

(Evisse note pour Fan et Ariane)

Mapping two-band problem for  $J_1 = J_2$  in each chain

I start from the report by Fan, line (31)

A chain is mapped onto the following hamiltonian

$$H_1 = \sum_k 2(J_1 + J_2) \cos(kl) a_k^\dagger a_k + (J_2 - J_1) i \sin(kl) (a_{-k}^\dagger a_k^\dagger + a_{-k} a_k)$$

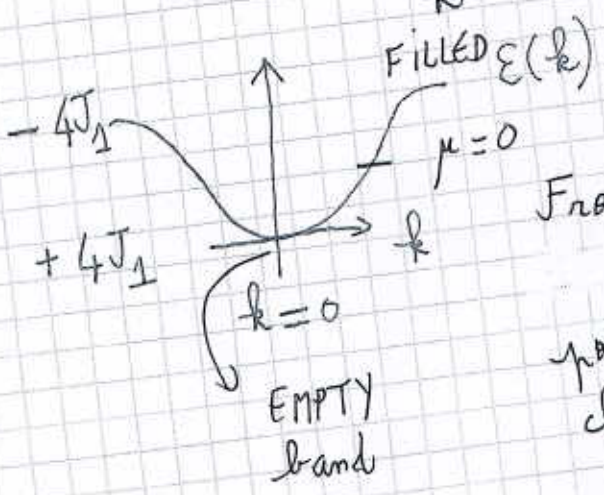
for  $J_1 = J_2$ , as expected we obtain a free fermion hamiltonian characterized by the following band structure

$$H_1 = \sum_k \epsilon(k) a_k^\dagger a_k$$

$$\epsilon(k) = 4J_1 \cos(kl)$$

$J_1 < 0$

$l$  is the lattice spacing



From the spin mapping with half filling, the chemical potential is ZERO in a given chain.

$$\langle a_{i\uparrow}^\dagger a_{i\uparrow} \rangle = \frac{1}{2}$$

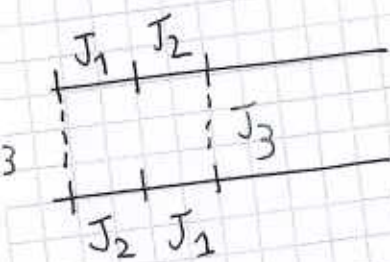
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The second chain has equal couplings  $J_1 = J_2$

$$H_2 = \sum_k \epsilon(k) b_k^\dagger b_k$$

$$\epsilon(k) = 4J_1 \cos(kl)$$

Let's consider the coupling between chains 2)



Let's consider first the case considered by Ariane suggesting that the "gapless" phase appears just on a line

$J_3$  term gives following our notations

$$H_{\perp} = \sum_{\substack{i \text{ even} \\ (\text{or } i \text{ odd})}} J_3 (\sigma_1^i)^z (\sigma_2^i)^z$$

if I use the Jordan-Wigner conventions in Eq. (29) used to diagonalize the spectrum in each chain, then we obtain

$$(\sigma_1^i)^z = (2 a_i^{\dagger} a_i - 1)$$

$$(\sigma_2^i)^z = (2 b_i^{\dagger} b_i - 1)$$

$$H_{\perp} = \sum_{\substack{i \text{ even} \\ (\text{or } i \text{ odd})}} J_3 (2 a_i^{\dagger} a_i - 1) (2 b_i^{\dagger} b_i - 1)$$

we observe two classes of terms

$$H_{\perp}^1 = \sum_{\substack{i \text{ even} \\ (\text{or } i \text{ odd})}} -s_{\mu} (a_i^{\dagger} a_i + b_i^{\dagger} b_i) \quad s_{\mu} = 2J_3 < 0$$

attractive interactions  
 $U = 4J_3 < 0$

$$H_{\perp}^2 = \sum_{\substack{i \text{ even} \\ (\text{or } i \text{ odd})}} U (a_i^{\dagger} a_i) (b_i^{\dagger} b_i)$$

Fourier transform 2M sites in each chain

$$a_i = \frac{1}{\sqrt{2M}} \sum_k e^{ikx_j} a_k$$

$$b_i = \frac{1}{\sqrt{2M}} \sum_k e^{ikx_j} b_k$$

$$-\frac{\delta\mu}{2M} \sum_{j \text{ even}} \sum_{k, k'} e^{i(k-k')x_j} a_k^\dagger a_{k'} \quad x_j = 2jl$$

$$\sum_j e^{i(k-k')(2jl)} \xrightarrow{\text{limit where } x=jl \rightarrow 0} \int_0^{+a} dx e^{i(k-k')2x} = I$$

The term at  $+a$  oscillates very rapidly and averages to zero

$$I = -\frac{1}{i(k-k')2}$$

decomposing in real & imaginary parts, the principal value gives a purely imaginary term that will be cancelled with the complex conjugate.

$$I \mapsto +\frac{\pi}{2} \delta(k-k')$$

The chemical potential type of terms gives:

$$-\frac{\delta\mu}{4M} \pi \sum_{k, k'} (a_k^\dagger a_k + b_k^\dagger b_k)$$

continuum limit

$$\sum_{k'} \rightarrow 4M \int \frac{dk'}{\pi}$$

(open boundary condition  $\cos(2Mk') = 0$ )

chemical potential term

$$- \delta\mu \sum_k (a_k^\dagger a_k + b_k^\dagger b_k)$$

$$k' = \frac{\pi(2m+1)}{2(2M)}$$

the bands will be empty approximately when

$$2J_3 = \delta\mu \approx 4J_1$$

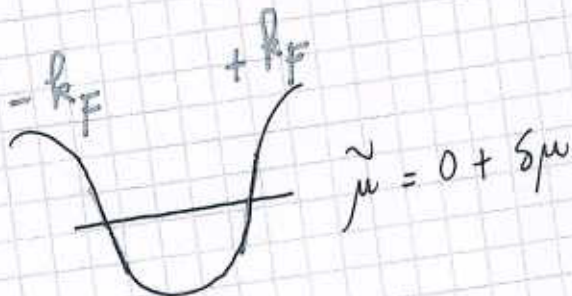
$$J_3 \approx 2J_1$$

(check factor 2)

conclusion 1: for  $J_4 = 0$  it seems we can increase  $J_3$  to a large value and this term is not important until  $J_3 \approx 2J_1$ .

$$H_{\perp}^2 = \sum_{\substack{i \text{ even} \\ (\text{or } i \text{ odd})}} u (a_i^\dagger a_i) (b_i^\dagger b_i)$$

$$u = (4J_3) \text{ ATTRACTIVE interactions}$$



we can linearize the energy spectrum and develop the "fields" around the Fermi points  $+k_F$  and  $-k_F$

$$a(x) = e^{-ik_F x} a_L(x) + a_R(x) e^{ik_F x} \quad 5)$$

L, R refer to left and right movers explicitly

the group velocity is negative for left movers and positive for right movers.

$$b(x) = e^{-ik_F x} b_L(x) + e^{ik_F x} b_R(x)$$

if we develop  $k$  close to  $k_F = \frac{\pi}{2l}$

$$\cos((k_F + q)l)$$

$$= -\sin\left(\frac{\pi}{2}\right) \sin(ql)$$

$$k = k_F + q$$

For the right mover  $q > 0$  (electron-like excitations)

in each chain, the band structure term or kinetic term is written as

$$H_0^a = v \int dk k (a_R^\dagger(k) a_R(k) - a_L^\dagger(k) a_L(k))$$

$\hookrightarrow k$  is the small displacement  $q$  above

$$v = -l 4J_1 > 0 \quad (\text{the velocity has dimension energy} \times \text{length})$$

the 2 bands have the same velocity:

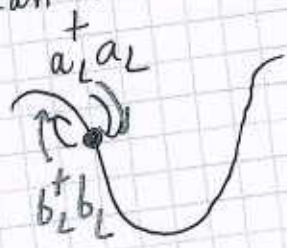
$$H_0^b = v \int dk k (b_R^\dagger(k) b_R(k) - b_L^\dagger(k) b_L(k))$$

this is reminiscent of Eq. (16) in arXiv: 0812.1581

Now, we can classify the interaction terms in  $H_{\perp}^2$  following the same "nomenclature"

$$H_{\perp}^2 = \sum_{\substack{i \text{ even} \\ (i \text{ odd})}} U (a_i^{\dagger} a_i) (b_i^{\dagger} b_i)$$

(note: at a general level terms involving only left movers or right movers can be shown to be NOT important for arguments of "gapped" phase)



the Fermi sea is not "affected"

TERMS of importance

$$J_{12} (a_R^{\dagger} a_R) (b_L^{\dagger} b_L)$$

or

$$J_{12} (a_L^{\dagger} a_L) (b_R^{\dagger} b_R)$$

the bare value of  $J_{12}$  is  $4J_3 < 0$

As we discussed on the board last time, there is potentially another term allowed here because the 2 bands have the same Fermi wave vector: REAL SPACE



$$\int dx \frac{1}{2} ((-1)^x + 1) b^{\dagger} \cdot (a_L^{\dagger} a_R) (b_R^{\dagger} b_L)$$

CONCLUSION 1:  $f_{12}$  cannot produce gapped phase

however, the new term  $b_{12}$  can (this can be shown using bosonization Eq. 25 of arXiv: 0812.1581 with  $(\theta_1 - \theta_2)$  being replaced by  $(\phi_1 - \phi_2)$ )

using bosonization, I get for one case

$$H = \sum_{i=1,2} \int dx \frac{v_i}{2} (\partial_x \phi_i)^2 + \frac{v_i}{2} \pi_i^2 + \int dx \frac{f_{12}}{2\pi} (\partial_x \phi_1) (\partial_x \phi_2)$$

(I assumed that  $(-1)^x$  averaged rapidly to 0)  $-\frac{b_{12}}{(2\pi)^2} \cos \sqrt{4\pi} (\phi_1 - \phi_2)$

The point then is to diagonalize the gradient terms  $\partial_x \phi_+$  &  $\partial_x \phi_-$  (Eq. (26) of arXiv: 0812.1581)

The important point is that all terms are attractive here

$$K_{\pm} = \sqrt{\frac{4\pi v_{\mp} \mp 2f_{12}}{4\pi v_{\pm} \pm 2f_{12}}}$$

$$f_{12} = 4J_3 < 0$$

$\lambda = 0$  in our case

It seems that  $K_+ > 1$   
 $K_- < 1$

CONCLUSION 2: the scaling dimension of the term  $b_{12}$  est  $(2K_-) < 2$ . It seems to me

that the term  $-\frac{b_{12}}{(2\pi)^2} \frac{1}{2} \cos \sqrt{4\pi} (\phi_1 - \phi_2)$  can be important

What is the physics here?

To minimize classically the energy, the ground state will lock the phases  $\phi_1 = \phi_2$   
 (the minus sign  $\phi_-$  acquires a mass:  
 $m^* \sim \Lambda |b_{12} \ell|^{1/2} 2K$

↳ bandwidth  $\approx 14J_1$

The phase is gapless because the mode + is gapless.

Physically since  $J_3 < 0$ , the "particles" in different chains attract each other (ferromagnetic coupling along z direction between chains)

the formed "charged" ( $2e$ ) particles can move freely



This gapless phase seems to be a superconductor with interchain superconductivity

this is similar to the phase we called inter-chain superconductivity.



The question then is why this gapless phase do not survive for  $J_1 \neq J_2$ .

The reason can be understood as follows; if  $J_1$  is not  $J_2$ , there is another "gapped" term in each chain stemming from

$$(J_2 - J_1) i \sin(kl) (a_{-k}^\dagger a_k^\dagger + a_{-k} a_k)$$

and similarly for chain b

This term is a 2-fermion term (as opposed to the  $J_3$  term which is a 4-fermion operator).

Therefore, in the regime where  $|J_2 - J_1|$  is small and  $J_3$  is small, the

and gives a fully gapped phase (2 gapped Kitaev models)

CONCLUSION for  $J_4 = 0$  at relatively small  $J_3$ , we predict a gapless "interchain SC" for  $J_1 = J_2$ , which turns into the Kitaev fully gapped phases

$A_x$  or  $A_y$  as soon as we deviate from the line.

\*\* NOTE for  $J_4 = J_3$  it seems that the physics for  $J_1 = J_2$  would not change much at small  $(J_3)$ . UNTIL  $J_3 \approx J_1$