. In the inertial range with $r \gg r_0$ (r_0 is the initial distance), from the dimensional argument we have¹

$$u(t) \sim (\epsilon_l t)^{1/2}, \quad r(t) \sim (\epsilon_l t^3)^{1/2}. \quad (1)$$

Statistical averaging over the probability distribution of ϵ_l , which is parametrically dependent on l , is denoted by $\langle \rangle$. Taking into account that integration over the sphere is a linear operation, we have

$$\langle \epsilon_l \rangle = \epsilon, \quad (2)$$

where ϵ is the mean rate of the energy dissipation. From (1) and (2) we immediately obtain relations,

$$\langle u^2(t) \rangle_L \sim \epsilon t, \quad \langle r^2(t) \rangle_L = l^2(t) \sim \epsilon t^3, \quad (3)$$

corresponding to the experimentally supported Richardson law.⁸ The important point is that second-order Lagrangian moments (3) are linearly dependent on ϵ and additional averaging over the fluctuations of ϵ does not change them. In this respect, the second-order Lagrangian moments play the same role of the strong point in the construction of models as the third-order Eulerian moment.¹ Let us note that the Richardson law in the Eulerian description acquires an intermittency correction (see Sec. IV). For high-order Lagrangian moments, we have from (1)

$$\langle u^n(t) \rangle_L \sim \langle \epsilon_l^{n/2} \rangle t^{n/2}, \quad \langle r^n(t) \rangle_L \sim \langle \epsilon_l^{n/2} \rangle t^{3n/2}, \quad (4)$$

where $l(t)$ is determined by (3).

Thus the effect of intermittency on turbulent diffusion is expressed in terms of the statistics of the dissipation field. In Sec. II the statistics of dissipation are described by using the concept of scale similarity of the breakdown coefficients (bdc). It is shown that this concept gives more information about the statistics of the system than simply a multifractal representation. The logarithmically periodic modulations are also considered in Sec. II and an analytical description of the form of these modulations for a specific model is presented in the Appendix. In Sec. III the influence of intermittency on turbulent diffusion is considered in more detail in connection with the imbedding of the theory of turbulence into the theory of Markov processes. In Sec. IV we consider the influence of intermittency on the spectral coefficient of turbulent viscosity, which can be used for large-eddy simulations of intermittent turbulent flows. The effect of molecular viscosity, which is important for a boundary layer, is taken into account. Conclusions and perspective are outlined in Sec. V.

II. SCALE SIMILARITY OF BREAKDOWN COEFFICIENTS AND MULTIFRACTALS

The concept of scale similarity of random fields was developed in Refs. 9 and 10 and we include in this section a brief account of some previous results, which are necessary for obtaining new results in this paper. For simplicity, we

will consider a one-dimensional section of the dissipation field, which is in accord with experimental reading in time (with the aid of the "frozen-flow" hypothesis). Thus ϵ_r is the dissipation (or similar quantity, see Ref. 10) averaged over the segment r . For a flow which is locally isotropic in scales less than a certain external scale L , the two- and three-dimensional statistical characteristics of dissipation have the same form [see exceptions in (32) and (35)].

We shall single out three segments inserted in one another with the lengths $l_* \ll r < \rho < l \ll L$ (l_* is the internal scale, defined below) and introduce corresponding breakdown coefficients (bdc),

$$q_{r,l} \equiv \epsilon_r / \epsilon_l \leq l/r, \quad (5)$$

$$q_{r,l} = q_{r,\rho} q_{\rho,l}. \quad (6)$$

In (5) we utilized the fact that the dissipation rate is non-negative. The scale similarity is determined by the following conditions: (i) probability distribution for bdc depends only on the ratio of the corresponding scales and (ii) $q_{r,\rho}$ and $q_{\rho,l}$ are statistically independent [instead of (ii) we can use a less restrictive condition, see below]. More general conditions, which take into account exact relative positions of segments, are considered in Ref. 10. For the moments of bdc, from conditions of scale similarity and (6), we have

$$a_p \left(\frac{l}{r} \right) = \langle q_{r,l}^p \rangle, \quad a_p \left(\frac{l}{r} \right) = a_p \left(\frac{\rho}{r} \right) a_p \left(\frac{l}{\rho} \right), \quad (7)$$

$$a_p(l/r) = (l/r)^{\mu(p)}, \quad \mu(0) = 0 \quad (8)$$

(in view of the arbitrariness of ρ and normalization of probability). In Refs. 9 and 10 it was shown that

$$\mu(1) = 0, \quad 0 < \mu(2) \equiv \mu < 1, \quad (9)$$

$$\mu(p + \delta) - \mu(p) \leq \delta \quad (\delta \geq 0), \quad (10)$$

$$\mu(p) \leq \mu + p - 2 \quad (p \geq 2). \quad (11)$$

Inequality (10) follows from (5), and (11) follows from (10) and (9). From (5) or (11) it also follows¹⁰ that probability density $w(q, l/r)$ for $q_{r,l}$ is uniquely defined by the set of $\mu(p)$ with integer p ($p = 0, 1, 2, \dots$). Thus we have

$$\begin{aligned} w\left(q, \frac{l}{r}\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \exp(-isq + isq_{r,l}) \rangle ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-isq) \left[\sum_{p=0}^{\infty} \frac{(is)^p}{p!} \left(\frac{l}{r}\right)^{\mu(p)} \right] ds. \end{aligned} \quad (12)$$

If function $\mu(p)$ has analytical continuation to the complex domain, then (12) can be written in the form (compare with Ref. 10)

$$w\left(q, \frac{l}{r}\right) = \frac{1}{2\pi q} \int_{-\infty}^{\infty} \exp\left[-is \ln q + \mu(is) \ln\left(\frac{l}{r}\right)\right] ds. \quad (13)$$

It can be shown that if (8) holds for arbitrary p (including imaginary p), then conditions of scale similarity [(i) and (ii)] follow. For the purposes of this paper it is sufficient if (8) holds for real p [instead of condition (ii)].

In constructing a model, we can make an assumption, say, about $w(q, 2)$. From (8) we obtain

$$\mu(p) = \log_2 \left(\int_0^2 q^p w(q, 2) dq \right). \quad (14)$$

Then, from (12) or (13) we have $w(q, l/r)$. If $w(q, 2)$ contains $\delta(q)$ or $\delta(q - 2)$ (or both), then the limits of integration in (14) are 0 — and 2 +.

Let us differentiate (14) with respect to p :

$$\mu'(p) = \langle q^p \ln q \rangle / \langle q^p \rangle \ln 2 < 1, \quad (15)$$

where $\langle \rangle$ means averaging with the probability density $w(q, 2)$ and we assume that $w(q, 2) \neq \delta(q - 2)$. Differentiation of (15) gives

$$\mu''(p) = \frac{\langle q^{2p} (\ln q)^2 \rangle \langle q^p \rangle - \langle q^p \ln q \rangle^2}{\langle q^p \rangle^2 \ln 2} > 0. \quad (16)$$

We will see below that inequalities (15) and (16) are important for the multifractal representation.

In Refs. 9 and 10 it was shown that if $\mu'(0)$ and $\mu''(0)$ are finite, then the characteristic function of the $\ln q_{r,l}$ tends to normal when $\ln(1/r) \rightarrow \infty$. However, we have this tendency only in the integral sense—in terms of characteristic functions. The probability density $w(q, l/r)$ does not tend to lognormal probability density. The moments (8) generally have nothing to do with the moments of the limiting lognormal distribution, which are defined by¹⁰

$$\mu_*(p) = \frac{1}{2} p \{ (p-1) [\mu_*(2) - 2\mu_*(1)] + 2\mu_*(1) \}, \quad (17)$$

$$2\mu_*(1) = 2\mu'(0) + \mu''(0), \quad \mu_*(2) = 2\mu'(0) + 2\mu''(0), \quad (18)$$

$$\mu_*(2) - 2\mu_*(1) = \mu''(0) > 0. \quad (19)$$

The strict inequality in (19) we obtain from (16), assuming $w(q, 2)$ is not a δ function ($q_{r,2r}$ is random). Formula (17) obviously violates (11) and even $\mu_*(1)$ and $\mu_*(2)$ do not have to coincide with $\mu(1)$ and $\mu(2)$ (see below). Let us note that these results have been misunderstood in Ref. 11.

We get statistical characteristics of ϵ_r assuming that¹⁰ $\epsilon_L \approx \epsilon$. In particular, for the moments we have

$$\langle \epsilon_r^p \rangle \sim \epsilon^p (L/r)^{\mu(p)}. \quad (20)$$

This formula may contain a constant depending on the large-scale structure of turbulent flow. In this sense, the statistical characteristics of bdc are more universal than statistical characteristics of dissipation fields and other fields (velocity, temperature, etc.). The information contained in formulas (10)–(19) and some results presented below, follow naturally from the concept of bdc and then, if it is necessary, they can be expressed in terms of multifractals (see below).

Let us now consider some examples, starting with the simple formula

$$\mu(p) = (p-1)\mu \quad (p > 0), \quad \mu(0) = 0, \quad 0 < \mu \equiv \mu(2) < 1. \quad (21)$$

This formula has been derived in Ref. 12 from a statistical model, based on the direct experimental investigation of intermittency.¹³ After this, the parameter μ has been connected with the so-called fractal dimension¹⁴ \mathcal{D} ($\mu = 3 - \mathcal{D}$) and formula (21) has been used by many authors (see, for example, Ref. 15 and references therein).

From (21) and (12) we obtain

$$w\left(q, \frac{l}{r}\right) = \left[1 - \left(\frac{r}{l}\right)^\mu \right] \delta(q) + \left(\frac{r}{l}\right)^\mu \delta\left[q - \left(\frac{l}{r}\right)^\mu\right]. \quad (22)$$

Physically this model means concentration of the dissipation in a small fraction of volume—asymptotically on a fractal set. This model is actually equivalent to the so-called β model,¹⁶ as was indicated in Ref. 16 and discussed in Ref. 1. The internal scale l_* for the dissipation field can be defined by the condition that the local Reynolds number in areas of high dissipation is of order 1. This gives^{16,17}

$$l_* = l_\nu \Re^{-3\mu/4(4-\mu)}, \quad l_\nu = \nu^{3/4} \epsilon^{-1/4}, \quad \Re = L^{4/3} \epsilon^{1/3} \nu^{-1}, \quad (23)$$

where ν is the molecular viscosity, l_ν is the classical Kolmogorov internal scale, and \Re is the Reynolds number.

A more refined mathematical model of intermittency has been presented in Ref. 17, where the background of relatively weak dissipation (with a nontrivial structure) was taken into account. Additionally, the logarithmically periodic modulations of the statistical characteristics of intermittency have been discovered in Ref. 17. These modulations now have preliminary experimental support¹⁸ and recently have been rediscovered in many fields of science (see, for example, Ref. 19 and the references therein). The form of these modulations provides much more information about the system than just a fractal dimension. In the Appendix we describe analytically the form of modulations for the model presented in Ref. 17.

If the system has a physically distinguished scale factor λ , then we can expect log-periodic modulations. Loosely speaking, if an unstable eddy in turbulent flow typically breaks up into two or three smaller eddies, but not into 10 or 20 eddies, then we can suspect existence of a preferable scale factor. We can also assume that $q_{r,\rho}$ and $q_{\rho,l}$ in (6) are statistically independent only if $\rho/r = \lambda^n$, $l/\rho = \lambda^m$, where n and m are integers. In this case, from (7) we obtain

$$q_p \left(\frac{l}{r} \right) = \left(\frac{l}{r} \right)^{\mu(p)} f_p \left[\ln \left(\frac{l}{r} \right) \right], \quad (24)$$

$$f_p(\ln x + \ln \lambda) = f_p(\ln x), \quad f_p(0) = 1, \quad (25)$$

where function f_p represents the log-periodic modulations. We will obtain the same result if we simply assume that power law (8) holds only for $l/r = \lambda^n$, where n is an arbitrary integer.

The model (21) and (22) can be called unifractal and corresponds to linear dependence $\mu(p)$ (we do not count the jump at $p = 0$). Below we will consider the multifractal representation of intermittency (in terms of singularity sets) for general $\mu(p)$. A very simple multifractal model (without log-periodic modulations) has been presented in Ref. 10:

$$w(q,2) = \frac{1}{2}, \quad 0 \leq q \leq 2. \quad (26)$$

From (14) we have¹⁰

$$\mu(p) = p - \log_2(p+1). \quad (27)$$

The probability density $w(q,l/r)$ calculated in Ref. 10 by using (13) is far from uniform density. This model has no adjustable parameters and describes very well the experimental data available in the early 1970s (see Ref. 10). After this, the data have been revised, mostly because of shifting the interval of similarity. Recent data¹⁸ do not match with (27) or with (21).

The natural multifractal generalization of model (21)

has the form

$$w(q,2) = \sum_{k=1}^n w_k \delta(q - q_k), \quad w_k > 0, \quad 2 \gg q_k \gg 0. \quad (28)$$

From (14) we obtain

$$\mu(p) = \log_2 \left(\sum_{k=1}^n w_k q_k^p \right). \quad (29)$$

Conditions $\mu(0) = 0$ and $\mu(1) = 0$ give

$$\sum_{k=1}^n w_k = 1, \quad \sum_{k=1}^n w_k q_k = 1.$$

A recently proposed model²⁰ (which matches very well with data obtained by the authors), corresponds to the particular case of (29) with $n = 2$ and $w_1 = w_2 = \frac{1}{2}$:

$$\mu(p) = \log_2(q_1^p + q_2^p) - 1 \quad (q_1 = 0.6, q_2 = 1.4). \quad (30)$$

From (30), (28), and (15)–(18) we obtain

$$\mu(1) = 0, \quad \mu(2) \approx 0.2141,$$

$$\mu_*(1) \approx 0.0037, \quad \mu_*(2) \approx 0.2663.$$

Thus the difference between $\mu_*(p)$ and $\mu(p)$ for $p = 1, 2$ definitely exists for this model. For the model (27) this difference is very significant.¹⁰

Another recent model,²¹ which is based on the concept of scale similarity and also agrees very well with the different sets of data, gives

$$\mu(p) = \log_{(6^{1/\mu})} [\Gamma(2p+1)/2^p], \quad \mu = 0.2, \quad (31)$$

where Γ is the gamma function. Let us note that we can extrapolate the probability for $q_{r,l}$ to the probability of the local dissipation and vorticity by taking the limit $r \rightarrow l_*$, $l \rightarrow L$ (see Refs. 1 and 10). However, the inverse procedure (used in Ref. 21)—the interpolation from the one-point distribution of, say, vorticity to the distribution of bdc—is not unique and we can run into trouble by violating the conditions of scale similarity. This is exactly the case with model (31), which contradicts (11) for large p (as was indicated in Ref. 21, but without reasoning, presented above).

The behavior of $\mu(p)$ for negative p depends on the model (14). For the unifractal model (22) we have a finite measure for zero q and $\mu(p)$ does not exist for negative p . For the model [(26) and (27)] there is limitation $p > -1$ and $\mu(p) \rightarrow \infty$ with $p \rightarrow -1$. Model (30) has no limitations for p and asymptotically $\mu(p) \sim |p|$ ($|p| \gg 1$). Generally, because of concavity (16), $\mu(p)$ increases with increasing $|p - p_m|$, where p_m corresponds to the negative minimum of $\mu(p)$ (also see Sec. III) and $\mu(p) > 0$ for $p < 0$.

Various characteristics of intermittency related to $\mu(p)$ have been used in the literature. One of these characteristics is the so-called generalized dimension²²

$$\mathcal{D}_p = d - \mu(p)/(p-1) \quad [\mathcal{D}_1 = d - \mu'(1)], \quad (32)$$

where d is the geometric dimension ($d = 1$ for the one-dimensional section of the field).

Another interesting representation of $\mu(p)$ is²³

$$\mu(p) = - \min_h [ph + c(h)]. \quad (33)$$

Here $c(h)$ is the fractal codimension of the set of points $S(h)$

for which the field has a singularity of order h . In terms of bdc it means

$$q_{r,l} \sim (r/l)^h. \quad (34)$$

Let us note that in Ref. 23 the set $S(h)$ was defined differently with the additional condition $c'(h) < 0$. We do not require this condition. Formula (33) corresponds to the Legendre transform²⁴ of $c(h)$. Codimension $c(h)$ represents the space factor, which was first introduced in the simple model^{12,13} and in terms of bdc [see (22)] equal to $(r/l)^\mu$, where μ is the codimension of the unifractal. For multifractal, instead of μ , we have a spectral codimension $c(h)$. By averaging expression $(r/l)^{ph+c(h)}$ over h and using the saddle-point method, we obtain (33). Related characteristics of singularity sets—the fractal dimension

$$f(\alpha) = d - c(\alpha - 1), \quad (35)$$

have been introduced in Ref. 25.

From (33) we have

$$c'(h_*) = -p, \quad c''(h) > 0, \quad (36)$$

$$\mu(p) = -ph_* - c(h_*), \quad (37)$$

where $h_*(p)$ is the solution of (36). The concavity of codimension (36) (or equivalent conditions) has been postulated in Refs. 23 and 25. By using bdc, we are able to prove this condition. From (37) and (36) we have

$$\mu'(p) = -h_*(p), \quad \mu'' = -h'_*(p), \quad c''(h_*)h'_*(p) = -1. \quad (38)$$

Inequalities (15) and (16) give

$$h > -1, \quad c''(h) > 0 \quad (39)$$

[the zero on the rhs of (16) is obtained only for a unifractal model (21)]. From (36)–(38) we see that (33) can be inverted:

$$c(h) = -\min_p [ph + \mu(p)], \quad (40)$$

$$c(h) = -hp_*(h) - \mu(p_*), \quad (41)$$

$$\mu'(p_*) = -h, \quad (42)$$

where $p_*(h)$ is the solution of (42). From (41), (42), (15), and (8) we obtain

$$p_*(h_m) = 0, \quad h_m = -\mu'(0) \\ = -\langle \ln q \rangle / \ln 2, \quad c(h_m) = 0, \quad (43)$$

where h_m corresponds to the minimum of $c(h)$. Therefore we proved another necessary condition: $c(h) \geq 0$.

III. THE INFLUENCE OF INTERMITTENCY ON TURBULENT DIFFUSION

From (1), (3), and (20) we obtain, for the mixed moments of relative velocity and distance between fluid particles,

$$\langle u^m(t)r^n(t) \rangle_L \sim \epsilon^{(m+n)/2} t^{(m+3n)/2} (T/t)^{(3/2)\mu(m+n)/2} \\ (T = L^{2/3} \epsilon^{-1/3}), \quad (44)$$

where T is the external time scale. In particular, for the unifractal model (21) we have

$$\langle u^m(t)r^n(t) \rangle_L \\ \sim \epsilon^{(m+n)/2} t^{(m+3n)/2} (T/t)^{(3/2)[(m+n)/2-1]\mu}. \quad (45)$$

We hope that these formulas will stimulate detailed experimental and numerical studies of relative diffusion in turbulent flows. In the meantime, we can use these results for advancing the theory.

In Ref. 1, the imbedding of the theory of turbulence into the theory of Markov processes with dependent increments of velocity is presented. One of the major goals of this theory is to be consistent with the Navier–Stokes equations and with the phenomena of intermittency. There are still some open questions. One of these questions is the following. For the inertial range we have [Eq. (11) in Ref. 1]

$$\frac{\partial}{\partial t} \langle u^2(t) \rangle_L = 2 \langle u_i(t)a_i(t) \rangle_L + C_1 \epsilon \quad (C_1 > 0). \quad (46)$$

Here $u_i(t)$ is the vector of relative velocity of fluid particles, the last term on the rhs of (46) represents random forcing (in the simplest model), C_1 is constant, and a_i represents the local relaxation. The relaxation, consistent with incompressibility and with a correct expression for the third-order structural tensor of the Eulerian velocity field (see Ref. 1), has the form

$$a_i(t) = -C\tau^{-1}[r(t)]u_i(t), \quad \tau(r) = \langle u^2(r) \rangle_E \epsilon^{-1}, \quad (47)$$

$$2(C-2) = C_1. \quad (48)$$

Here τ is the characteristic time for the motion with the scale r , $r(t)$ is the distance between fluid particles, $u(r)$ is the Eulerian velocity difference between two points separated by distance r , and $\langle \rangle_E$ means the averaging over the Eulerian ensemble of velocity fields. The question is whether or not this relaxation prevents the Richardson law (3). In other words, we have to check whether or not the first term on the rhs of (46) is smaller than the second term. We can verify this by numerical simulations of trajectories (Ref. 1). However, it is interesting to check this analytically by using the result (44), obtained from a different approach. Let us do this.

From the Kolmogorov formula

$$u(r) \sim \epsilon^{1/3} r^{1/3} \quad (49)$$

and (20), we have

$$\langle u^n(r) \rangle \sim \epsilon^{n/3} r^{n/3} (L/r)^{\mu(n/3)}. \quad (50)$$

From (50), (47), (1), and (20), after simple calculations, we arrive at the expression

$$\langle u_i(t)a_i(t) \rangle_L \sim -\epsilon(t/T)^{\mathcal{H}}, \quad (51)$$

$$\mathcal{H} = \frac{3}{2} \left[\mu \left(\frac{2}{3} \right) - \mu \left(\frac{2}{3} + \frac{\mu(\frac{2}{3})}{2} \right) \right]. \quad (52)$$

The unifractal model (21) gives $\mathcal{H} = \mu^2/4 > 0$. Thus, in the inertial range ($t \ll T$), the local relaxation will not prevent the increase of relative velocity. The models (27), (30), and (31) give the same result: $\mathcal{H} > 0$. For the lognormal model

$$\mu(p) = p(p-1)(\mu/2), \quad 0 < \mu < 1 \quad (53)$$

[which contradicts (11), but can be used as an approximation for small p], we also obtain $\mathcal{H} > 0$. Generally, because

of the conditions $\mu(0) = \mu(1) = 0$ and $\mu''(p) > 0$, we have $\mu(\frac{2}{3}) < 0$. From all models and from experimental data, we know that the intermittency correction to the "two-thirds law" is small. Thus (52) can be written in the form

$$\mathcal{H} \approx -\frac{2}{3}\mu(\frac{2}{3})\mu'(\frac{2}{3}). \quad (54)$$

From the conditions indicated above, it follows that function $\mu(p)$ has a negative minimum at some point p_m between 0 and 1. If $p_m < \frac{2}{3}$, as it is for all considered models, then $\mu'(\frac{2}{3}) > 0$ and $\mathcal{H} > 0$.

In recent papers^{26,27} there are conjectures (based on the unifractal model) that the intermittency correction to the "two-thirds law" has a different sign; the energy spectra in the inertial range is flattened by intermittency and depends on molecular viscosity. Experimental data on dissipation fields²⁰ agree very well with the concept of scale similarity of bdc; in particular, they indicate that $\mu(\frac{2}{3}) < 0$. Experimental data on the velocity moments¹⁸ correspond to (50) with the same $\mu(p)$ as in (20), at least for high-order moments,²⁰ when intermittency corrections are significant. This means that the Kolmogorov formula (49) is correct for these moments. The above indicated conjectures imply that the Kolmogorov formula (49) is incorrect for the moment of second order.

Having in mind that there is no intermittency correction for the moment of third order, it is very desirable to have detailed experimental verification of small intermittency corrections in (50) for $n < 3$.

IV. PARAMETRIZATION OF SMALL-SCALE INTERMITTENT TURBULENCE

For the large-eddy numerical simulations of turbulent flows it is necessary to have a parametrization of small-scale turbulence. We propose in this section a simple parametrization, which takes into account the effect of intermittency.

First of all, we need an expression for the so-called spectral coefficient of turbulent viscosity $\nu(k)$, depending on the wavenumber k . We will obtain this expression in the spirit of the renorm group,²⁸⁻³⁵ but without the ϵ expansion around the dimension $d = 4$ and without the fictitious random forces (Gaussian and δ correlated in time).

In the range of wavenumbers larger than L^{-1} , we can express $\nu(p)$ in terms of $\nu(k)$, p, k , and ϵ , keeping in mind that $k \gg p$ and $\nu(k)$ depends on the molecular viscosity ν :

$$\nu(p) = \nu(k)\psi[p/k, \epsilon/k^4\nu^3(k)], \quad \psi(1, x) \equiv 1. \quad (55)$$

Let us differentiate (55) with respect to p and put $p = k$. We obtain

$$\nu'(k) = \frac{\nu(k)}{k} \phi\left(\frac{\epsilon}{k^4\nu^3(k)}\right), \quad \phi(k) = \frac{\partial\psi(y, x)}{\partial y} \Big|_{y=1}. \quad (56)$$

In order to specify $\phi(x)$, we assume that it is calculated in the lowest nontrivial approximation, which involves the source of energy, characterized by ϵ . In this approximation, the rhs of (56) should be proportional to ϵ , which implies that

$$\phi(x) = -4cx/3, \quad (57)$$

where c is constant. From (56) and (57), with the condition

$\nu(\infty) = \nu$, we have

$$\nu(k) = (c\epsilon/k^4 + \nu^3)^{1/3}. \quad (58)$$

It is difficult to calculate the constant c without additional assumptions. In the renorm-group approach to the turbulence, the Gaussian and δ correlated in time random forces have been used with the isotropic spectrum

$$\mathcal{F}_{ij}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2) B k^{-3}, \quad (59)$$

where constant B has the same dimensions as ϵ . We may think that these forces are the most suitable for the description of self-similar turbulence and for calculation of the constant c . However, for Gaussian, δ correlated in time forces, we have the exact relation³⁶:

$$\epsilon = 2\pi \int_0^\infty \mathcal{F}_{ii}(k) k^2 dk, \quad (60)$$

and this integral diverges for the spectrum (59). In order to obtain finite dissipation, we have to cut spectrum (59) on both ends:

$$\epsilon = 4\pi B \ln(k_{\max}/k_{\min}), \quad (61)$$

which breaks the similarity. We will not try to deal with the representation of the Navier-Stokes equation in terms of divergent spectral series nor calculate constant c analytically, keeping in mind that the effect of intermittency leads to an additional constant, accumulating with c .

We can rewrite (58) in the space representation:

$$\nu(r) = (c\epsilon r^4 + \nu^3)^{1/3} \quad (rk \sim 1). \quad (62)$$

In the inertial range from (62) we obtain the Richardson law in the Eulerian representation⁸:

$$\nu(r) = c^{1/3} \epsilon^{1/3} r^{4/3}. \quad (63)$$

It is natural to describe the effect of intermittency by substitution ϵ_r instead of ϵ in (62) and (63). In the inertial range, by using (20), we have

$$\langle \nu(r) \rangle \sim \epsilon^{1/3} r^{4/3} (L/r)^{\mu(1/3)}. \quad (64)$$

Generally,

$$\langle \nu(r) \rangle = \int_0^{L/r} (c\epsilon r^4 q + \nu^3)^{1/3} w\left(q, \frac{L}{r}\right) dq, \quad (65)$$

where $w(q, L/r)$ is the probability density for the bdc, $q, L \approx \epsilon_r / \epsilon$. For the unifractal model (21) and (22), we obtain

$$\langle \nu(r) \rangle = \left[1 - \left(\frac{r}{L}\right)^\mu\right] \nu + \left(\frac{r}{L}\right)^\mu \left[c\epsilon r^4 \left(\frac{L}{r}\right)^\mu + \nu^3 \right]^{1/3}. \quad (66)$$

For the model (30), formula (13) with $L/r = 2^n$ (n is an integer) gives

$$w(q, 2^n) = 2^{-n} \sum_{k=0}^n \binom{n}{k} \delta(q - q_1^{n-k} q_2^k). \quad (67)$$

From (65) and (67) we have

$$\langle \nu(r) \rangle = 2^{-n} \sum_{k=0}^n \binom{n}{k} (c\epsilon r^4 q_1^{n-k} q_2^k + \nu^3)^{1/3}. \quad (68)$$

For nonhomogeneous, statistically stationary turbulent flow, the mean dissipation rate ϵ and external scale L generally depend on the space coordinates. For the unstratified flow over the flat boundary, $L \sim z$, where z is the distance from the boundary. If $\epsilon(z)$ is unknown for large-eddy simu-

lation, it can be connected with the strain rate of the resolved velocity field \bar{v}_i :

$$\epsilon = 2\langle v(r) \rangle s^2(r), \quad s^2 = s_{ij}^2, \quad (69)$$

$$s_{ij}(r) = \frac{1}{2} \left(\frac{\partial \bar{v}_i}{\partial x_j} + \frac{\partial \bar{v}_j}{\partial x_i} \right).$$

For the inertial range, from (64) and (69), we obtain

$$\langle v(r) \rangle \sim r^2 s(r) (L/r)^{3/2\mu(1/3)}. \quad (70)$$

This is the generalization of the Smagorinski scheme³⁷ with the correction for intermittency. Because $\mu(\frac{1}{3}) < 0$ (see Sec. III), this correction reduces the turbulent viscosity. It may partly explain why the numerical coefficient in the formula, analogous to (70) but without an intermittency correction, is usually chosen small (of order 0.07).³⁸ Close to the boundary, the molecular viscosity becomes important and the more general formulas, presented above, can be useful.

V. CONCLUSION AND PERSPECTIVE

The concept of scale similarity of breakdown coefficients (bdc) is shown to be more informative than simply a multifractal representation of intermittency in terms of singularity sets. In the present paper we applied this concept to the intermittent turbulent diffusion and to the parametrization of small-scale intermittent turbulence for large-eddy simulation. Additionally, log-periodic modulations in the presence of a physically distinguished scale factor, have been considered. The concept of scale similarity of bdc can be applied not only for turbulence, but also for a variety of physical systems that display the phenomena of intermittency.

The important unsolved problem is whether the singularities are real or just a way of representation of intermittency. In the recent paper by the author,³⁹ the hierarchy of closed systems of ordinary differential equations, which describe the formation of singularities for the Euler and for the Navier–Stokes equations, is derived. The analytical solutions presented in Ref. 39 suggest that singularities are real, at least for the Euler equations. We plan to unify this dynamical approach with the statistical description in terms of bdc and Markov processes in order to achieve a deeper understanding of the nature of turbulent flows.

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APPENDIX: LOGARITHMICALLY PERIODIC MODULATIONS

According to the mathematical model of intermittency,¹⁷ let us consider the system of random points x_{k_1} , distributed by the Poisson law with a mean density σ . We connect with each of the points x_{k_1} a system of points $x_{k_2} = x_{k_1} + l_1 \theta_{k_1 i}$ ($i = 1, \dots, m$) where $\theta_{k_1 i}$ are mutually independent stochastic variables with the same probability density $\rho(\theta)$. In the transition from the stage s to the stage $s + 1$, we let

$$x_{k_{s+1}} = x_{k_s} + l_s \theta_{k_s i} \quad (i = 1, \dots, m), \quad l_s = l_1 \lambda^{s-1}, \quad (A1)$$

$$0 < \lambda < m\lambda < 1. \quad (A2)$$

The similarity is expressed in having the same number, and distribution law of the variables $\theta_{k_s i}$ for the same scale reduction factor λ .

Let us define a stationary stochastic function $y_s(x)$ consisting of a sequence of pulses with centers at points x_{k_s} :

$$y_s(x) = \langle y \rangle \alpha_s^{-1} \int I_s \left(\frac{x-x'}{l_s} \right) z_s(x') dx', \quad (A3)$$

$$\alpha_s = \alpha_1 \alpha^{s-1}, \quad \alpha_1 = \sigma l_1, \quad \alpha = m\lambda < 1, \quad (A4)$$

$$z_s(x) = \sum_{k_s} \delta(x - x_{k_s}), \quad \langle z_s \rangle = \sigma m^{s-1}, \quad (A5)$$

$$\int I_s(\theta) d\theta = 1, \quad \langle y_s \rangle = \langle y \rangle, \quad (A6)$$

where $z_s(x)$ is a stochastic density function. We interpret the statistical characteristics of a cascade-intermittent stochastic function (in the given construction) as the limit to which the corresponding statistical characteristics of the function $y_s(x)$ tend as $s \rightarrow \infty$.

From (A3)–(A6) we obtain the following expression for the spectrum of the random function $y_s(x)$:

$$F_s(p) = \langle y \rangle^2 \sigma^{-1} \Psi_s(p) \mathcal{F}_s(pl_s), \quad (A7)$$

$$\mathcal{F}_s(q) = \left| \int I_s(\theta) \exp\{iq\theta\} d\theta \right|^2, \quad (A8)$$

$$\mathcal{F}_s(0) = 1, \quad \mathcal{F}_s(q) < 1, \quad (A9)$$

$$\Psi_1(p) = 1, \quad \Psi_s(0) = 1, \quad (A10)$$

where $\Psi_s(p)$ is the normalized spectrum of $z_s(x)$. From (A1) and (A5), after some simple calculations (see Ref. 17), we obtain the recursion formula:

$$\Psi_{s+1}(p) = \Psi_s(p) g(pl_s) + \gamma^s [1 - g(pl_s)] \quad (\gamma = m^{-1}), \quad (A11)$$

$$g(q) = \left| \int \rho(\theta) \exp\{iq\theta\} d\theta \right|^2, \quad g(0) = 1, \quad g(q) < 1. \quad (A12)$$

With successive application of (A11), taking (A1) and (A10) into account, we have

$$\Psi_{s+1}(p) = \prod_{k=1}^s g(pl_k \lambda^{k-1}) + \sum_{k=1}^s \gamma^k [1 - g(pl_k \lambda^{k-1})] \prod_{n=k+1}^s g(pl_n \lambda^{n-1}) \quad (A13)$$

Introducing the notation

$$\eta(pl_1) = \lim_{s \rightarrow \infty} \Psi_s(p), \quad G(q) = \prod_{k=1}^{\infty} g(q\lambda^{k-1}), \quad (A14)$$

we obtain the following series from (A13):

$$\eta(q) = (1 - \gamma) \sum_{k=0}^{\infty} \gamma^k G(q\lambda^k). \quad (A15)$$

This series is equivalent to the functional equation

$$\eta(q) = \gamma \eta(q\lambda) + (1 - \gamma) G(q). \quad (A16)$$

By substituting

$$\eta(q) = q^{-1+\mu} \phi(\ln q / \ln \lambda) \quad (0 < \mu = \ln(m\lambda) / \ln \lambda < 1) \quad (\text{A17})$$

into (A16), we obtain

$$\phi(z) = \phi(z+1) + \chi(z), \quad (\text{A18})$$

$$\chi(z) = (1-\gamma)\gamma^z G(\lambda^z). \quad (\text{A19})$$

Function (A19) decreases exponentially when $z \rightarrow \infty$, because $\gamma < 1$, $\lambda < 1$, and $G(0) = 1$. The solution of (A18) has the form

$$\phi(z) = \sum_{k=0}^{\infty} \chi(z+k). \quad (\text{A20})$$

We assume that $g(q) \rightarrow 0$ when $q \rightarrow \infty$. It can be shown in this case that $G(q)$ falls off more rapidly at infinity than any power of q . Therefore, when $z \rightarrow \infty$, from (A20) we have

$$\phi(z) \rightarrow \phi_*(z) = \sum_{k=-\infty}^{\infty} \chi(z+k), \quad \phi_*(z+1) = \phi_*(z). \quad (\text{A21})$$

If we assume that the shapes of the pulses do not vary too much with increasing s , formulas (A1) and (A9) imply that the last factor on the rhs of (A7) is equal to unity in the limit. We finally obtain

$$F(p) = \lim_{s \rightarrow \infty} F_s(p) = \langle y \rangle^2 \sigma^{-1} (pl_1)^{-1+\mu} \phi_*(\ln pl_1 / \ln \lambda) \quad (pl_1 \gg 1). \quad (\text{A22})$$

According to the previous formulas, the form of the log-periodic modulations $\phi^*(z)$ depends on the probability density $\rho(\theta)$. For the three-dimensional isotropic random field $y(\mathbf{x})$ the analogous construction gives the same result (A22) for the spectra, integrated over the sphere in wave space. It will be interesting in the future to calculate (analytically or numerically) the forms of log-periodic modulations for the high-order moments of $y(\mathbf{x})$ with particular $\rho(\theta)$.

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