

Correlation function along the quantum critical line

$T_3 = 0$  bosonization study predict ①  
 $\sigma^z \mapsto (a^\dagger a)$   $\langle \sigma_i^z \sigma_j^z \rangle$

when we switch on  $b_{12}$ , we open a mass term  $\approx \frac{1}{|i-j|^2}$

$$m^* \approx \frac{\Delta}{|4J_1|} |b_{12} b| \frac{1}{2-2K_-} \quad K_- < 1$$

with the bosonization for chain 1

$\sigma_1^z \mapsto (a_{1\uparrow}^\dagger a_{1\uparrow})$   $\langle \sigma_1^z(x) \sigma_1^z(0) \rangle \approx \frac{1}{x^2}$

correlation function along x and y directions

$\langle \sigma_1^+(x) \sigma_1^-(0) \rangle \approx \langle a_{1\uparrow}^\dagger(x) a_{1\uparrow}(0) \rangle$

We keep the most dominant term in Jordan-Wigner string

$a_{1\uparrow}(x) \approx e^{i(\theta_1 \pm \phi_1)(x)}$

when we open a gap in the mode  $(\phi_1 - \phi_2)$ , then we need to re-write  $\phi_1$  and  $\theta_1$  in terms of the gapped mode  $\phi_- = \frac{\phi_1 - \phi_2}{\sqrt{2}}$  and gapless mode,

$\phi_+ = \frac{\phi_1 + \phi_2}{\sqrt{2}}$  and similarly for the  $\theta$  mode. The

mass term in the mode  $\phi_-$  makes the correlation

Junction ~~with~~ ~~the~~ ~~anti~~ ~~symmetric~~ mode as follows:

$$\langle e^{i\phi_1(x)} e^{-i\phi_1(y)} \rangle \approx \text{const} \quad (2)$$

$$\langle e^{i\theta_1(x)} e^{-i\theta_1(y)} \rangle \approx \exp\left(-\frac{|x-y|}{\xi}\right)$$

$\xi \sim m^{-1} \quad (m, \text{mass})$

(These correlation functions come from the anti-symmetric mode correlation functions).

We then conclude that the correlation functions along  $x$  or  $y$  direction decay very rapidly.

$$\langle \sigma_1^+(x) \sigma_1^-(0) \rangle \approx \exp\left(-\frac{x}{\xi}\right)$$

The correlation function in the  $z$  direction evolves slightly as follows:

$$\phi_1 = \frac{\phi_+ + \phi_-}{\sqrt{2}} \quad \sigma_1^z \approx \partial_x \phi_1 \mapsto \partial_x(\phi_+ + \phi_-)$$

$\phi_-$  is pinned to a "classical" value zero and therefore

$$\langle \sigma_1^z(x) \sigma_1^z(0) \rangle \approx \langle \partial_x \phi_+(x) \partial_x \phi_+(0) \rangle$$

$$\approx \frac{1}{x^{2K_+}}$$

$$K_+ > 1$$

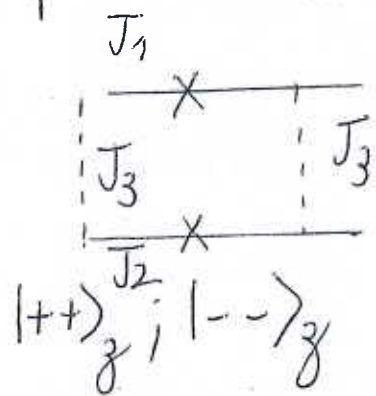
Moreover, increasing  $|J_3|$  means increasing  $|J_{12}|$

Therefore, we conclude that for very large  $J_3$  we obtain " $K_+ = +\infty$ " which means very short-range correlations in the  $z$  direction. (3)

Picture in the limit  $|J_3| \gg |J_1|$ .

$$\langle \sigma_j^z \sigma_{j+k}^z \rangle \propto \exp - \frac{|z_j - z_{j+k}|}{\xi} \\ \xi \propto |J_3|^{-1}$$

perturbation theory in  $J_1$  and in  $J_2$



Let's start with sites  $\{i \& i+1\}$  in  $i$ , say we start with a bond

$$\begin{aligned} & \begin{array}{c} |++\rangle_z^i \\ |--\rangle_z^i \end{array} \\ & \swarrow \quad \searrow \\ & \text{chain 1} \quad \text{chain 2} \\ & J_1 (\sigma_1^x)^i (\sigma_1^x)^{i+1} \left[ \begin{array}{c} |++\rangle_z^i \\ |--\rangle_z^i \end{array} \otimes \left[ \begin{array}{c} |++\rangle_z^{i+1} \\ |--\rangle_z^{i+1} \end{array} \right] \right] \\ & \quad \downarrow \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & J_1 (\sigma_1^x)^{i+1} \left[ \begin{array}{c} | - + \rangle_z^i \\ | - - \rangle_z^i \end{array} \otimes \left[ \begin{array}{c} |++\rangle_z^{i+1} \\ |--\rangle_z^{i+1} \end{array} \right] \right] \\ & = J_1 \left[ \begin{array}{c} | - + \rangle_z^i \\ | - - \rangle_z^i \end{array} \right] \otimes \left[ \begin{array}{c} | - + \rangle_z^{i+1} \\ | - - \rangle_z^{i+1} \end{array} \right] \end{aligned}$$

Now let's focus on the application of  $J_2$  in the (4) lower chain:

$$J_2 (\sigma_2^y)^i (\sigma_2^y)^{i+1} \left[ |-\rangle_z^i \right] \otimes \left[ |-\rangle_z^{i+1} \right]$$

$$\downarrow$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$J_2 i (\sigma_2^y)^{i+1} \left[ |-\rangle_z^i \right] \otimes \left[ |-\rangle_z^{i+1} \right]$$

$$\downarrow$$

$$i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= J_2 \left[ |-\rangle_z^i \right] \otimes \left[ |-\rangle_z^{i+1} \right] J_3 \begin{matrix} \text{X coupling} \\ \vdots \\ \text{X} \\ \vdots \\ \text{X} \\ \text{Y coupling} \end{matrix} \frac{J_1 J_2}{J_3}$$

Conclusion: To second order in perturbation after applying  $(J_1 J_2)$  to the state  $|+\rangle_z^i$  does not polarize the spins in the  $(i+1)$  site. We have the same chance to have "+" or "-"

This is in agreement with the bosonization that predicts shorter & shorter "correlation lengths" ( $R_T \rightarrow +\infty$ ) by increasing  $|J_3|$ .

Ground state at half-filling

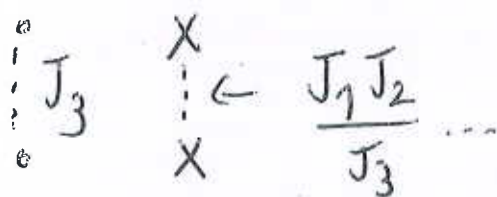
$$|GS\rangle = \prod_i |v v\rangle_z^i$$

$\downarrow$   
 All bonds

$v_i = \pm 0 \text{ or}$   
 (macroscopic quantum degeneracy)

Correlation length "zero"

The cartoon is



On each site, we have the 2 allowed configurations  $|++\rangle_z$  and  $|--\rangle_z$

$$\langle \sigma_i^z \sigma_j^z \rangle \approx \exp - \frac{|i-j|}{\xi}$$

$\xi \approx a$  (As written in the notes page 10)

Note on superconductivity

Physically, Heisenberg type processes come from second-order super-exchanges

$$\sigma_i = c_{i\alpha}^\dagger \sigma_\alpha c_{i\beta}$$

$(c_{i\uparrow}, c_{i\downarrow})$  fermions

Mott phase  $u \gg t \ll a$

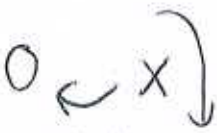
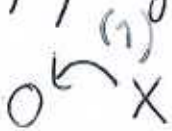
We can then build a  $t$ - $J$  model

(6)

$$H = -t \sum_{\substack{i, i+1 \\ \alpha = \uparrow, \downarrow}} \sum_{\substack{\alpha \\ \alpha}} C_{1, \alpha}^{\dagger}(i+1) C_{1, \alpha}(i) + \underbrace{J_3}_{J_3} \sigma_1^z \sigma_2^z$$

$$J_3 \rightarrow -\infty$$

(1,2) refers to chain 1 and 2



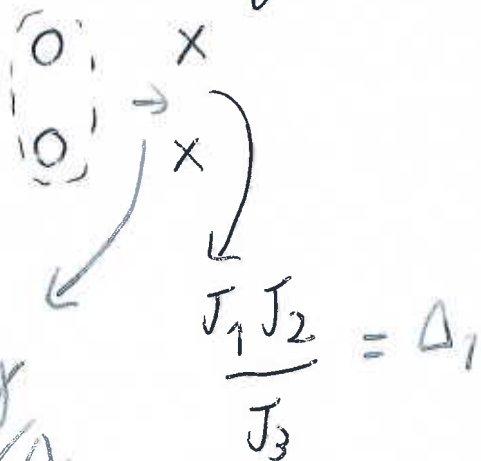
$$\frac{J_1 J_2}{J_3}$$

perturbation theory

Process (1) & (2) is allowed

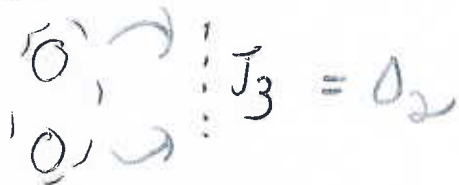
1<sup>st</sup> order processes in  $t$  gives zero because destroy a "magnetic bond".

To second order in  $t$ , the hole pair moves with different hopping amplitudes:



hopping  $\approx t^2 / \Delta_1$

then



hopping  $\approx t^2 / \Delta_2$

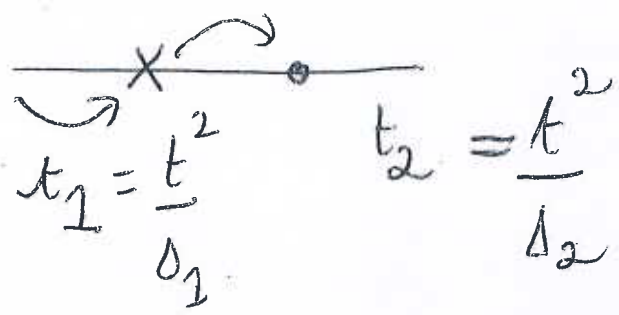
hole pairs are two fermion operators (Boson) (7)

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad c_{1\alpha} c_{2\alpha} \begin{pmatrix} |\uparrow, \uparrow\rangle \\ |\downarrow, \downarrow\rangle \end{pmatrix}$$

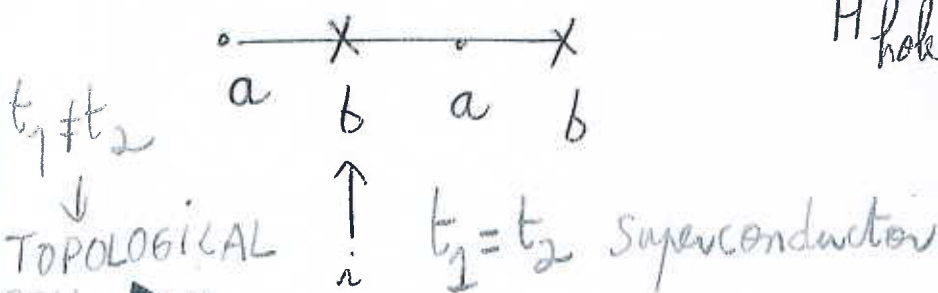
in the dilute limit, hole pairs behave as hard-core bosons [we cannot put "2" on the same site]

boson  
By construction, on each site allowed states are:  
 $|\uparrow\rangle = c_{\uparrow}^{\dagger} |0\rangle$   
 $|\downarrow\rangle = c_{\downarrow}^{\dagger} |0\rangle$   
 One fermion or  $|0\rangle$   
 $|\uparrow\downarrow\rangle$  suppressed by "Mott"

similarly to the usual ladder, the model of hole pairs is a "free fermion model" (dilute limit)



this is similar to 2 inequivalent sites band structure of spinless fermions (hole pair)



$$H_{\text{hole}} = -t_1 \sum_i a_{i\uparrow}^{\dagger} b_i + -t_2 \sum_i a_{i\uparrow}^{\dagger} b_{i+1} + \text{h.c.}$$