Spin Liquids and BCS Hamiltonian with Majorana Fermions in Kitaev Ladders

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Introduction

We first emphasize the connection between magnetism and Majorana fermions, by giving a reminder on the representations of a spin-$\frac{1}{2}$ particle, in terms of Pauli matrices, fermionic operators and Majorana fermions. A single spin-$\frac{1}{2}$ fermion, in the quantum mechanics language, is represented by a Lie algebra, for example the Pauli matrices $\{\sigma^x, \sigma^y, \sigma^z\}$ or $\{\sigma^+, \sigma^-, \sigma^z\}$, with $\sigma^x = \sigma^+ + \sigma^-$ and $\sigma^y = \frac{1}{i}(\sigma^+ - \sigma^-)$. Using the Jordan-Wigner transformation, one can describe the spin with fermionic operators $\hat{a}^\dagger$,$\hat{a}$, s.t. :

$$\sigma^+ = \hat{a}^\dagger \, , \sigma^- = \hat{a} \, , \sigma^z = 1 - 2\hat{a}^\dagger\hat{a}$$

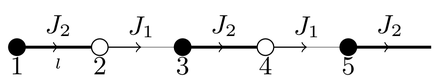
One can also choose to replace the fermionic operator $\hat{a}$ by two Majorana fermions η_1 and η_2, verifying $\eta_{1,2} = \eta_{1,2}^\dagger$, such that :

$$\hat{a} = \eta_1 - i\eta_2 \, , \hat{a}^\dagger = \eta_1 + i\eta_2$$

Note that these Majorana operators can be understood as qubits located on the site of the spin. Experimentally, such qubits are still difficult to manipulate, but there is ongoing research on this topic.
Here, we study a single spin chain system and a two-legged spin ladder system in the Majorana language, which enables us to obtain an exact diagonalization and study the energy spectrum and phase diagram of these models. We remind the mapping between the single spin chain and the BCS Hamiltonian, and show the connection between the two-legged spin ladder and interchain supraconductivity.

1D Magnetic Kitaev Chain

Model and exact diagonalization



$$H = \sum_{j=2m-1} J_1\sigma_j^x\sigma_{j+1}^x + J_2\sigma_{j+1}^y\sigma_{j+2}^y$$

Figure 1: Magnetic Kitaev chain and Hamiltonian in spin representation ; the interaction between nearest neighbors is attractive : $J_1, J_2 < 0$.

Fermionic representation

The quantum spin operators are replaced by fermionic operators $\hat{a}^\dagger, \hat{a}$ using Jordan-Wigner :

$$\begin{cases} \sigma_j^x = (\hat{a}_j^\dagger + \hat{a}_j)e^{i\pi\sum_{\{i\}\in string} a_i^\dagger a_i} \\ \sigma_j^y = -i(\hat{a}_j^\dagger - \hat{a}_j)e^{i\pi\sum_{\{i\}\in string} a_i^\dagger a_i} \\ \sigma_i^z = 2\hat{a}_j^\dagger\hat{a}_j - 1 = \begin{cases} 1, \text{if } |\uparrow\rangle_z \\ -1, \text{if } |\downarrow\rangle_z \end{cases} \end{cases}$$

In this language :

$$H = \sum_{j=2m-1} J_1(\hat{a}_j^\dagger - \hat{a}_j)(\hat{a}_{j+1}^\dagger + \hat{a}_{j+1}) - J_2(\hat{a}_{j+1}^\dagger + \hat{a}_{j+1})(\hat{a}_{j+2}^\dagger - \hat{a}_{j+2}) \quad (1)$$

Majorana representation

1 fermion operator on site $j \equiv 2$ Majorana fermions :

$$\hat{c}_j = \begin{cases} i(\hat{a}_j^\dagger - \hat{a}_j) \, , j = 2m \\ \hat{a}_j^\dagger + \hat{a}_j \, , j = 2m-1 \end{cases} , \hat{d}_j = \begin{cases} \hat{a}_j^\dagger + \hat{a}_j \, , j = 2m \\ i(\hat{a}_j^\dagger - \hat{a}_j) \, , j = 2m-1 \end{cases}$$

In this language :

$$H = -i\sum_{j=2m-1} J_1\hat{c}_j\hat{c}_{j+1} - J_2\hat{c}_{j+1}\hat{c}_{j+2}$$

$\Rightarrow$ involves only the $\hat{c}$ operators ; the $\hat{d}$ operators are unpaired.

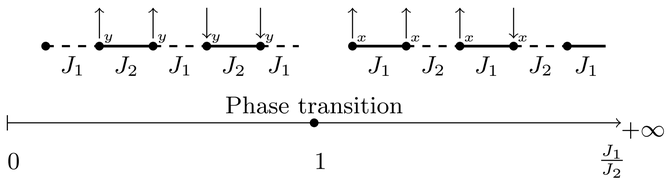
Mapping to BCS Hamiltonian

Fourier transform of (1) :

$$H = \sum_k 2(J_1 + J_2)\cos(kl)\hat{a}_k^\dagger\hat{a}_k + i(J_1 - J_2)\sin(kl)(\hat{a}_{-k}^\dagger\hat{a}_k^\dagger + \hat{a}_{-k}\hat{a}_k) \equiv H_{BCS}$$

Gap : $\Delta_k = i(J_1 - J_2)\sin(kl)$.
Energy spectrum : $\epsilon(k) = \pm\sqrt{J_1^2 + J_2^2 + 2J_1J_2\cos(2kl)}$

Phase Diagram



$$\frac{J_1}{J_2} \gg 1 : \langle\sigma_j^x\sigma_{j+k}^x\rangle \propto e^{-\frac{|r_j - r_{j+k}|}{\xi}}$$

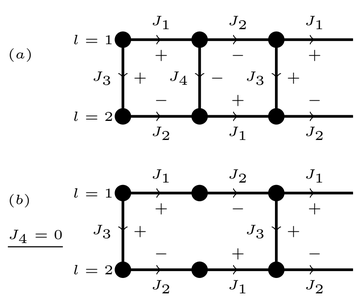
$$\frac{J_1}{J_2} \approx 0 : \langle\sigma_j^y\sigma_{j+k}^y\rangle \propto e^{-\frac{|r_j - r_{j+k}|}{\xi}}$$

$$\xi \propto |J_1 - J_2|^{-1}$$

Figure 2: The gapped phases at $\frac{J_1}{J_2} > 1$ and $\frac{J_1}{J_2} < 1$ are spin liquids, characterized by a coherence lenght $\xi \propto |J_1 - J_2|^{-1}$. There is a phase transition at $J_1 = J_2$.

Two Legged Spin Ladder

Model and exact diagonalization



$$H = H_1 + H_2 + H_3$$

$$H_1 = \sum_{j=2m-1} J_1\sigma_{j,1}^x\sigma_{j+1,1}^x + J_2\sigma_{j+1,1}^y\sigma_{j+2,1}^y$$

$$H_2 = \sum_{j=2m-1} J_1\sigma_{j+1,2}^x\sigma_{j+2,2}^x + J_2\sigma_{j,2}^y\sigma_{j+1,2}^y$$

$$H_3 = \sum_{j=2m-1} J_3\sigma_{j,1}^z\sigma_{j,2}^z + J_4\sigma_{j+1,1}^z\sigma_{j+1,2}^z \, , m \in [\![1, M]\!]$$

Figure 3: 2 coupled magnetic Kitaev chains (ladder), in (a) the general case and (b) in the limit $J_4 = 0$; the interactions are attractive : $J_1, J_2, J_3, J_4 < 0$.

Note 1: the phase diagram for the case $J_3 = J_4$ in the ladder (a) has been studied by [1]
Note 2: the 2D version of the model (b), the so called honeycomb lattice (see fig. 4), has been solved exactly by Kitaev [2]

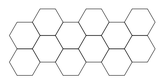


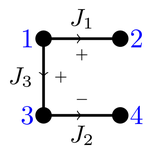
Figure 4: Honeycomb lattice.

Majorana representation

$$c_{j,l} = \begin{cases} i(a_{j,l}^\dagger - a_{j,l}) \, , j + l = 2m \\ a_{j,l}^\dagger + a_{j,l} \, , j + l = 2m-1 \end{cases} , d_{j,l} = \begin{cases} a_{j,l}^\dagger + a_{j,l} \, , j + l = 2m \\ i(a_{j,l}^\dagger - a_{j,l}) \, , j + l = 2m-1 \end{cases} \quad (2)$$

With the appropriate $\mathbf{Z}_2$ gauge choice, the Hamiltonian becomes :

$$H = -i\sum_{s\lambda,t\mu} J_{s\lambda,t\mu}c_{s\lambda}c_{t\mu}$$



Here, instead of labelling the sites with two integers, (j,l), $j \in [\![1, M]\!]$, $l \in \{1, 2\}$, we changed the notation to (s, λ), where $s \in [\![1, M]\!]$ denotes the cell index, and $\lambda \in [\![1, 4]\!]$ denoting the position of the site in a cell (see figure on the left).

In Fourier space :

$$H = -i\sum_{k,\lambda,\mu} X^T \begin{pmatrix} 0 & \alpha & \beta & 0 \\ -\alpha^* & 0 & 0 & 0 \\ -\beta^* & 0 & 0 & -\alpha^* \\ 0 & 0 & \alpha & 0 \end{pmatrix} X, \begin{cases} \alpha = J_1e^{ikl} + J_2e^{-ikl} \\ \beta = J_3e^{il} \end{cases} , X = \begin{pmatrix} c_{-k,1} \\ c_{-k,2} \\ c_{-k,3} \\ c_{-k,4} \end{pmatrix}$$

Phase diagram

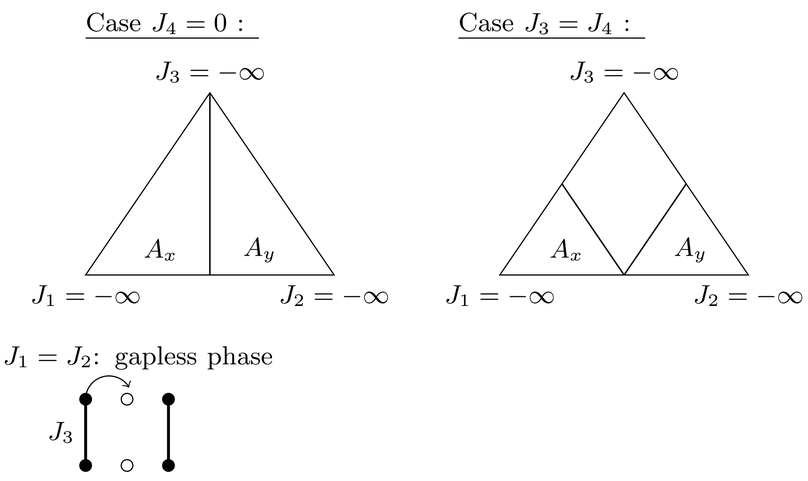


Figure 5: Left : $J_4 = 0$; there are two gapped phases A_x and A_y at $\frac{J_1}{J_2} > 1$ and $\frac{J_1}{J_2} < 1$ respectively (spin liquids) ; the $J_1 = J_2$ line is gapless for all J_3 (interchain SC). Right : $J_3 = J_4$ case ; the three gapped phases are separated by two gapped phases (black lines) ; the phases A_x, A_y are similar to those from the $J_3 = 0$ case.

Conclusion and Perspectives

We used the Majorana representation to study the two legged spin ladder. In the $J_3 = J_4$ limit, we found results consistent with [1]. We also obtained the full phase diagram for the $J_3 = 0$ case ("1D honeycomb"), and we predict an interchain supraconductivity behaviour in the gapless phase at $J_1 = J_2$, which disappears as soon as $J_1 \neq J_2$ (A_x, A_y phases described by Kitaev [2].
Further perspectives include the study of the dopped two legged spin ladder, and of the decorated Kagome lattice.

References

[1] T. Xiang. X. Y. Feng, G.-M. Zhang. Topological characterization of quantum phase transitions in a spin-1/2 model. *Phys. Rev. Lett.*, 98:(8):087204, 2007.

[2] A. Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321, 2006.