

Fluctuation Statistics in Networks: a Stochastic Path Integral Approach

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We investigate the statistics of fluctuations in a classical stochastic network of nodes joined by connectors. The nodes carry generalized charge that may be randomly transferred from one node to another. Our goal is to find the time evolution of the probability distribution of charges in the network. The building blocks of our theoretical approach are (1) known probability distributions for the connector currents, (2) physical constraints such as local charge conservation, and (3) a time-scale separation between the slow charge dynamics of the nodes and the fast current fluctuations of the connectors. We integrate out fast current fluctuations and derive a stochastic path integral representation of the evolution operator for the slow charges. The statistics of charge fluctuations may be found from the saddle-point approximation of the action. Once the probability distributions on the discrete network have been studied, the continuum limit is taken to obtain a statistical field theory. We find a correspondence between the diffusive field theory and a Langevin equation with Gaussian noise sources, leading nevertheless to non-trivial fluctuation statistics. To complete our theory, we demonstrate that the cascade diagrammatics, recently introduced by Nagaev, naturally follows from the stochastic path integral. By generalizing the principle of minimal correlations, we extend the diagrammatics to calculate current correlation functions for an arbitrary network. One primary application of this formalism is that of full counting statistics (FCS), the motivation for why it was developed in the first place. We stress however, that the formalism is suitable for general classical stochastic problems as an alternative approach to the traditional master equation or Doi-Peliti technique. The formalism is illustrated with several examples: both instantaneous and time averaged charge fluctuation statistics in a mesoscopic chaotic cavity, as well as the FCS and new results for a generalized diffusive wire.

PACS numbers: 73.23.b, 02.50.r, 05.40.a, 72.70.+m

I. INTRODUCTION

Consider an exclusive night-club with a long line at the entrance. A bouncer is at the front of the line to keep out the rif-raf. At every time step, a person is accepted inside the club with probability p , or rejected with probability $1 - p$. Inside the club, people stay for a while and eventually leave. At every time step, the probability a person leaves is q . We want to answer a question such as “what is the probability that Q people leave the club after t time steps?”.

Assuming that p and q remain constant, the situation is simple and we can easily solve the relevant probabilistic problem. However, in realistic situations this rarely happens: the management wants to make money. If the club is almost empty, they instruct the bouncer to be less discriminating, while if the club is almost full, the bouncer is to be more discriminating. Thus, p becomes a function of the number of people in the club. People will be more likely to leave if the club is very crowded, so q is also a function of the number of people inside the club. The problem posed now is much more difficult because of the presence of feed-back: the elementary processes change in response to the cumulative effect of what they have accomplished in the past.

This simple example captures all the basic features of the problems we wish to consider. Although the example was given with people, the actors in the probability game may be any quantity such as charge, energy, heat or particles, which we will refer to simply as generalized

charge. Similarly, the night club can be a mesoscopic chaotic cavity,¹ a birth-death process,² a biological membrane channel,³ etc.

Historically, general stochastic problems are solved with the master equation. The time rate of change of the probability to be in a particular state is given in terms of transition rates to other states. This approach has had great success and leads naturally to the Fokker-Planck and Langevin equations.⁴ However, once the master equation is given, the solution is often quite difficult to obtain.

This paper takes a different approach. Rather than beginning with a master equation describing the probability of all processes happening in a unit of time, we make several assumptions from which we can reformulate the problem. Although these assumptions limit the applicability of the theory, when they apply, the problems are much easier to solve. The assumptions are:

- The system we are interested in is a composite system made out of constituent parts. In the night club example, the system is made up of three physical regions: outside the front door, the interior of the club, and outside the back door. The decomposition of a larger system into smaller interacting parts is only meaningful for us if there is a separation of time scales. This means that the charge inside the constituent parts changes on a slower time scale than the fluctuations at the boundaries. In the night club example, this simply means that the average time a person spends in the club will be

much longer than the typical time needed to enter the door.

- Taken alone, the parts of the composite system have a finite number of simple properties or parameters. The only property of the night club that was relevant for the problem was the total number of people in it at any given time. The important element of the line out in front is that it never runs out. All other details are irrelevant.
- In the limit where all parts of the network are very large (so that the elementary transport processes do not affect themselves in the short-run), the transport probability distributions between elements are known. In the night club example, the probability of getting Q people through the front door after t time steps (given a constant, large number of people inside) is easy to find, because we have assumed that the elementary probability p does not change from trial to trial. The transport probability distribution is simply the binomial distribution⁴ where the probability p is a function of the (approximately unchanging) number of people inside. The back door distribution is obtained in the same way.
- There are conservation laws that govern the probabilistic processes. No matter what probability distributions we have, there are certain rules that must be obeyed. The net number of people that enter, stay, and leave the club must be a constant. This means that the time rate of change of the club's occupancy is given by the people-current in minus the people-current out. The people in the line outside are a special case. There is in principle always a replacement, so moving one person inside the club doesn't affect the properties of the line.

Now, the strategy is to use this information as the starting point to find transport statistics for the combined interacting system. The main result derived is a path integral expression for the conditional probability (taking conservation laws into account) for starting and ending with a given amount of charge at each location after some time has passed. From this conditional probability, specific quantities such as transport statistics through the system, fluctuation statistics of charge at a particular location and the like may be found.

One primary application of this formalism is that of full counting statistics (FCS),^{5,6} the motivation for why it was developed in the first place.⁷ FCS describes the fluctuations of currents in electrical conductors. It gives the distribution of the probability that a certain number of electrons pass a conductor in certain amount of time. Mean current flow and shot noise¹ correspond to the first and second cumulant of this distribution. The full distribution (defined by all cumulants) provides a full characterization of the transport properties of a electrical conductor in the long time limit. In the

past, FCS was mainly addressed with quantum mechanical tools such as the scattering theory^{5,6,8,9} of coherent conductors, the circuit theory based on Keldysh Green functions,^{10,11,12,13} or the nonlinear σ model.¹⁴ However, a number of works realized that for semi-classical systems with a large number of conductance channels, shot noise may be calculated without accounting for the phase coherence of the electron.^{15,16,17,18} These works treat the basic sources of noise quantum mechanically, but calculate the spread of the noise throughout the conductor classically. For specific conductors like diffusive wires and chaotic cavities, this idea has been extended to the calculation of third and fourth cumulants via the cascade principle^{19,20} and to the full generating function of FCS.^{7,21,22,23} In the present work, we consider an abstract model instead of any particular example and develop the mathematical foundations of the proposed semi-classical procedure to obtain FCS. We introduce and investigate networks of elements with known transport statistics and show how the FCS of the entire network can be constructed systematically.

The formalism we present is related to a different approach in non-equilibrium statistical physics called the Doi-Peliti technique.²⁴ The idea is that once the basic master equation governing the time evolution of probability distributions is given, it may be interpreted as a Schrödinger equation which may be cast into a second-quantized language. This quantum problem is then converted into a quantum mechanical path integral (often obeying bosonic or fermionic statistics) from which one may take the continuum limit and use a field theory renormalization group approach with diagrammatic perturbation expansion.²⁵ This approach is useful in many situations far from equilibrium and has several parallels to our approach. It has been pointed out that this technique is in some sense the classical limit of the quantum mechanical Keldysh formalism,²⁶ the same tool used in the past to calculate FCS, so this gives another connection with the subject matter we are concerned with.

There are several advantages of our approach. First, we skip the master equation step. If the probability distributions of the connector fluctuations are given, we may immediately construct network distributions. Second, from a computational view, our formulation of the problem is much simpler than starting from first principles for situations where the ingredients we need are available, and results are much easier to obtain than beginning with the master equation alone. Thirdly, our formulation also applies to situations where temporal transition probabilities may be large. Finally, the formalism's physical origin is clear, so the needed mathematical objects are well motivated.

The rest of the paper is organized as follows. In Sec. II, we introduce and develop the general theory. After reviewing elements of probability theory, we derive the stochastic path integral for a network of nodes as well as explore the relationship to the master equation and Doi-Peliti formalism. In Sec. III, the continuum limit

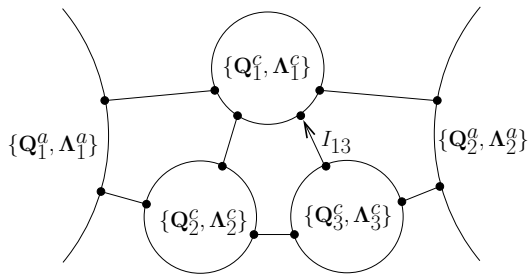


FIG. 1: An arbitrary network. Each node has charge and counting variables $\{\mathbf{Q}_\alpha, \mathbf{\Lambda}_\alpha\}$. The nodes transfer charge via currents $I_{\alpha\beta}$ through the connectors. The absorbed counting fields $\{\mathbf{\Lambda}_\alpha^a\}$ are constants by definition of the absorbed charges \mathbf{Q}_α^a (see text). Each node may have an arbitrary number of different charge species, $\mathbf{Q}_\alpha = \{Q_\alpha^1, Q_\alpha^2, \dots, Q_\alpha^j\}$.

is taken to derive a stochastic field theory and link our formalism with the Langevin equation point of view. In Sec. IV, we develop diagrammatic rules to calculate cumulants of the current distribution as well as current correlation functions for an arbitrary network. Sec. V gives several applications of the theory to different physical situations. We solve the field theory for the mesoscopic wire and demonstrate universality in multiple dimensions as well as present new results for the conditional occupation function and probability distribution. We also consider the problem of charge fluctuation statistics (both instantaneous and time-averaged) in a mesoscopic chaotic cavity. Sec. VI contains our conclusions.

II. GENERAL FORMALISM

Once we have the basic elements of our theory (the generalized charges), we must specify some spatial structure that they move around on. As we noted in the introduction, the essential structure needed to state the problem are simply points we refer to as nodes, joined by connectors. This defines a network (see Fig. 1). The state of each node α is described by one (effectively continuous) charge Q_α ,²⁷ and \mathbf{Q} is the charge vector describing the charge state of the network. The node's state may be changed by transport: flow of charges between nodes takes place via the connectors carrying currents $I_{\alpha\beta}$ from node α to node β . The variation of these charges Q_α is given by

$$Q_\alpha(t + \Delta t) - Q_\alpha(t) = \sum_\beta Q_{\alpha\beta}, \quad (1)$$

where the transmitted charges $Q_{\alpha\beta}(t) = \int_0^{\Delta t} dt' I_{\alpha\beta}(t+t')$ are distributed according to $P_{\alpha\beta}(Q_{\alpha\beta}(t))$. The fact that the probabilities $P_{\alpha\beta}(t)$ also depend on the charges $\mathbf{Q}(t)$ is one source of the difficulty of the problem.

Assuming that the probability distributions $P_{\alpha\beta}$ (which depend parametrically on the state of nodes α and β) of the transmitted charges $Q_{\alpha\beta}$ are known, we seek the

time evolved probability distribution $\Gamma(\mathbf{Q}, t)$ of the set of charges \mathbf{Q} for a given initial distribution $\Gamma(\mathbf{Q}, 0)$. In other words, one has to find the conditional probability (which we refer to as the evolution operator) $U(\mathbf{Q}, \mathbf{Q}', t)$ such that

$$\Gamma(\mathbf{Q}, t) = \int d\mathbf{Q}' U(\mathbf{Q}, \mathbf{Q}', t) \Gamma(\mathbf{Q}', 0). \quad (2)$$

We assume that there is a separation of time scales, $\tau_0 \ll \tau_C$, between the correlation time of current fluctuations, τ_0 , and the slow relaxation time of charges in the nodes, τ_C . As we will show in the next section, this separation of time scales allows us to derive a stochastic path integral representation for the evolution operator,

$$U(\mathbf{Q}_f, \mathbf{Q}_i, t) = \int \mathcal{D}\mathbf{Q}\mathcal{D}\mathbf{\Lambda} \exp\{S(\mathbf{Q}, \mathbf{\Lambda})\}, \quad (3a)$$

$$S(\mathbf{Q}, \mathbf{\Lambda}) = \int_0^t dt' [-i\mathbf{\Lambda} \cdot \dot{\mathbf{Q}} + (1/2) \sum_{\alpha\beta} H_{\alpha\beta}(\mathbf{Q}, \lambda_\alpha - \lambda_\beta)], \quad (3b)$$

where the vector $\mathbf{\Lambda}$ has components λ_α : node variables conjugated to the Q_α that impose charge conservation in the network.

In the following, we define the functions $H_{\alpha\beta}$ as the generating functions of the fast currents between nodes α and β . On the time scale $\Delta t \gg \tau_0$, the currents through isolated connectors are Markovian, so that all cumulants (irreducible correlators which are denoted by double angle brackets) of the transmitted charge $\langle\langle (Q_{\alpha\beta})^n \rangle\rangle$ are linear in Δt . Following the standard notation in mesoscopic physics,²⁸ we define the current cumulants $\langle\langle (\tilde{I}_{\alpha\beta})^n \rangle\rangle$ as the coefficients in

$$\langle\langle (Q_{\alpha\beta})^n \rangle\rangle = \Delta t \langle\langle (\tilde{I}_{\alpha\beta})^n \rangle\rangle, \quad (4)$$

where the tilde symbol has been introduced to distinguish the bare currents of each connector (the sources of noise) from the physical currents $I_{\alpha\beta}$ flowing through that same connector when it is placed into the network. Then the generators $H_{\alpha\beta}$ are defined via the equation

$$\langle\langle (\tilde{I}_{\alpha\beta})^n \rangle\rangle = \left. \frac{\partial^n H_{\alpha\beta}(\mathbf{Q}, \lambda_{\alpha\beta})}{(i\partial\lambda_{\alpha\beta})^n} \right|_{\lambda_{\alpha\beta}=0}, \quad (5)$$

and thus contain complete information about the statistics of the noise sources. The $\lambda_{\alpha\beta}$ [eventually to be replaced with $\lambda_\alpha - \lambda_\beta$ in Eq. (3)] is the generating variable for the current $\tilde{I}_{\alpha\beta}$. The notion of current cumulants is useful because they are the time independent objects, and thus have a time independent generators, Eq. (5). The generators $H_{\alpha\beta}(\mathbf{Q}, \lambda_{\alpha\beta})$ depend in general on the full vector \mathbf{Q} and not just on the generalized charges of the neighboring nodes Q_α and Q_β . This may serve to incorporate long range interactions between distant nodes.

The charge $Q_{\alpha\beta}$ transferred through the connectors [characterized by Eqs. (4,5)] may be discrete. However, the charge in the nodes Q_α is treated as an effectively continuous variable in Eqs. (1-3). This is justified if many charges in the node participate in transport. Formally, this limit allows a saddle-point evaluation of the propagator (3a).

A. Derivation of the Path Integral.

To derive the path integral Eq. (3), we follow the usual procedure²⁹ and first discretize time, $t = n\Delta t$ to derive an expression for U that is valid for propagation over one time step Δt . Because of the separation of time scales $\tau_0 \ll \tau_C$, we can consider Δt as an intermediate time scale,

$$\tau_0 \ll \Delta t \ll \tau_C. \quad (6)$$

The left inequality, $\tau_0 \ll \Delta t$, implies that the transmitted charges $Q_{\alpha\beta}$ are Markovian.⁴ This means that charges transmitted in separate time intervals are uncorrelated with each other. While it is not necessary to specify the source of the current correlation in the general formulation, it is worth noting two examples. In a mesoscopic point contact, the correlation time τ_0 has the interpretation of the time taken by an electron wavepacket to pass the point contact. In chemical dynamics, it could be the time taken for a long molecule in solution to traverse a filter.

In a time Δt , the probability that charge $Q_{\alpha\beta}$ is transmitted between nodes α and β can be written as the Fourier transform of the exponential of a generating function $S_{\alpha\beta}$:

$$P_{\alpha\beta}(Q_{\alpha\beta}, \Delta t) = \int \frac{d\lambda_{\alpha\beta}}{2\pi} \exp\{-i\lambda_{\alpha\beta}Q_{\alpha\beta} + S_{\alpha\beta}(\lambda_{\alpha\beta})\}. \quad (7)$$

The definition of the cumulant of transmitted charge is

$$\langle\langle (Q_{\alpha\beta})^n \rangle\rangle = \left. \frac{\partial^n S_{\alpha\beta}(\lambda_{\alpha\beta})}{(i\partial\lambda_{\alpha\beta})^n} \right|_{\lambda_{\alpha\beta}=0}. \quad (8)$$

The Markovian assumption implies that the probability of transmitting charge $Q_{\alpha\beta}$ in time Δt followed by charge $Q'_{\alpha\beta}$ in time $\Delta t'$ through any connector is given by the product of independent probability distributions. This implies that the probability of transmitting charge $Q_{\alpha\beta}$ in time $\Delta t + \Delta t'$ may be calculated by finding all ways of independently transferring charge $Q'_{\alpha\beta}$ in the first step and $Q_{\alpha\beta} - Q'_{\alpha\beta}$ in the second step,

$$P(Q_{\alpha\beta}, \Delta t + \Delta t') = \int dQ'_{\alpha\beta} P(Q_{\alpha\beta} - Q'_{\alpha\beta}, \Delta t') \times P(Q'_{\alpha\beta}, \Delta t), \quad (9)$$

which takes the form of a convolution of probabilities. Applying a Fourier transform to both sides of

Eq. (9) with argument $\lambda_{\alpha\beta}$ decouples the convolution into product of the two Fourier transformed distributions. Eq. (7) implies $S_{\alpha\beta}(\Delta t + \Delta t', \lambda_{\alpha\beta}) = S_{\alpha\beta}(\Delta t, \lambda_{\alpha\beta}) + S_{\alpha\beta}(\Delta t', \lambda_{\alpha\beta})$. It then immediately follows that the generating function must be linear in time. Therefore, a time independent $H_{\alpha\beta}$ may be introduced: $S_{\alpha\beta} = \Delta t H_{\alpha\beta}$. The linear dependence of $S_{\alpha\beta}$ on time implies that all charge cumulants (8) will be proportional to time. Therefore, we define the time independent current cumulants, Eq. (5).

Different connectors are clearly uncorrelated for $\Delta t \ll \tau_C$, which indicates that the total probability distribution of transmitted charges is a product of the independent probabilities in each connector:³⁰

$$P[\{Q_{\alpha\beta}\}] = \prod_{\alpha>\beta} P_{\alpha\beta}[Q_{\alpha\beta}, \Delta t]. \quad (10)$$

Thus far, the analysis is only valid for times much smaller than τ_C . For this case, the charges in the nodes will only slightly change. Since we wish to consider longer times, we need to take into account the fact that charge transfer between different nodes will be correlated as charge piles up inside the nodes. This may be accounted for by imposing charge conservation Eq. (1) during the time interval with a delta function,

$$\begin{aligned} & \delta(Q_\alpha - Q'_\alpha - \sum_\beta Q_{\alpha\beta}) \\ &= \int \frac{d\lambda_\alpha}{2\pi} \exp\{-i\lambda_\alpha[Q_\alpha - Q'_\alpha - \sum_\beta Q_{\alpha\beta}]\}. \end{aligned} \quad (11)$$

Here, Q'_α is the charge in the node before the time interval while Q_α is the charge accumulated in the node after the time interval is over. In Eq. (11), λ_α (referred to as a *counting variable*) plays the role of a Lagrange multiplier. The propagator is obtained by multiplying the constraint (11) and the independent probability distribution (10). Representing the probabilities in their Fourier form (7) then yields

$$\begin{aligned} \tilde{U}(\mathbf{Q}, \mathbf{Q}', Q_{\alpha\beta}, \Delta t) &= \prod_\alpha \int \frac{d\lambda_\alpha}{2\pi} \prod_{\alpha>\beta} \int \frac{d\lambda_{\alpha\beta}}{2\pi} \exp(S), \\ S &= -i \sum_\alpha \lambda_\alpha (Q_\alpha - Q'_\alpha - \sum_\beta Q_{\alpha\beta}) \\ &+ \sum_{\alpha>\beta} [-i\lambda_{\alpha\beta}Q_{\alpha\beta} + \Delta t H_{\alpha\beta}(\mathbf{Q}', \lambda_{\alpha\beta})]. \end{aligned} \quad (12)$$

The full propagator $\tilde{U}(\mathbf{Q}, \mathbf{Q}', Q_{\alpha\beta}, \Delta t)$ still keeps track of each individual connector contribution $Q_{\alpha\beta}$. We now integrate out the fast fluctuations to obtain the dynamics of the slow variables. This may be done by using the identity $\sum_\alpha \lambda_\alpha \sum_\beta Q_{\alpha\beta} = \sum_{\alpha>\beta} (\lambda_\alpha Q_{\alpha\beta} + \lambda_\beta Q_{\beta\alpha})$ and $Q_{\alpha\beta} = -Q_{\beta\alpha}$. The integration over $Q_{\alpha\beta}$ gives a delta function of argument $\lambda_{\alpha\beta} - (\lambda_\alpha - \lambda_\beta)$, so that the $\lambda_{\alpha\beta}$

integrals may be trivially done. We obtain

$$U(\mathbf{Q}, \mathbf{Q}', \Delta t) = \prod_{\alpha} \int \frac{d\lambda_{\alpha}}{2\pi} \exp \left\{ -i \sum_{\alpha} \lambda_{\alpha} (Q_{\alpha} - Q'_{\alpha}) + \Delta t \sum_{\alpha > \beta} H_{\alpha\beta}(\mathbf{Q}', \lambda_{\alpha} - \lambda_{\beta}) \right\}. \quad (13)$$

This is the general result for the one step propagator. If any two nodes are unconnected, $H_{\alpha\beta}$ is zero.

An important comment is in order: because $H_{\alpha\beta}$ changes slightly over the time period, which in turn affects the probability of transmitting charge through the contacts, it is not clear at what part of the time step $H_{\alpha\beta}$ should be evaluated. This ambiguity exists because our theory is not microscopic. Rather, it takes the microscopic noise generators as an input. This ambiguity gives the freedom of stochastic quantization.³¹ The same problem also occurs in quantum mechanical path integrals, and its source there is an ambiguity in operator ordering.³² As we are interested in the large transport-

ing charge limit, $\gamma \gg 1$, and evaluate the integrals in leading order saddle-point approximation, this ambiguity will not affect the results.⁷ For calculations beyond the large transporting charge limit, the canonical variables \mathbf{Q} and $\mathbf{\Lambda}$ need to be properly ordered, which can only be done with a microscopic theory. For example, the master equation discretized in time as discussed in Sec. IID requires the placement of $\mathbf{\Lambda}$ operators in front of \mathbf{Q} operators, since the generating functions $H_{\alpha\beta}$ of the transition probabilities depend on the state of the system at the beginning of the time period.

To extend the propagator (13) to longer times $t = n\Delta t$, we use the composition property of the evolution operator (also known as the Chapman-Kolmogorov equation⁴). This requires separate $\{Q_{\alpha}\}$ integrals at each time step, so that for n time steps there will be $n - 1$ integrals over \mathbf{Q} , while each of the n one-step propagators comes with its own $\mathbf{\Lambda}$ integral, $\mathbf{\Lambda} = \{\lambda_{\alpha}\}$. Inserting our expression for the Δt step propagator Eq. (13), we find

$$U(\mathbf{Q}_f, \mathbf{Q}_i, t) = \int d\mathbf{\Lambda}_0 \prod_{k=1}^{n-1} \int d\mathbf{Q}_k d\mathbf{\Lambda}_k \exp \left[\sum_{k=0}^{n-1} -i\mathbf{\Lambda}_k \cdot (\mathbf{Q}_{k+1} - \mathbf{Q}_k) + \Delta t H(\mathbf{Q}_k, \mathbf{\Lambda}_k) \right], \quad (14a)$$

with

$$H(\mathbf{Q}_k, \mathbf{\Lambda}_k) = \sum_{\alpha > \beta} H_{\alpha\beta} [Q_{\alpha,k}; \lambda_{\alpha,k} - \lambda_{\beta,k}], \quad (14b)$$

where we have introduced the notations $d\mathbf{Q}_k = \prod_{\alpha} dQ_{\alpha,k}$ and $d\mathbf{\Lambda}_k = \prod_{\alpha} (d\lambda_{\alpha,k}/2\pi)$. We are now in a position to take the continuous time limit. Writing $\mathbf{Q}_{k+1} - \mathbf{Q}_k = \Delta t \dot{\mathbf{Q}}$, which is valid because the charge in any node changes only slightly over the time scale Δt , the action of this discrete path integral has the form $S = \Delta t \sum_{k=1}^n S_k$, which goes over into a time integral in the continuous limit. Using the standard path integral notation $\int \mathcal{D}\mathbf{Q}\mathcal{D}\mathbf{\Lambda} = \int d\mathbf{\Lambda}_0 \prod_{k=1}^{n-1} \int d\mathbf{Q}_k d\mathbf{\Lambda}_k$, and invoking the symmetry $H_{\alpha\beta}(\lambda_{\alpha} - \lambda_{\beta}) = H_{\beta\alpha}(\lambda_{\beta} - \lambda_{\alpha})$ we recover Eq. (3). The only explicit constraint on the path integral comes with the charge configurations at the start and finish, \mathbf{Q}_i and \mathbf{Q}_f . We also note that $H_{\alpha\beta}$ depends on any external parameters such as voltages or chemical potentials driving the charge \mathbf{Q} .

In the simplest case of one charge and counting variable, the form of the path integral is the same as the (Euclidian time) path integral representation of a quantum mechanical propagator in phase space with position coordinate Q and momentum coordinate λ .³² The differences with the quantum version are that the propagator evolves probability distributions, not amplitudes (similarly to Ref. 25), as well as the fact that the ‘‘Hamiltonian’’ $H = (1/2) \sum_{\alpha\beta} H_{\alpha\beta}(\mathbf{Q}, \lambda_{\alpha} - \lambda_{\beta})$ is not really a

Hamiltonian, but rather a current cumulant generating function and therefore is not Hermitian in general. Even so, because of the similarity we shall refer to H as the Hamiltonian from now on.

B. Absorbed Charges, Boundary Conditions and Correlation Functions.

A useful special case occurs when one has absorbed charges. These are charges that vanish into (or are injected from) absorbing nodes without altering the system dynamics. In mesoscopics for example, the absorbing nodes are metallic reservoirs. Formally, we divide the charges into those that are conserved and those that are absorbed: $\mathbf{Q} = \{\mathbf{Q}^c, \mathbf{Q}^a\}$, where the subset of absorbed charges $\mathbf{Q}^a = \{Q_{\alpha}^a\}$ does not appear in $H_{\alpha\beta}$. We do the same for the corresponding counting variables: $\mathbf{\Lambda} = \{\mathbf{\Lambda}^c, \mathbf{\Lambda}^a\}$. Because $H_{\alpha\beta}$ does not depend on \mathbf{Q}^a , these charges may be integrated out by integrating the action by parts,

$$i \int_0^t dt' \mathbf{\Lambda}^a \cdot \dot{\mathbf{Q}}^a = -i \int_0^t dt' \mathbf{Q}^a \cdot \dot{\mathbf{\Lambda}}^a + i(\mathbf{\Lambda}_f^a \cdot \mathbf{Q}_f^a - \mathbf{\Lambda}_i^a \cdot \mathbf{Q}_i^a), \quad (15)$$

and then functionally integrating over \mathbf{Q}^a to obtain $\delta(\dot{\mathbf{\Lambda}}^a)$, where δ is a functional delta function. This immediately constrains the $\mathbf{\Lambda}^a$ to be constants of motion

so the functional integration over Λ^a becomes a normal integration, $\mathcal{D}\Lambda^a \rightarrow d\Lambda^a$. The absorbed kinetic terms in the action may then be integrated to obtain

$$U(\mathbf{Q}_f, \mathbf{Q}_i, t) = \int d\Lambda^a \int \mathcal{D}\mathbf{Q}^c \mathcal{D}\Lambda^c \exp\{S(\mathbf{Q}, \Lambda)\}, \quad (16a)$$

$$S(\mathbf{Q}, \Lambda) = \int_0^t dt' [-i\Lambda^c \cdot \dot{\mathbf{Q}}^c + (1/2) \sum_{\alpha\beta} H_{\alpha\beta}(\lambda_\alpha - \lambda_\beta)] - i\Lambda^a \cdot (\mathbf{Q}_f^a - \mathbf{Q}_i^a). \quad (16b)$$

Often one is interested in the probability to transmit some amount of charge through each of the absorbing nodes. By applying a Fourier transform to Eq. (16a) with respect to $\mathbf{Q}^a(t) - \mathbf{Q}^a(0)$ we remove the last term in Eq. (16b) and obtain the path integral representation for the characteristic function Z which generates current moments at every absorbing node

$$Z(\Lambda^a) = \int \mathcal{D}\mathbf{Q}^c \mathcal{D}\Lambda^c \exp\{S(\mathbf{Q}, \Lambda)\}, \quad (17a)$$

$$S(\mathbf{Q}, \Lambda) = \int_0^t dt' [-i\Lambda^c \cdot \dot{\mathbf{Q}}^c + (1/2) \sum_{\alpha\beta} H_{\alpha\beta}(\lambda_\alpha - \lambda_\beta)]. \quad (17b)$$

Note that the counting variables Λ^a enter the action (17b) only as a set of constant parameters. The initial condition in the path integral (17) is given by the initial charge states $\mathbf{Q}^c(0)$. There is a choice of the final condition: by fixing the final $\mathbf{Q}^c(t)$ one obtains the distribution of the conserved charge subject to this constraint, while by fixing $\Lambda^c(t)$ the corresponding characteristic function is obtained. The choice of $\Lambda^c(t) = 0$ in Eq. (17) gives the characteristic function of the absorbed charge under the condition that the conserved charge is not being monitored, i.e. the final charge state is integrated over. Therefore $\ln Z$ becomes the generator of the FCS, defining the charge cumulants at the absorbing node,

$$\langle\langle [Q_\alpha^a(t) - Q_\alpha^a(0)]^n \rangle\rangle = \left. \frac{\partial^n \ln Z}{\partial (i\lambda_\alpha^a)^n} \right|_{\Lambda^a=0}. \quad (18)$$

In the long time limit, this quantity is proportional to time, independent of the details of the boundary conditions.

Alternatively, in the short time limit one may calculate irreducible correlation functions of absorbed and conserved current fluctuations, $\mathbf{I} = \dot{\mathbf{Q}}$. These correlation functions can be obtained by extending the time integral in (3b) to infinity, introducing sources³² in the action, $S \rightarrow S + i \int dt \chi(t) \cdot \mathbf{I}(t)$, and applying functional derivatives with respect to χ . Repeating the steps leading to Eqs. (17), we find that variables λ_α in the Hamiltonian in Eq. (17b) have to be shifted $\lambda_\alpha \rightarrow \lambda_\alpha + \chi_\alpha$. Then, the irreducible current correlation function is given by

$$\langle\langle I_{\alpha_1}(t_1) \cdots I_{\alpha_n}(t_n) \rangle\rangle = \left. \frac{\delta^n \ln Z[\chi]}{\delta i\chi_{\alpha_1}(t_1) \cdots \delta i\chi_{\alpha_n}(t_n)} \right|_{\chi=0}. \quad (19)$$

With these correlation functions, one may calculate for example the frequency dependence of current cumulants.³⁴

C. The Saddle Point Approximation.

If the Hamiltonian has some dimensionless large prefactor, then the path integral (3) may be evaluated using the saddle point approximation, which is justified below. At the saddle point, (where the first variation of the action vanishes), we can write equations of motion analogous to the Hamiltonian equations of classical mechanics:

$$i\dot{\mathbf{Q}}^c = \frac{\partial}{\partial \Lambda^c} H(\mathbf{Q}^c, \Lambda), \quad i\dot{\Lambda}^c = -\frac{\partial}{\partial \mathbf{Q}^c} H(\mathbf{Q}^c, \Lambda), \quad (20)$$

where $H(\mathbf{Q}^c, \Lambda) = (1/2) \sum_{\alpha\beta} H_{\alpha\beta}(\mathbf{Q}^c; \lambda_\alpha - \lambda_\beta)$. There may be many saddle point solutions in general, and one has to sum over all of them. Eqs. (20) are solved subject to the temporal boundary conditions and generally describe the relaxation of the conserved charges from the initial state to a stationary state $\{\bar{\mathbf{Q}}^c, \bar{\Lambda}^c\}$ on a time scale given by τ_C , the dynamical time scale of the nodes. These stationary coordinates are functions of any external parameters as well as the (constant) absorbed counting variables Λ^a . In the saddle point approximation, the action takes the form $S = S_{sp} + S_{fluc}$.⁷ The term S_{sp} is the contribution to the action from the solution of the equations (20), which describes the evolution of the system from the initial to the final state. The term S_{fluc} describes fluctuations around the saddle point and is suppressed compared to the saddle-point contribution, if the Hamiltonian has a large prefactor (in analogy to the \hbar -expansion of quantum mechanics). Physically, the validity condition for the saddle point approximation is that there should be many (transporting) charge carriers in the nodes. For times longer than the charge relaxation time of the node, the dominant contribution is from the stationary state only, where the saddle-point part of the action is simply linear in time:

$$S_{sp}(\bar{\mathbf{Q}}, \bar{\Lambda}) = tH(\bar{\mathbf{Q}}, \bar{\Lambda}), \quad t \gg \tau_C. \quad (21)$$

The linear time dependence of Eq. (21) indicates that the dynamics are Markovian on a long time scale. It is the fact that the contribution S_{sp} emerges in a dominant way which makes the approach given here a powerful tool to analyze the counting statistics of transmitted charge.

We now discuss the large parameter that justifies the saddle point approximation. The boundary conditions on the charge in the absorbing nodes fix a (dimensionless) charge scale of the system, γ . All charges in the network are scaled accordingly, $\mathbf{Q} \rightarrow \gamma\mathbf{Q}$. We make the assumption that there is a one parameter scaling of the Hamiltonian, $H \rightarrow \gamma H$. The time is also scaled by τ_C , the time scale of charge relaxation in the nodes. The dimensionless action is now $S = \gamma \int_0^{t/\tau_C} dt' (-i\dot{\mathbf{Q}}\lambda + \tau_C H)$.

The saddle point action is proportional to $\gamma t/\tau_C$, while the fluctuation contribution will be of order t/τ_C . We note that the parameter γ is related to (though not necessarily the same as) the separation of time scales, τ_C/τ_0 , needed to derive the path integral. For the mesoscopic conductors considered in the example section VB of this paper, the charge scale is set by the maximum number of semiclassical states on the cavity involved in transport, $\gamma = \Delta\mu N_F \gg 1$, the bias times the density of states at the Fermi level. On the other hand, for the chaotic cavity, $\tau_C/\tau_0 = \gamma/(G_L + G_R)$, where $G_{L,R} \gg 1$ are the dimensionless conductances of the left and right point contact.

D. Relation to the master Equation and Doi-Peliti Technique.

The evolution operator $U(\mathbf{Q}, \mathbf{Q}', t)$ may be interpreted as a Green function of a differential equation which determines the propagation in time of an initial probability distribution $\Gamma(\mathbf{Q})$. In the theory of stochastic processes, such a differential equation is called a master equation. A natural question that arises is the relationship of the formalism presented here to other approaches to stochastic problems.

The most general type of Markovian master equation for discrete states and discrete time is of the form

$$\dot{\Gamma}_n(t_{k+1}) = \sum_m P_{nm}(t_{k+1}, t_k) \Gamma_m(t_k), \quad (22)$$

where $\Gamma_m(t_k)$ is the probability to be in state m at time t_k and P_{nm} is the transition probability from state m to state n . The state is described by a vector $n = (n_1, \dots, n_N)$ whose components are the charges n_α of each node α . The Markovian assumption implies that $t_{k+1} - t_k = \Delta t$ is greater than the correlation time, τ_0 . If we further assume that the probability to make a transition to another state is small, $P_{nm} \ll 1$ for $n \neq m$, so that the transition probability is only linear in Δt , a transition rate $W_{nm} = P_{nm}/\Delta t$ may be defined. It then follows that we may write a differential master equation,

$$\dot{\Gamma}_n(t) = \sum_m [W_{nm} \Gamma_m(t) - W_{mn} \Gamma_n(t)]. \quad (23)$$

Eq. (23) is the starting point for the Doi-Peliti technique,²⁴ where one formally maps the space of physical states to the Fock space of states $|n\rangle = (a_1^\dagger)^{n_1} \dots (a_N^\dagger)^{n_N} |0\rangle$, where n is the number of charges. The entire state of the system is expressed by a vector $|\Psi\rangle = \sum_n \Gamma_n |n\rangle$ which weights the states $|n\rangle$ with their probabilities Γ_n . Thus, the master equation (23) may be interpreted as a many-body Schrödinger equation where the rates W_{mn} are incorporated into a Hamiltonian in a second-quantized form. One may then write a coherent-state path integral over the variables a , and a^\dagger for this

many-body quantum system and perform perturbation expansions along with the renormalization group.²⁵ This procedure eventually involves taking the continuum limit so the discrete charge states become continuous.

Let us now consider how our formalism is related to the master equation or the Doi-Peliti technique. According to the results of Sec. II, our stochastic path integral, Eq.(3) solves the continuum variable version of Eq. (22) with the transition probabilities given by the one step propagator $U(\mathbf{Q}, \mathbf{Q}', \Delta t)$. *In general, the transition probabilities are neither small nor linear in time for $\Delta t > \tau_0$.* It is instructive nevertheless to consider the special case of processes where $H\tau_0 \ll 1$, when we can expand the one-step propagator (13) to first order in Δt ,

$$U(\mathbf{Q}, \mathbf{Q}', \Delta t) \approx \delta(\mathbf{Q} - \mathbf{Q}') + \Delta t \int d\Lambda e^{-i\Lambda \cdot (\mathbf{Q} - \mathbf{Q}')} H(\mathbf{Q}', \Lambda). \quad (24)$$

Defining the Fourier transform of the generating function as $\tilde{H}(\mathbf{Q}, \mathbf{Q}')$, the differential equation governing the evolution of a probability distribution of charges $\Gamma(\mathbf{Q})$ is then

$$\dot{\Gamma}(\mathbf{Q}, t) = \int d\mathbf{Q}' \tilde{H}(\mathbf{Q}, \mathbf{Q}') \Gamma(\mathbf{Q}', t). \quad (25)$$

Comparison with the continuous version of the master equation (23),

$$\dot{\Gamma}(\mathbf{Q}, t) = \int d\mathbf{Q}' [W(\mathbf{Q}, \mathbf{Q}') \Gamma(\mathbf{Q}', t) - W(\mathbf{Q}', \mathbf{Q}) \Gamma(\mathbf{Q}, t)], \quad (26)$$

indicates that \tilde{H} is related to W . The Hamiltonian may be expressed in terms of the transition kernel³³ as,

$$H(\mathbf{Q}', \Lambda) = \int d\mathbf{Q} [e^{i(\mathbf{Q} - \mathbf{Q}') \cdot \Lambda} - 1] W(\mathbf{Q}, \mathbf{Q}'), \quad (27)$$

where the normalization of probability is expressed by $H(\mathbf{Q}', 0) = 0$. Eq. (27) is an important result, because it allows the conversion of the master equation (26) into the stochastic path integral (3).

We would like to stress that our formalism is not simply equivalent to the differential master equation (26) (and therefore the Doi-Peliti technique), but that it allows the treatment of a complementary class of problems. Our formalism assumes effectively continuous charge, and thus cannot resolve effects due to the discreteness of charge on the nodes. Such effects are present in the master equation (23). In contrast, the differential master equation assumption, $H\tau_0 \ll 1$ (which simply states that transition probabilities are small in the time interval τ_0) is not required. Our formalism is especially important when this is not the case, *i.e.* $H\tau_0 \sim 1$.

This is illustrated by the simple example from mesoscopes of two metallic reservoirs connected by a single electron barrier with hopping probability p and bias $\Delta\mu$ at zero temperature. For a time interval Δt larger than the correlation time $\tau_0 = \hbar/\Delta\mu$ (the time scale for

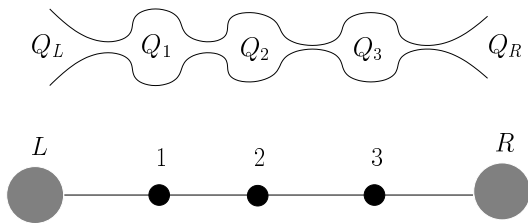


FIG. 2: A one dimensional lattice of nodes connected on both ends to absorbing reservoirs. This situation could represent a series of mesoscopic chaotic cavities connected by quantum point contacts.

an electron wavepacket to transverse the barrier), $\Delta t/\tau_0$ electrons approach the barrier and either are transmitted or reflected. Mathematically, this is a classical binomial process with the generator

$$S = (\Delta t/\tau_0) \ln[1 + p(e^{ie\lambda} - 1)]. \quad (28)$$

As this action is the starting point of many mesoscopic implementations of the formalism, it is an important example. Since the action is proportional to the large parameter $\Delta t/\tau_0 > 1$, for $p \sim 1$ the expansion of $\exp(S)$ to first order in Δt is strictly forbidden, effectively not allowing a first order differential master equation. Only in the limit $p \ll 1$, (*i.e.* when Eq. (28) describes a Poissonian process) may the logarithm be expanded to first order. This suggests that Eq. (26) describes the slow dynamics of systems whose fast transitions are Poissonian in nature. A more general type of dynamics such as the binomial distribution may only be found using the continuous charge state master equation in discrete time (22).

III. THE FIELD THEORY

From the stochastic network, Fig. 1, it is straightforward to go to spatially continuous systems as the spacing between the nodes is taken to zero. The goal is to introduce a Hamiltonian functional $h(\rho, \lambda)$ whose arguments are the charge density ρ and the counting field functions λ , that are themselves functions of space and time. We may then replace $(1/2) \sum_{\alpha, \beta} H_{\alpha, \beta} \rightarrow \int dz h(\rho, \lambda)$. Our description is local, so in the model each node is only connected to its nearest neighbors. We first derive the one dimensional field theory with one charge species in detail, and then generalize to multiple dimensions and charge species.

Consider a series of identical, equidistant nodes separated by a distance Δz . This nodal chain could represent a chain of chaotic cavities, Fig. 2, in a mesoscopic context.^{35,36} The sum over α and β becomes a sum over each node in space connected to its neighbors. The action

for this arrangement is

$$S = \int_0^t dt' \sum_{\alpha} \{-\lambda_{\alpha} \dot{Q}_{\alpha} + H(Q_{\alpha}, Q_{\alpha-1}; \lambda_{\alpha} - \lambda_{\alpha-1})\}, \quad (29)$$

where for simplicity we have chosen real counting variables, $i\lambda_{\alpha} \rightarrow \lambda_{\alpha}$. The imaginary counting variables will be restored at the end of the section. The only constraint made on H is that probability is conserved, $H(\lambda_{\alpha} - \lambda_{\alpha-1}) = 0$ for $\lambda_{\alpha} = \lambda_{\alpha-1}$. We now derive a lattice field theory by formally expanding H in $\lambda_{\alpha} - \lambda_{\alpha-1}$ and $Q_{\alpha} - Q_{\alpha-1}$. Only differences of the counting variables will appear in the series expansion, while we must keep the full Q dependence of the Hamiltonian. If there are $N \gg 1$ nodes in the lattice, for fixed boundary conditions the difference between adjacent variables, $\lambda_{\alpha} - \lambda_{\alpha-1}$ and $Q_{\alpha} - Q_{\alpha-1}$ will be of order $1/N$, and therefore provides a good expansion parameter. The expansion of the Hamiltonian (29) to second order in the difference variables gives

$$H = \frac{\partial H}{\partial \lambda_{\alpha}} (\lambda_{\alpha} - \lambda_{\alpha-1}) + \frac{1}{2} \frac{\partial^2 H}{\partial \lambda_{\alpha}^2} (\lambda_{\alpha} - \lambda_{\alpha-1})^2 + \frac{\partial^2 H}{\partial Q_{\alpha} \partial \lambda_{\alpha}} (Q_{\alpha} - Q_{\alpha-1})(\lambda_{\alpha} - \lambda_{\alpha-1}), \quad (30)$$

where the expansion coefficients are evaluated at $\lambda_{\alpha} = \lambda_{\alpha-1}$ and $Q_{\alpha} = Q_{\alpha-1}$ and are functions of $Q_{\alpha-1}$. Terms involving only differences of $Q_{\alpha} - Q_{\alpha-1}$ are zero because $H(\lambda_{\alpha} - \lambda_{\alpha-1}) = 0$ for $\lambda_{\alpha} = \lambda_{\alpha-1}$. All terms in Eq. (30) need explanation. First, the expression $\partial H/\partial \lambda_{\alpha}$ is the local current at zero bias (because the charges in adjacent nodes are equal) which will usually be zero. There may be circumstances where this term should be kept,³⁷ but we do not consider them here. The term $\partial^2 H/\partial Q_{\alpha} \partial \lambda_{\alpha} = -G(Q_{\alpha-1})$ is the linear response of the current to a charge difference. Hence, G is the generalized conductance³⁸ of the connector between nodes α and $\alpha - 1$. $\partial^2 H/\partial \lambda_{\alpha}^2 = C(Q_{\alpha-1})$ is the current noise through the same connector because H is the generator of current cumulants.

We are now in a position to take the continuum limit by replacing the node index α with a coordinate z , introducing the fields $Q(z), \lambda(z)$, and making the expansions

$$\lambda_{\alpha} - \lambda_{\alpha-1} \rightarrow \lambda' \Delta z + (1/2) \lambda'' (\Delta z)^2 + \mathcal{O}(\Delta z)^3, \quad (31a)$$

$$Q_{\alpha} - Q_{\alpha-1} \rightarrow Q' \Delta z + (1/2) Q'' (\Delta z)^2 + \mathcal{O}(\Delta z)^3. \quad (31b)$$

The action may now be written in terms of intensive fields by scaling away Δz ,

$$H \rightarrow h(\rho, \lambda) \Delta z, \quad Q_{\alpha} \rightarrow \rho(z) \Delta z, \\ G_{\alpha} (\Delta z)^2 \rightarrow D(\rho), \quad C_{\alpha} \Delta z \rightarrow F(\rho), \quad (32)$$

and taking the limit $\sum_{\alpha} H \rightarrow \int dz h(\rho, \lambda)$. One may check that expanding the Hamiltonian to higher than second order in Δz will result in terms suppressed by powers

of $\Delta z/L$ and consequently vanish as $\Delta z \rightarrow 0$. This scaling argument for the field theory is analogous to Van Kampen's size expansion.³⁹ Though the lattice spacing Δz does not appear in the continuum limit, it provides a physical cut-off for any ultra-violet divergences that might appear in a loop expansion.

These considerations leave the one dimensional action as

$$S = - \int_0^t dt' \int_0^L dz \left[\lambda \dot{\rho} + D \rho' \lambda' - \frac{1}{2} F (\lambda')^2 \right]. \quad (33)$$

Here D is the local diffusion constant and F is the local noise density which are discussed in detail below. It is very important that these two functionals D, F are all that is needed to calculate current statistics. Classical field equations may be obtained by taking functional derivatives of the action with respect to the charge and counting fields: $\delta S / \delta \rho(z) = \delta S / \delta \lambda(z) = 0$ to obtain the equations of motion,

$$\dot{\lambda} = - \frac{1}{2} \frac{\delta F}{\delta \rho} (\lambda')^2 - D \lambda'', \quad \dot{\rho} = [-F \lambda' + D \rho']'. \quad (34)$$

From the charge equation, one can see immediately that the term inside the derivative may be interpreted as a current density so that local charge conservation is guaranteed. We have to solve these coupled differential equations subject to the boundary conditions

$$\begin{aligned} \rho(t, 0) &= \rho_L(t), & \rho(t, L) &= \rho_R(t), \\ \lambda(t, 0) &= \lambda_L(t), & \lambda(t, L) &= \lambda_R(t), \end{aligned} \quad (35)$$

where $\rho_L(t)$, $\rho_R(t)$, $\lambda_L(t)$, and $\lambda_R(t)$ are arbitrary time dependent functions. Functions $\rho_L(t)$ and $\rho_R(t)$ are the charge densities at the far left and right end of the system which may be externally controlled. Functions $\lambda_L(t)$ and $\lambda_R(t)$ are the counting variables of the absorbed charges at the far left and right end which count the current that passes them.

Once Eqs. (34) are solved subject to the boundary conditions (35), the solutions $\rho(z, t)$ and $\lambda(z, t)$ should be substituted back into the action (33) and integrated over time and space. The resulting function, $S_{sp}[\rho_L(t), \rho_R(t), \lambda_L(t), \lambda_R(t), t, L]$ is the generating function for time-dependent cumulants of the current distribution. Often, the relevant experimental quantities are the stationary cumulants. These are given by neglecting the time dependence, finding static solutions, $\dot{\rho} = \dot{\lambda} = 0$, and imposing static boundary conditions. Similarly to section IID, we can also introduce sources $\int dt dz \chi(z, t) \rho(z, t)$ and calculate density correlation functions.

To estimate the contribution of the fluctuations to the action, it is useful to define dimensionless variables. The boundary conditions ρ_L , and ρ_R provide the charge density scale ρ_0 in the problem, so we define $\rho(z) = \rho_0 f(z)$, where $f \sim 1$ is an occupation. We furthermore rescale $z \rightarrow Lz$, and $t \rightarrow \tau_D t$, where $\tau_D = L^2/D$ is the diffusion

time, thus obtaining

$$S = -L\rho_0 \int_0^t dt' \int_0^1 dz' \left[\lambda \dot{f} + f' \lambda' - \frac{F}{2D\rho_0} (\lambda')^2 \right]. \quad (36)$$

We assume that the combination $F/D\rho_0$ is of order 1. From Eq. (36), the dimensionless large parameter is $\gamma = \rho_0 L \gg 1$, i.e. the number of transporting charge carriers. As in Sec. IIC, the saddle point contribution is of order $\gamma t / \tau_D$, while the fluctuation contribution is of order t / τ_D .

Repeating this derivation in multiple dimensions with N charge species $\rho = \{\rho_i(\mathbf{r})\}$ and counting fields $\Lambda = \{\lambda_i(\mathbf{r})\}$, $i = 1, \dots, N$ yields the action

$$S = - \int_0^t dt' \int_{\Omega} d\mathbf{r} \left[\Lambda \dot{\rho} + \nabla \Lambda \hat{D} \nabla \rho - (1/2) \nabla \Lambda \hat{F} \nabla \Lambda \right], \quad (37)$$

where tensor notation is used and we have introduced $\hat{F}_{ij} = \partial_{\lambda_i} \partial_{\lambda_j} h$ and $\hat{D}_{ij} = -\partial_{\rho_i} \partial_{\rho_j} h$ as general matrix functionals of the field vector ρ and coordinate \mathbf{r} which should be interpreted as noise and diffusion matrices. If the medium is isotropic, then the vector gradients simply form a dot product. It should be emphasized that the vectors appearing are vectors of different species of charge fields, as all node delimitation has been accounted for in the spatial integration. The functional integral now runs over all field configurations that obey the imposed boundary conditions at the surface $\partial\Omega$. Classical field equations may be formally obtained by taking functional derivatives of the action with respect to the charge and counting fields as in the 1D case.

As in any field theory, symmetries of the action play an important role because they lead to conserved quantities. We first note that the Hamiltonian $h(\rho, \nabla \rho, \nabla \Lambda)$ is a functional of $\nabla \Lambda$ alone with no Λ dependence. This symmetry is analogous to gauge invariance, and leads to the equation of motion

$$\dot{\rho} + \nabla \cdot \mathbf{j} = 0, \quad \mathbf{j} = -\hat{D} \nabla \rho + \hat{F} \nabla \Lambda, \quad (38)$$

which can be interpreted as conservation of the conditional current \mathbf{j} . The next symmetry is related to the invariance under a shift in the space and time coordinates $\{\delta \mathbf{r}, \delta t\}$. This symmetry leads to equations analogous to the conservation of the local energy/momentum tensor.⁴⁰ We do not explicitly give this quantity because it is rather cumbersome in the general case. However, for the stationary limit (where $\dot{\rho}$ and $\dot{\lambda}$ vanish) and for symmetric diffusion and noise tensors, the one charge species conservation law is relatively simple and is given by

$$\sum_m \nabla_m T_{mn} = 0, \quad (39a)$$

$$T_{mn} = j_m (\nabla_n \lambda) - (\nabla_n \rho) (\hat{D} \nabla \lambda)_m - h \delta_{mn}. \quad (39b)$$

For the special case of a one dimensional geometry, the Hamiltonian itself is the conserved quantity (see Sec. VA).

In the continuum limit, all terms of higher order in Λ are suppressed so that the action is quadratic in the Λ variables. This fact may be viewed as a consequence of the central limit theorem and confirms the observation made by Nagaev that local noise in the mesoscopic diffusive wire (see Sec. V A) is Gaussian.¹⁹ To further clarify the physical meaning of D and F , and also to make connection with previous work,³² we restore the complex variables, $\Lambda \rightarrow i\Lambda$, and make a Hubbard-Stratonovich transformation by introducing an auxiliary vector field ν ,

$$\begin{aligned} & \exp\{-(1/2)\nabla\Lambda\hat{F}\nabla\Lambda\} \\ & = (\det\hat{F})^{-\frac{1}{2}} \int \mathcal{D}\nu \exp\{-(1/2)\nu\hat{F}^{-1}\nu + i\nu\nabla\Lambda\}. \end{aligned} \quad (40)$$

We may then integrate out the Λ variables, taking account of the boundary terms to obtain,

$$\begin{aligned} U & = \exp\left\{\int_0^t dt' \int_{\partial\Omega} ds \cdot (i\Lambda^a \mathbf{J})\right\} \int \mathcal{D}\rho \mathcal{D}\nu \delta(\dot{\rho} + \nabla \cdot \mathbf{J}) \\ & \times (\det\hat{F})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \int_0^t dt' \int_{\Omega} d\mathbf{r}' \nu \hat{F}^{-1} \nu\right\}, \end{aligned} \quad (41)$$

where the δ above is a functional delta function, imposing the Langevin equation

$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0, \quad (42a)$$

$$\mathbf{J} = -\hat{D}\nabla\rho + \nu, \quad (42b)$$

with a current noise source ν , whose correlator⁴¹ is given by

$$\langle \nu(\mathbf{r}, t) \nu(\mathbf{r}', t') \rangle = \delta(t - t') \delta(\mathbf{r} - \mathbf{r}') \hat{F}(\rho). \quad (42c)$$

\mathbf{J} may be interpreted as the physical current density [not to be confused with the conditional current density (38)] so that local current conservation is guaranteed, and the $(\det\hat{F})^{-1/2}$ serves to normalize the ν probability distribution. The role of the boundary term is to count the current \mathbf{J} flowing out of the boundary with the counting variable Λ^a , which serves as a Lagrange multiplier. This formula gives an immediate translation between the Langevin approach and full counting statistics, a connection not previously known. The algorithm is as follows:

1. Given a Langevin equation of the form (42), write the average of the boundary term with source Λ^a as a path integral (41) over noise and density fields.⁴²
2. Introduce an auxiliary field Λ that takes on the value Λ^a at the boundaries and represents the delta function in Eq. (41) imposing current conservation (42a) in Fourier form.
3. Integrate out the Gaussian noise to obtain an action of the form of Eq. (37).
4. Find where the first variation of the action is zero and solve the equations of motion subject to the boundary conditions.

5. Insert the solutions back into the action, and do the space and time integrals. The answer is the current cumulant generating function.

IV. PERTURBATION THEORY

We have shown in Sec. II C that a large number of participating elementary charges justifies the saddle point approximation for the generator of counting statistics. While the generator may sometimes be found in closed form,⁷ in general, it has no compact expression and the cumulants should be found separately at every order. This may be done by expanding $S_{sp}(Q, \lambda, \chi)$ as a series in χ and solving the saddle point equations to a given order in χ directly. However, there is another approach for evaluating the higher cumulants, the cascade diagrammatics representing higher-order cumulants in terms of the lower ones. It has been introduced by Nagaev in the context of mesoscopic charge statistics in the diffusive wire¹⁹ and later extended to the chaotic cavity,²⁰ but without proof. The basic idea is that lower order cumulants mix in to yield corrections to the bare fluctuations of higher order cumulants. This method was used successfully in Ref. 43 to explain the recent experiment of Ref. 44. In this section, we demonstrate that these rules follow naturally from the stochastic path integral in the same way as Feynman diagrams follow from the quantum mechanical functional integral. In Sec. IV C we present another (simpler) method for computing cumulants based completely on differential operators obtained from the Hamiltonian equations of motion. In Sec. IV D we generalize the cascade diagrammatics to an arbitrary network, and to the case of time-dependent correlators.

A. The Principle of Minimal Correlations.

To motivate the cascade diagrammatics, we refer to a specific physical system (see the inset of Fig. 8), the mesoscopic chaotic cavity.¹ For the purposes of this section, the cavity is a conserving node carrying charge Q , the electronic reservoirs correspond to the left and right are absorbing nodes, and the two point contacts are the connectors described by Hamiltonians H_L, H_R (see Fig. 3). Although a detailed description of this system is given in Sec. V B, we would like to mention that the mesoscopic cavity is described by an electron distribution function f , which is fluctuating around its mean value, f_0 . The actual electrical charge in the cavity Q and the occupation f are related via the large parameter γ through $Q = \gamma(f - f_0)$, where $\gamma = \Delta\mu N_F \gg 1$ (the density of states at the Fermi energy N_F times the bias $\Delta\mu$) is the maximum possible number of electrons on the cavity which contribute to the transport (see Sec. II C).

The cascade approach builds on the principle of minimal correlations developed in Ref. 18: The point contacts create bare noise $\langle\langle \tilde{I}_L^2 \rangle\rangle = \partial^2 H_L / (\partial i\lambda_L)^2$, and $\langle\langle \tilde{I}_R^2 \rangle\rangle =$

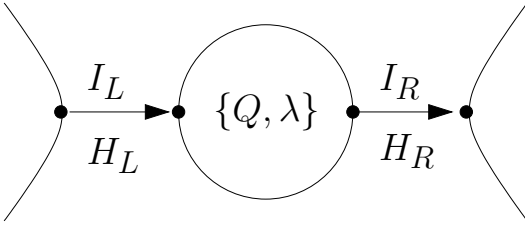


FIG. 3: Network representing a chaotic cavity. The state of the internal node is described by the variable Q , the charge on the cavity. The statistics of the connectors are characterized by the two generating functions $H_{L,R}$.

$\partial^2 H_R / (\partial i \lambda_R)^2$ with no correlation, $\langle \langle \tilde{I}_L \tilde{I}_R \rangle \rangle = 0$ [see Eq. (5)]. However, for times longer than the average dwell time of electrons in the cavity, the current conservation requirement imposes “minimal correlations” on the fluctuations of the physical currents I_L and I_R , which can be expressed in the form of the Langevin equations,

$$I_L = \tilde{I}_L - G_L Q, \quad I_R = \tilde{I}_R + G_R Q, \quad (43)$$

where $\tilde{I}_{L,R}$ are now the sources of bare noise, $G_{L,R}$ are the generalized conductances of the left and right point contact, and Q is the fluctuating charge in the cavity. Current conservation of the physical currents, $I_L = I_R = I$, can now be used to obtain

$$I = \frac{G_R \tilde{I}_L + G_L \tilde{I}_R}{G_L + G_R}, \quad Q = \frac{\tilde{I}_L - \tilde{I}_R}{G_L + G_R}. \quad (44)$$

Combining powers of I and Q and averaging over the bare noise, we obtain the minimal correlation result for arbitrary cumulants $\langle \langle Q^k I^l \rangle \rangle_m$. In particular, using $\langle \langle \tilde{I}_L \tilde{I}_R \rangle \rangle = 0$, we find the second cumulant of current is^{17,18}

$$\langle \langle I^2 \rangle \rangle = \langle \langle I^2 \rangle \rangle_m = \frac{G_R^2 \langle \langle \tilde{I}_L^2 \rangle \rangle + G_L^2 \langle \langle \tilde{I}_R^2 \rangle \rangle}{(G_L + G_R)^2}, \quad (45)$$

where the subscript m denotes the minimal correlation result. We stress that the bare correlators $\langle \langle \tilde{I}_{L,R}^2 \rangle \rangle$ are fully determined by the average occupation function f_0 of the cavity.

This example demonstrates that a simple redefinition of the current fluctuations makes it straightforward to find the noise. Therefore, it came as a surprise⁴⁵ that the minimal correlation approach is not sufficient to correctly obtain higher-order cumulants of current. The reason for the failure of the minimal correlation approach has been found recently by Nagaev,¹⁹ who showed that from the third order cumulant on, there are “cascade corrections” to the minimal correlation result, which may be interpreted as “noise of noise”. For example, the third cumulant of current through the mesoscopic cavity,²⁰

$$\langle \langle I^3 \rangle \rangle = \langle \langle I^3 \rangle \rangle_m + 3 \langle \langle I Q \rangle \rangle_m \frac{\partial}{\partial Q} \langle \langle I^2 \rangle \rangle_m, \quad (46)$$

contains a contribution from fluctuations of the charge in the cavity that couples back into the current fluctuations. The factor of 3 comes from the fact that there are 3 independent currents that the charge fluctuation may be correlated with. For higher cumulants, there will be more cascade corrections that may be represented in a diagrammatic form.^{19,20}

B. Derivation of Diagrammatic Rules.

We now present a derivation of these diagrammatic rules for a single node attached between two absorbing nodes. Generalizations to an arbitrary network will subsequently be given in Sec. IV D. As we have shown in Sec. II C, the charge scale imposed by the boundary conditions, γ , gives a dimensionless large parameter which justifies the saddle point approximation of the path integral, so that fluctuations around the saddle point are suppressed by $1/\gamma$. In the diagrammatic language, we will show that loop diagrams are suppressed by the same factor $1/\gamma$. The diagrammatic approach given here is based on perturbation theory originally developed in quantum mechanics.²⁹

Consider the path integral expression of the generating function for the charge absorbed in the left (L) and right (R) node:

$$Z(\chi_L, \chi_R) = \int \mathcal{D}Q \mathcal{D}\lambda \exp \left\{ \int_0^t dt' [-i\dot{Q}\lambda + H(Q, \lambda, \chi_L, \chi_R)] \right\}, \quad (47)$$

where $H = H_L(Q, \lambda - \chi_L) + H_R(Q, \chi_R - \lambda)$. The perturbation theory is formulated as follows. First, the external counting variables are set to zero, $\chi_L = \chi_R = 0$. The Hamiltonian $H \rightarrow H_L(Q, \lambda) + H_R(Q, -\lambda)$ has a stationary saddle point located at $\{Q_0, \lambda_0\}$ that we wish to define as the origin of coordinates. The probability distributions of transferred charge are normalized, so

$$\partial_Q^n H_{L,R}(Q, \lambda)|_{\lambda=0} = 0, \quad \forall n. \quad (48)$$

In particular, $\partial_Q H_L(\lambda)|_{\lambda=0} = \partial_Q H_R(\lambda)|_{\lambda=0} = 0$, and therefore $\lambda_0 = 0$. Next, $\partial_{i\lambda} H(\lambda)|_{\lambda=0} = \langle I_L(Q) \rangle - \langle I_R(Q) \rangle = 0$, since H_L and H_R are the generators of the left and right current respectively. Therefore, Q_0 is fixed as the charge in the node such that left and right connector currents are equal on average. The stability of the saddle point is guaranteed by the fact that the bare noise correlators, $\langle \langle \tilde{I}_{L,R}^2 \rangle \rangle$, are positive. The derivatives $\partial_{i\lambda} \partial_Q H_L = -G_L$, $\partial_{i\lambda} \partial_Q H_R = -G_R$ define the generalized conductance of each connector, where the current flows from left to right in both connectors.

The principle of minimal correlation plays an important role in the cascade diagrammatics. We will show that this principle is equivalent to exploiting certain freedoms in the path integral in order to postpone the cascade corrections to third and higher order cumulants. In

the long-time limit, $t \gg \tau_C$ (where $1/\tau_C = G_L + G_R$ is the relaxation rate of the charge in the node), the absorbed current is conserved, $I_R = I_L$. Therefore, the current through the node can be defined as weighted average of the left and right connector currents $I = (1-v)I_L + vI_R$, where v is arbitrary constant. The corresponding counting variable χ is introduced by substituting $\chi_R = v\chi$ and $\chi_L = (v-1)\chi$. Consider now the second derivative

$$\left. \frac{\partial^2 H}{\partial i\chi \partial Q} \right|_{\chi=0} = (v-1)G_L + vG_R. \quad (49)$$

We may set it to zero by fixing $v = G_L/(G_L + G_R)$. This is equivalent to imposing conservation of current fluctuations as in Eq. (44). If we consider further the derivative

$$\left. \frac{\partial^2 H}{\partial i\lambda \partial Q} \right|_{\chi=0} = -(G_L + G_R), \quad (50)$$

we have the freedom to scale λ to make the right hand side of Eq. (50) equal to -1 [this scaling only alters the χ independent prefactor of Eq. (47)]. The Hamiltonian takes the new form

$$H = H_L \left(Q, \frac{G_R \chi + \lambda}{G_L + G_R} \right) + H_R \left(Q, \frac{G_L \chi - \lambda}{G_L + G_R} \right). \quad (51)$$

We refer to these new variables as minimal correlation coordinates and will see that they simplify the diagrammatic expansion.

Define $\delta Q(t) = Q(t) - Q_0$ and $\delta \lambda(t) = \lambda(t) - \lambda_0$. If we expand the Hamiltonian in a power series in χ , δQ , and $\delta \lambda$, the terms linear in δQ and $\delta \lambda$ vanish at the saddle point, as well as the $(\delta Q)^2$ coefficient by Eq. (48) with $n = 2$. As argued above, in the minimal correlation coordinates, $\partial_{i\lambda} \partial_Q H(Q_0, \lambda_0) = -1$. With these transformations, we may split the action S as

$$S = S_0 + \int_0^t dt' V(t'), \quad S_0 = -i \int_0^t dt' \delta \lambda (\tau_C \delta \dot{Q} + \delta Q), \quad (52)$$

where V represents the rest of the H power series and will be treated perturbatively. It should be emphasized that V is a general nonlinear function of $\delta \lambda$, so unlike most quantum examples, the full momentum dependence must be kept.

In order to formulate the perturbation theory, we add two sources, J and K to the action, $S \rightarrow S + \int dt' [J\delta Q + iK\delta \lambda]$, so that any average of a function of the variables $\delta Q, \delta \lambda$ may be evaluated by taking functional derivatives with respect to the sources J , and K , and then setting the sources to zero. In particular, for the generating function we can write

$$\begin{aligned} Z(\chi) &= \int \mathcal{D}Q \mathcal{D}\lambda \exp \left\{ \int_0^t dt' V(\delta Q, \delta \lambda, \chi) \right\} \exp \left\{ S_0 + \int_0^t dt' [J\delta Q + iK\delta \lambda] \right\} \Big|_{J,K=0} \\ &= \exp \left\{ \int_0^t dt' V \left(\frac{\delta}{\delta J}, \frac{\delta}{\delta iK}, \chi \right) \right\} \int \mathcal{D}Q \mathcal{D}\lambda \exp \left\{ S_0 + \int_0^t dt' [J\delta Q + iK\delta \lambda] \right\} \Big|_{J,K=0} \end{aligned} \quad (53)$$

Using S_0 from Eq. (52) we evaluate the integral over Q and λ and obtain:

$$Z(\chi) = \exp \left\{ \int_0^t dt' V \left(\frac{\delta}{\delta J}, \frac{\delta}{\delta iK}, \chi \right) \right\} W(J, K) \Big|_{J,K=0}, \quad (54)$$

where the functional $W(J, K)$ is

$$W(J, K) = \exp \left\{ \int \int_0^t dt' dt'' J(t') D(t', t'') K(t'') \right\}. \quad (55)$$

The operator $D = (\tau_C \partial_t + 1)^{-1}$ is the retarded propagator, and may be found explicitly by inverting the kernel in frequency space,

$$\begin{aligned} D(t, t') &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{-i\tau_C \omega + 1} \\ &= \tau_C^{-1} \Theta(t-t') \exp[-(t-t')/\tau_C]. \end{aligned} \quad (56)$$

It describes the relaxation of the charge $Q(t)$ to the stationary state Q_0 with the rate $1/\tau_C = G_L + G_R$.

Expanding the exponential in Eq. (54) and taking the $t \gg \tau_C$ limit, we arrive at the following expression for the n th current cumulant

$$\begin{aligned} \langle \langle I^n \rangle \rangle &= t^{-1} \frac{\delta^n}{\delta (i\chi)^n} \sum_{m=1}^{\infty} \frac{1}{m!} \left[\int_0^t dt' V \left(\frac{\delta}{\delta J}, \frac{\delta}{\delta iK}, \chi \right) \right]^m \\ &\quad \times W(J, K) \Big|_{\substack{\chi=J=K=0 \\ \text{connected}}}. \end{aligned} \quad (57)$$

According to the linked cluster expansion,³² by considering $\ln Z(\chi)$ rather than $Z(\chi)$, we have eliminated all disconnected terms. In order to compare with the results of Ref. 20, we introduce a new notation by defining

$$\partial_Q^j \langle \langle Q^k I^l \rangle \rangle_m \equiv \partial_Q^j \partial_{i\lambda}^k \partial_{i\chi}^l V(Q_0, \lambda_0, \chi = 0). \quad (58)$$

Here $\langle \langle Q^k I^l \rangle \rangle_m$ is the irreducible correlator expressed in terms of the noise sources, i.e. the minimal correlation cumulant. In this notation, the expansion of V in a Tay-

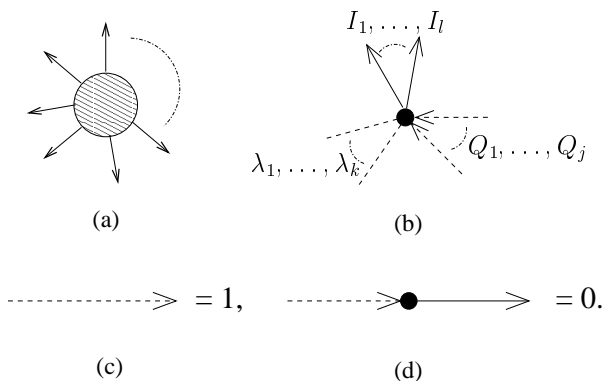


FIG. 4: (a) An n -point current cumulant. (b) The vertex connecting l external lines with j internal Q lines and k internal λ lines. (c) The propagator connecting λ to Q , equal to 1 in the stationary limit. (d) The vanishing vertex $\partial_Q \langle I \rangle$ in minimal correlation coordinates.

lor series of all variables takes the form:

$$V(\delta Q, \delta \lambda, \chi) = \sum_{j,k,l} \frac{1}{j!k!l!} \partial_Q^j \langle \langle Q^k I^l \rangle \rangle_m \times [\delta Q(t)]^j [i\delta \lambda(t)]^k [i\chi]^l. \quad (59)$$

Inserting the expansion Eq. (59) into the formula for the current cumulants Eq. (57) gives the formal solution to the problem. From the form of $W(J, K)$ and V , we can immediately read off the diagrammatic rules with the internal lines given by the propagators (56), and the expansion coefficients $\partial_Q^j \langle \langle Q^k I^l \rangle \rangle_m$ playing the role of vertices.

The following simplifications can be done before the rules are finally formulated. First, it is straightforward to see that loop diagrams are suppressed by powers of γ^{-1} . Indeed, according to our single-parameter scaling assumption, the action (52) has a large prefactor γ , which can be explicitly displayed, $S \rightarrow \gamma S$, by rescaling the charge, $Q \rightarrow \gamma Q$. Then it becomes clear that each propagator D , represented by an internal line, comes with a factor of γ^{-1} . Each vertex comes from V and therefore has a factor of γ . If a diagram has I internal lines, E external legs, V vertices and L loops, it will come with a total γ power of $V - I$. Furthermore, Euler's formula tells us that $V + L - I = 1$. Therefore, diagrams with no loops ("tree" diagrams) come with a power of γ , while loop diagrams are suppressed by the number of loops, γ^{1-L} . From now on we will concentrate on tree-level diagrams, since they represent current cumulants at the level of the saddle-point approximation.

Second, in the long time limit, $t \gg \tau_C$, each propagator (56) integrated over time gives 1. As a result, since every vertex is connected to at least one other vertex, all the time integrals together simply give a factor of t , and the time dependence cancels on the right hand side of the Eq. (57). There are no time integrals in the vertices and the propagators just give a factor of 1 as in Ref. 20. We are now able to formulate the diagrammatic rules for high-order current cumulants:

1. The n th order cumulant $\langle \langle I^n \rangle \rangle$ is a connected n -point function of n external legs I represented by solid arrows (see Fig. 4a).
2. The external legs must be connected by using vertices (see Fig. 4b) and linking internal dashed lines to internal dashed arrows.
3. The vertices $\partial_Q^j \langle \langle I^l Q^k \rangle \rangle_m$ are represented by a circle with l external legs, k internal outgoing dashed lines, and j internal incoming dashed arrows (see Fig. 4b).
4. Multiply each diagram by the number of inequivalent permutations (NIP).

Formally, the vertices $\partial_Q^j \langle \langle I^l Q^k \rangle \rangle_m$ are the expansion coefficients in (59). However, it is important to note that they can also be easily evaluated by solving the Langevin equations (43) and expressing the minimal correlation cumulants $\langle \langle I^l Q^k \rangle \rangle_m$ in terms of cumulants of the noise sources, $\langle \langle \tilde{I}_L^{l+k} \rangle \rangle$ and $\langle \langle \tilde{I}_R^{l+k} \rangle \rangle$. Some vertices are zero, $\partial^p H / \partial Q^p(Q_0, \lambda_0)|_{\chi=0} = 0$ because of probability conservation, but other may or may not be zero depending on the physical system. Here, the advantage of the minimal correlation coordinates is made clear: the vertex $\partial_Q \langle \langle I \rangle \rangle_m = 0$, and therefore any diagram that contains this vertex is zero (see Fig. 4d).

To obtain the overall prefactor of a diagram, one can write out all the numerical constants and count the number of different ways of producing the same diagram.³² For example, there is the $n!$ from the χ derivatives, the $1/m!$ from the Taylor series of e^V , a binomial coefficient from expanding V^m , and the $1/(j!k!l!)$ from every vertex with $j+k+l$ attachments for the different lines. To compensate these factors, we have to do the combinatorics of the number of equivalent terms: interchange the vertices, find the number of different placements of lines on a vertex, etc. Often, the number of permutations of the n external legs will cancel the $m!$, and the $j!k!l!$ number of permutations of the internal legs attaching to the vertex will cancel that factor arising from the Taylor expansion.

Rather than making this expansion, there is a simpler method which exploits these cancellations given by counting the number of inequivalent permutations of the diagram (NIP). The NIP of the diagram is defined by how many ways the external legs of the diagram may be relabeled, such that the diagram is not topologically equivalent under deformation of the external legs. In other words, a diagram with n external legs has $n!$ ways of labeling them. If this diagram with a given labeling of the legs may be topologically deformed to give the diagram back with a different labeling, these two sets of labelings are equivalent permutations. If we write out all the different labelings the external legs can have, and cross out every labeling that is an equivalent permutation of another, then the number of labelings that remain is the NIP. This number is most easily found by dividing $n!$ by the number of equivalent permutations of the diagram.

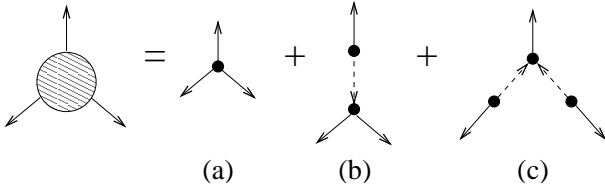


FIG. 5: Tree level contributions to the third cumulant of transmitted current.

The number of equivalent permutations of the diagram is also called the symmetry factor of the diagram.

We illustrate these two approaches with the third cumulant, see Fig. 5. With the simplifications discussed above, these diagrams may be written as

$$\begin{aligned} \langle\langle I^3 \rangle\rangle &= \langle\langle I^3 \rangle\rangle_m + 3\langle\langle I Q \rangle\rangle_m \frac{\partial}{\partial Q} \langle\langle I^2 \rangle\rangle_m \\ &+ 3\langle\langle I Q \rangle\rangle_m^2 \frac{\partial^2}{\partial Q^2} \langle\langle I \rangle\rangle_m. \end{aligned} \quad (60)$$

Note that diagram (c) does not appear in Ref. 20, because it happens to vanish for the chaotic cavity [see also Eq. (46)]. Referring to the formula (57), the contributions in Eq. (60) are from $m = 1, 2, 3$ respectively. Each diagram must have a χ^3 term in the expansion. We first show the combinatorial method to obtain the prefactor: Diagram (a) has a factor of $1/3!$ from the number of permutations of the χ variables, canceling the $3!$ from the χ derivatives. Diagram (b) has a factor of $1/2!$ from the number of permutations of the χ variables, a factor of $1/2!$ from the Taylor series of the exponential, a factor of 2 from the binomial expansion of V^2 , and the $3!$ from the χ derivatives, leaving a factor of 3. Diagram (c) has a factor of $1/3!$ from the Taylor series of the exponential, a factor of 3 from the binomial expansion of V^3 , a factor of $1/2!$ from the number of permutations of the δQ variables, a factor of 2 from the functional derivatives acting on W , and the $3!$ from the χ derivatives, leaving a factor of 3. The NIP is simpler to derive: We divide the number of permutations of the external legs, $m!$, by the number of equivalent permutation of the elements of the diagram that leave it unchanged. The number of equivalent permutations of diagrams (a,b,c) are $3!, 2!, 2!$, leaving the overall factors 1, 3, 3.

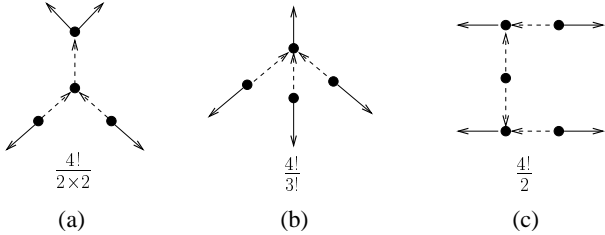


FIG. 6: Three examples of diagrams contributing to the fourth cumulant.

The computation of these diagrammatic contributions is best understood by a little practice on some examples. Consider three of the diagrams that contribute to the fourth cumulant drawn in Fig. 6. The diagrams symbolically represents the combinations:

$$(a) = \frac{\partial}{\partial Q} \langle\langle I^2 \rangle\rangle_m \frac{\partial^2}{\partial Q^2} \langle\langle Q \rangle\rangle_m \langle\langle I Q \rangle\rangle_m^2, \quad (61a)$$

$$(b) = \frac{\partial^3}{\partial Q^3} \langle\langle I \rangle\rangle_m \langle\langle I Q \rangle\rangle_m^3, \quad (61b)$$

$$(c) = \langle\langle Q^2 \rangle\rangle_m \left(\frac{\partial^2}{\partial Q^2} \langle\langle I \rangle\rangle_m \right)^2 \langle\langle I Q \rangle\rangle_m^2. \quad (61c)$$

To figure out the numerical prefactors, we divide $4!$ (4 is the number of external legs) by the symmetry factor of the diagram. We first consider the symmetry factor of (a): The upper two legs may be flipped, and the lower two legs may be independently flipped where the dotted arrows join without altering the topology of the diagram. Therefore, the symmetry factor is $2 \times 2 = 4$, and the NIP is $4!/4 = 6$. Moving on to diagram (b), the three lower legs may be permuted amongst themselves to give a symmetry factor $3!$, and therefore the NIP is $4!/3! = 4$. Finally, diagram (c) may be flipped about its center for a symmetry factor of 2, giving a NIP of $4!/2 = 12$.

C. Operator approach.

In the stationary limit, $t \gg \tau_C$, the action takes the form $S = tH(Q, \lambda, \chi)$ so that the evaluation of the cumulant generating function reduces to finding the stationary point of the Hamiltonian H as a function of variables λ and Q . This can be done by solving equations $\partial_Q H = 0$ and $\partial_\lambda H = 0$. The generating function is then obtained by substituting the solutions $\{\bar{Q}, \bar{\lambda}\}$ into the Hamiltonian. In the previous section we have shown that this problem can be solved using path integral methods, and the solution can be represented diagrammatically. In the next section we will exploit the full strength of the path integral formalism in order to generalize the diagrammatics to an arbitrary network, and for the case of time-dependent charges. However, in the stationary limit, the conceptual simplicity of the problem of finding the stationary point of the function H indicates that there should exist a simple iterative procedure for evaluating the cumulants up to a given order. In this section we use classical mechanics methods to prove that this is indeed the case.

We first make the variable transformation $i\lambda \rightarrow \lambda$, and $i\chi \rightarrow \chi$, so that the Hamiltonian becomes a real function. For $\chi = 0$ the saddle point is located at $\{Q_0, \lambda_0\}$. For non-zero χ the saddle point moves to a new position $\{\bar{Q}, \bar{\lambda}\}$, which depends on χ , and the Hamiltonian $H(\bar{Q}, \bar{\lambda}, \chi)$ becomes the generator of cumulants of the current,

$$\langle\langle I^n \rangle\rangle = d^n H(\bar{Q}, \bar{\lambda}, \chi) / d\chi^n |_{\chi=0}. \quad (62)$$

By expressing the total χ derivative in terms of partial derivatives, the average current can be written as

$$\langle I \rangle = (\partial_\chi + Q' \partial_Q + \lambda' \partial_\lambda) H(Q, \lambda, \chi)|_{\{\chi=0, Q_0, \lambda_0\}}, \quad (63)$$

where $Q' = dQ/d\chi$, $\lambda' = d\lambda/d\chi$ are χ dependent. We wish to eliminate the functions Q' and λ' and to express the cumulant in terms of the partial derivatives of H . This is done by applying a total derivative to the equations of motion: $[\partial_Q H]' = [\partial_\lambda H]' = 0$ and leads to two equations for Q' and λ' which may be solved,

$$Q' = \frac{\{\partial_\lambda H, \partial_\chi H\}}{\{\partial_Q H, \partial_\lambda H\}}, \quad \lambda' = -\frac{\{\partial_Q H, \partial_\chi H\}}{\{\partial_Q H, \partial_\lambda H\}}, \quad (64)$$

where $\{A, B\}$ is the Poisson bracket, defined as $\{A, B\} = \partial_\lambda A \partial_Q B - \partial_Q A \partial_\lambda B$. The solutions have to be inserted into the Eq. (63).

The advantage of this representation is clear: Now the right hand side of the Eq. (63) (before taking the $\chi = 0$ saddle point) depends only on variables λ , Q , and χ . Therefore, we can apply the procedure again in order to express the high-order cumulant in terms of partial derivatives. This procedure solves the problem by giving a single operator,

$$D = \partial_\chi + \frac{\{\partial_\lambda H, \partial_\chi H\} \partial_Q - \{\partial_Q H, \partial_\chi H\} \partial_\lambda}{\{\partial_Q H, \partial_\lambda H\}}, \quad (65)$$

which, being applied n times to a given Hamiltonian H and evaluating the resulting expression at the $\chi = 0$ saddle point, gives cumulants of current:

$$\langle\langle I^n \rangle\rangle = D^n H(Q, \lambda, \chi)|_{\{\chi=0, Q_0, \lambda_0\}}. \quad (66)$$

This approach is obviously more simple compared to the diagrammatic method, since in the diagrammatics, after drawing all of the diagrams, they have to be evaluated individually by taking many partial derivatives of the Hamiltonian and evaluating them at the $\chi = 0$ saddle point. With this new approach, given the Hamiltonian H , the operator D may be constructed (65) and with a mathematical program, an arbitrary cumulant may be easily computed (66).

It is easy to see the importance of the minimal correlation coordinates in this solution. After applying D several times, the derivative quotient rule generates a large number of denominators, $\{\partial_Q H, \partial_\lambda H\} = (\partial_Q \partial_\lambda H)(\partial_\lambda \partial_Q H) - (\partial_Q \partial_Q H)(\partial_\lambda \partial_\lambda H)$. At $\chi = 0$, as we argued previously, $\partial_Q \partial_Q H = 0$, and it is possible to change coordinates so that $\partial_Q \partial_\lambda H = -1$. As a result, the denominator in (66) is equal to 1, which greatly simplifies the expansion. Finally, we would like to stress that the operator approach, introduced in this section for the one node case, can be easily generalized to a network.

D. Network Cascade Diagrammatics: Correlation Functions.

Consider now a general network. In the Sec. IV B, we saw that the dominant contribution to Eq. (47) arises

from tree-level diagrams. On time scales $t \gg \tau_C$, the time dependence drops out, and the current cumulants are static. We now generalize the diagrammatic rules presented in the section IV B to investigate time- and node-dependent correlation functions of conserved and absorbed charges, Eq. (19). To define the network, we must arbitrarily label the current flow, yielding a directed network. By doing so we fix the signs of the elements $H_{\alpha\beta} = -H_{\beta\alpha}$ of the Hamiltonian. In particular, the elements of the generalized conductance matrix \hat{G} ,

$$G_{\alpha\beta} = \frac{\partial^2 H}{\partial(i\lambda_\alpha \partial Q_\beta)}, \quad (67)$$

(evaluated at $\mathbf{Q} = \mathbf{Q}_0$, $\mathbf{\Lambda} = 0$) are negative or positive depending on the chosen direction. If we segregate absorbing (a) and conserving (c) nodes, the conductance matrix \hat{G} may be put in block form. Two of them, the blocks \hat{G}_{cc} (real symmetric) and \hat{G}_{ac} will be relevant. This gives us the necessary tool to define the generalized minimal correlation coordinates. We consider the frequency dependent response by letting the evolution time extent to infinity, and introduce the time Fourier transform of the variables $\{\mathbf{Q}^c, \mathbf{\Lambda}^c, \chi^c, \chi^a\}$, where the vector $\{\chi^c, \chi^a\}$ is a time-dependent source term introduced to produce correlation functions of the conserved and absorbed currents [see Eq. (19)].

Following the steps of section IV B, we again split the action into two parts, $S = S_0 + \int dt V$, where

$$\begin{aligned} S_0 &= i \int dt [-\mathbf{\Lambda}^c \dot{\mathbf{Q}}^c + \mathbf{\Lambda}^c \hat{G}_{cc} \mathbf{Q}^c + (\chi^c \hat{G}_{cc} + \chi^a \hat{G}_{ac}) \mathbf{Q}^c] \\ &= i \iint \frac{d\omega d\omega'}{2\pi} [\mathbf{\Lambda}^c(i\omega' + \hat{G}_{cc}) \mathbf{Q}^c \\ &\quad + (\chi^c \hat{G}_{cc} + \chi^a \hat{G}_{ac}) \mathbf{Q}^c] \delta(\omega + \omega'), \end{aligned} \quad (68)$$

and where we have dropped the δ in front of the variables for simplicity. As in Sec. IV B, the generalized minimal correlation coordinates are defined by shifting and rescaling the $\mathbf{\Lambda}^c$ variables in order to eliminate the χ variables in Eq. (68). However, because χ is now a vector, the proportionality factor must be a frequency dependent matrix,

$$\mathbf{\Lambda}^c(\omega) \rightarrow \hat{D}^\dagger(\omega) [\mathbf{\Lambda}^c(\omega) + \hat{G}_{cc}^\dagger \chi^c(\omega) + \hat{G}_{ca}^\dagger \chi^a(\omega)]. \quad (69)$$

Here $\hat{D}(\omega)$ is the matrix network propagator,

$$\hat{D}(\omega) = -(i\omega \hat{E} + \hat{G}_{cc})^{-1}, \quad (70)$$

and \hat{E} is the identity matrix. It is straightforward to verify that after the shift, the functional $\int dt V$ becomes the generator of cumulants of minimal correlation currents, i.e. of the currents which are solutions of the Langevin equations:

$$I_\alpha^c = -i\omega Q_\alpha^c = -i\omega \sum_{\beta\gamma} D_{\alpha\beta}(\omega) \tilde{I}_{\beta\gamma} \quad (71a)$$

$$I_\alpha^a = \sum_{\beta\gamma\alpha'} G_{\alpha\alpha'} D_{\alpha'\beta}(\omega) \tilde{I}_{\beta\gamma} + \sum_\gamma \tilde{I}_{\alpha\gamma}, \quad (71b)$$

where $\tilde{I}_{\alpha\beta}$ are the bare noise sources as defined in Eq. (5). We finally rescale $\chi^c(\omega) \rightarrow \chi^c(\omega)/(i\omega)$ in order to replace conserved currents with charges, $\mathbf{I}^c \rightarrow \mathbf{Q}^c$.

The total action now acquires the following form

$$S = (2\pi i)^{-1} \int d\omega \mathbf{\Lambda}^c(-\omega) \mathbf{Q}^c(\omega) + \int dt V \left[\mathbf{Q}^c, \hat{D}^\dagger(\mathbf{\Lambda}^c + \chi^c) + \hat{D}^\dagger \hat{G}_{ca}^\dagger \chi^a, \chi^a \right], \quad (72)$$

where the simplified form of the $\mathbf{\Lambda}$ argument of V follows after composing the various transformations. Following the plan of the previous section, we replace the charge and counting variables $\{\mathbf{Q}(\omega), \mathbf{\Lambda}(\omega)\}$ by functional derivatives with respect to the charge and counting

sources $\{\mathbf{J}(\omega), \mathbf{K}(\omega)\}$, and take the V term outside of the functional integral. The functional integrals may now be performed to obtain

$$W(\mathbf{J}, \mathbf{K}) = \exp \left\{ \iint \frac{d\omega d\omega'}{2\pi} \mathbf{J}(\omega') \mathbf{K}(\omega) \delta(\omega + \omega') \right\}. \quad (73)$$

The perturbation V must now be expanded in a Taylor series with respect to all variables. The time dependence only appears through the variables themselves, so the expansion coefficients will be time independent, with the exception of the propagator $D_{\alpha\beta}(\omega)$ multiplying the counting variables.

$$V = \sum_{\{\alpha, j_\alpha, k_\alpha, l_\alpha\}=0}^{\infty} \frac{\delta^{j_1+\dots+j_n}}{\delta(Q_1^c)^{j_1} \dots \delta(Q_n^c)^{j_n}} \langle\langle (I_1^a)^{l_1} \dots (I_r^a)^{l_r} (Q_1^c)^{i_1} \dots (Q_q^c)^{i_q} (Q_1^c)^{k_1} \dots (Q_p^c)^{k_p} \rangle\rangle_m \times \frac{(\chi_1^a)^{l_1}}{l_1!} \dots \frac{(\chi_r^a)^{l_r}}{l_r!} \times \frac{(\chi_1^c)^{i_1}}{i_1!} \dots \frac{(\chi_q^c)^{i_q}}{i_q!} \times \frac{\lambda_1^{k_1}}{k_1!} \dots \frac{\lambda_p^{k_p}}{k_p!} \times \frac{(Q_1^c)^{j_1}}{j_1!} \dots \frac{(Q_n^c)^{j_n}}{j_n!}. \quad (74)$$

As in the one node case, the vertices $\delta_{Q_\alpha^c} \langle I_\beta^a \rangle$ vanish. We note again that the notation chosen for the expansion coefficients in Eq. (74) connects the formalism described here with the Langevin equation point of view. The minimal correlation cumulant $\langle\langle \dots \rangle\rangle_m$ may be calculated either by the expansion procedure described by Eqs. (72,74), or by expressing the physical currents and charges in terms of the current source cumulants by solving the Langevin equations for currents and charges, given by Eq. (71).

The n th order irreducible correlator $\langle\langle I_1^a(\omega_1) \dots Q_n^c(\omega_n) \rangle\rangle$ may be expressed as a tree-level diagram with n external lines representing absorbed currents I_α^a and conserved charges Q_α^c . Every vertex is local in time, so if there are p legs at a vertex, each is assigned an independent frequency, while the time integral imposes overall frequency conservation, $\delta(\sum_i \omega_i)$. The cascade rules are generalized as follows:

1. Every vertex represents the object

$$\delta_{Q_1^c(\omega_1)} \dots \delta_{Q_l^c(\omega_l)} \langle\langle I_{l+1}^a(\omega_{l+1}) \dots Q_n^c(\omega_n) \rangle\rangle_m,$$

which is multiplied by a δ -function conserving overall frequency, $\delta(\sum_{i=1}^n \omega_i)$.

2. The minimal correlation cumulants $\langle\langle I_{l+1}^a(\omega_{l+1}) \dots Q_n^c(\omega_n) \rangle\rangle_m$ may be evaluated by expressing them in terms of cumulants of sources $\langle\langle I_{\alpha\beta}^n \rangle\rangle$ via the solutions (71) of the Langevin equations, or by Eq.(74) if the Hamiltonian is known.

3. The internal dashed arrow goes from $Q_\alpha^c(\omega)$ to $\delta_{Q_\alpha^c(\omega)}$. It conserves the node index α and the frequency ω .⁴⁶
4. External lines for absorbed currents and conserved charges originate from $I_\alpha^a(\omega)$ or $Q_\alpha^c(\omega)$ of the vertices. They conserve the node index and the frequency.
5. Sum over all internal node indices, and integrate over all internal frequencies to remove all but one of the frequency delta functions.
6. The result has to be multiplied by the total number of inequivalent permutations.

The cascade rules are easily extended to the field theory (see Sec. III). The functional analog to the inverse conductance matrix is the operator

$$\hat{G}^{-1}(\mathbf{r} - \mathbf{r}') \equiv \frac{\delta^2 h}{\delta\lambda(\mathbf{r}) \delta\rho(\mathbf{r}')} = -\delta(\mathbf{r} - \mathbf{r}') \nabla \hat{D} \nabla. \quad (75)$$

The diffusion propagator $(i\omega + \hat{G}^{-1})^{-1}$ can be used to solve the Langevin equations (42) for the density $\rho(\omega, \mathbf{r})$ and current $I(\omega)$ in order to evaluate minimal correlation cumulants. We would like to stress that these cumulants are limited to second order only, because in the diffusion limit the noise sources are Gaussian. The summation over node indices is replaced with an integration over the coordinate \mathbf{r} .

V. APPLICATIONS

The formalism presented above is intentionally abstract and general. This is to facilitate maximum applicability and not to tie it to a particular field. However, it is important to give concrete examples. For this reason, we give a detailed treatment of two problems. As a first problem, we consider the saddle-point equations of the 1D field theories for D and F being arbitrary functions of the density ρ [see Eq. (33)]. We apply the results of this analysis to the transport in a diffusive mesoscopic wire at zero temperature, rederive the FCS generating function of the transmitted charge obtained in Refs. 8 and 10, and give new results. We also prove the conjecture made in Ref. 22 that the current noise of the diffusive symmetric exclusion process at half-filling is Gaussian, i.e. all high-order cumulants of transmitted charge vanish. In the end of the Sec. V A we generalize our results to multi-dimensional diffusion models and prove the universality of their transport statistics. As a second problem, we address the statistics of charge fluctuations in a mesoscopic chaotic cavity. We explicitly find the probability distributions for different physical configurations.

A. FCS for one-dimensional field theories. The mesoscopic diffusive wire.

Before demonstrating our solution for the FCS of the mesoscopic diffusive wire specifically, we first consider the general 1D field theory with the action (33). In the stationary limit, $\dot{\rho} = \dot{\lambda} = 0$ the action can be written as

$$S = t \int_0^L dz \left[-D\rho'\lambda' + \frac{1}{2}F(\lambda')^2 \right]. \quad (76)$$

The stationary saddle-point equations

$$(F\lambda' - D\rho')' = 0, \quad 2D\lambda'' + \frac{\delta F}{\delta \rho}(\lambda')^2 = 0, \quad (77)$$

can be partially integrated leading to the following two equations:

$$D\rho' = \pm \sqrt{\mathcal{I}^2 - 2\mathcal{H}F}, \quad (78a)$$

$$\lambda' = 2\mathcal{H}/(\mathcal{I} - D\rho'). \quad (78b)$$

The two integration constants $\mathcal{I} = -D\rho' + F\lambda'$ and $\mathcal{H} = -D\rho'\lambda' + (F/2)(\lambda')^2$ are the conserved (conditional) current and the Hamiltonian density, respectively. These conservation laws follow from the symmetries of our 1D field theory [see Eqs. (38) and (39) and the surrounding discussion]. Thus we obtain the following result for the action (76),

$$S = tL\mathcal{H}. \quad (79)$$

The equations (78) and (79) represent the formal solution of the FCS problem for 1D diffusion models with $D(\rho)$ and $F(\rho)$ being arbitrary functions of ρ . The following procedure has to be done in order to obtain the cumulant generating function $S(\chi)$ of the transmitted charge:

1. The differential equation (78a) has to be solved for $\rho(z)$ with the boundary conditions $\rho(z)|_{z=0} = \rho_L$ and $\rho(z)|_{z=L} = \rho_R$. The constant \mathcal{I} should be expressed through the constants ρ_L , ρ_R , and \mathcal{H} .
2. Next, $\rho(z)$ is substituted into Eq. (78b) which is integrated to obtain $\lambda(z)$ with the boundary conditions $\lambda_L = 0$ and $\lambda_R = \chi$.
3. Finally, using the solution for $\lambda(z)$ the constant \mathcal{H} is expressed in terms of ρ_L , ρ_R , χ , and substituted into the action (79).

We note that by expressing \mathcal{H} and χ in terms of \mathcal{I} , we may also formally obtain the logarithm of the current distribution,

$$\ln P(I) = S(\mathcal{I}) - t\mathcal{I}\chi(\mathcal{I}), \quad \mathcal{I} \rightarrow I, \quad (80)$$

as a result of the stationary phase approximation for the integral $P(I) = \int d\chi \exp[S(\chi) - tI\chi]$ and because $\partial\mathcal{H}/\partial\chi = \mathcal{I}/L$.

As an example of the 1D field theory, we consider the FCS of the electron charge transmitted through the mesoscopic diffusive wire. When the chemical potential difference $\Delta\mu = \mu_L - \mu_R > 0$ is applied to the wire, the electrons flow from the left lead to the right lead with the average current $I_0 = e^{-1}G\Delta\mu$, where G is the conductance of the wire. The elastic electron scattering causes non-equilibrium fluctuations of the current. At zero temperature, and for noninteracting electrons (the cold electron regime), the zero-frequency current noise power has been found^{15,47,48} to be equal to $\langle\langle I^2 \rangle\rangle = (1/3)eI_0$, i.e. the noise is suppressed compared to the Poissonian value. The suppression factor 1/3 was shown to be universal,^{49,50} i.e. it does not depend on the character of the disorder or on the shape of the wire. The FCS of the transmitted charge has been studied in Refs. 8 and 10 using quantum-mechanical methods, and recently in Ref. 22 using a classical method with the following result for the generating function of cumulants of the dimensionless charge Q/e :

$$S(\chi) = (tI_0/e) \operatorname{arcsinh}^2 \left[\sqrt{\exp(\chi) - 1} \right]. \quad (81)$$

Here we will rederive this result using our classical method.

On the classical level, the electrons in the diffusive wire are described by the distribution function $f(z)$. Under transport conditions (and at zero temperature), this distribution $f(z)$ varies from $f_L = 1$ in the left lead to $f_R = 0$ in the right lead. Starting from the Langevin equation³² as described in Sec. III or, alternatively, taking the continuum limit for the series of mesoscopic

cavities,³⁶ we arrive at the action (76) with the form⁵¹

$$S = (tI_0/e) \int_{-1/2}^{1/2} dz [-f'\lambda' + f(1-f)(\lambda')^2], \quad (82)$$

where we have rescaled the coordinate z , $\rho(z)$ has been replaced with the distribution $f(z)$, and where $D = 1$, and $F = 2f(1-f)$ up to the overall constant I_0/e . This form of F is quite general for fermionic systems. It originates from the Pauli blocking factors, i.e. the transition probability is proportional to the probability that the initial state is populated times to probability that the final state is empty.¹⁵ Applying now the procedure described in the beginning of this section, we solve the saddle-point equations and find the fields f and λ ,

$$f(z, \chi) = \frac{1}{2} \left[1 - \frac{\sinh(2\alpha z)}{\sinh \alpha} \right], \quad (83a)$$

$$\lambda(z, \chi) = 2 \operatorname{arctanh} [\tanh(\alpha/2) \tanh(\alpha z)], \quad (83b)$$

$$\alpha = \operatorname{arcsinh} \left[\sqrt{\exp(\chi) - 1} \right], \quad (83c)$$

where $\mathcal{H} = \alpha^2$, so that according to the Eq. (79) we immediately obtain the result (81).

The logarithm of the current distribution $\ln[P(I)]$ can be now found from the equation (80). We obtain the following result:

$$\ln[P(I)] = -(tI_0/e)[2\alpha \coth \alpha \ln(\cosh \alpha) - \alpha^2], \quad (84)$$

where α has to be expressed in terms of $\mathcal{I} = I/I_0$ by solving the equation

$$\alpha \coth \alpha = I/I_0. \quad (85)$$

The last equation has real positive solutions, $0 < \alpha < \infty$, for $I > I_0$, and pure imaginary solutions $\alpha = i\beta$ with $0 < \beta < \pi/2$, for $I < I_0$. The distribution $P(I)$ is strongly asymmetric around the average current $I = I_0$ (see Fig. 7). It has the following asymptotics: $\ln P = -(tI_0/e)[\mathcal{I}^2 - (2 \ln 2)\mathcal{I}]$, for $\mathcal{I} = I/I_0 \gg 1$, i.e. P has a Gaussian tail, and $\ln P = -(\pi^2/4)(tI_0/e)$, for $I = 0$.

We also plot the conditional electron occupation $f(z, I)$, Eq. (83a), for different values of the normalized current I/I_0 . There are several interesting points to stress. (i) For large currents, $I > I_0$, the function f drops mostly at the ends of the wire, while for small currents, $I < I_0$, the drop of f is mostly concentrated in the center of the wire. This effect has a simple explanation. At the end points of the wire, $z = \pm 1/2$, the occupation $f(z)$ is fixed independent of the particular value of the current I . On the other hand, its derivative takes the value $f' = -\mathcal{I} = -I/I_0$ at $z = \pm 1/2$, which can be easily verified using Eqs. (83a) and (85). As a result, $f(z)$

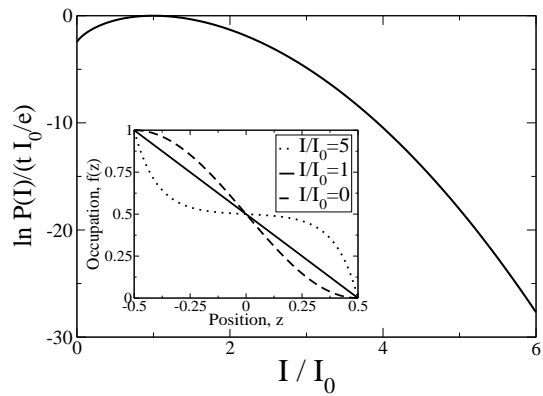


FIG. 7: The logarithm of the distribution of the current through a mesoscopic diffusive wire as a function of the ratio I/I_0 of the current to its average value I_0 . The distribution is strongly asymmetric, with the Gaussian tail at $I \gg I_0$. Inset: The electron occupation f inside the wire as a function of the rescaled coordinate z , under the condition that the average current $I = I_0$, no current $I = 0$, and large current $I = 5I_0$ has been measured.

deviates from its linear behavior, $f(z) = 1/2 - z$, characteristic of the average value of current, $\mathcal{I} = 1$. The actual reason for this effect is that according to Eq. (42b) the total current $\mathcal{I} = -f' + \nu$ contains a contribution from the source of noise, ν . The greatest contribution is concentrated at the center of the wire, where the noise power $F = 2f(1-f)$ has its maximum, while it vanishes at the ends of the wire. Since the current \mathcal{I} is conserved, f' has to be redistributed in such a way as to partially compensate the effect of the source ν . (ii) Fluctuations of f are strongly suppressed at the ends of the wire, which is imposed by the boundary conditions, and at the center of the wire, as a result of the discrete symmetry, $\{z \rightarrow -z; f \rightarrow 1 - f\}$. (iii) Eq. (85) has additional solutions with $\beta > \pi/2$. These solutions are not physical however, since f becomes negative or larger than 1 leading to $I < 0$, which is impossible at $T = 0$.

Returning to the saddle-point equations (77), we note that if $\delta F/\delta \rho = 0$ for a particular density ρ_0 , then the fields $\rho(z) = \rho_0$, and $\lambda(z) = \chi z/L$ solve these equations. The fluctuations of the current become Gaussian with the noise power $\langle\langle I^2 \rangle\rangle = F(\rho_0)/L$. This generalizes and proves the conjecture made in Ref. 22 that the noise of the diffusive symmetric exclusion process is Gaussian at half-filling, $f = 1/2$.

As a final remark we note that the whole class of multi-dimensional field theories,

$$S = t \int_{\Omega} d\mathbf{r} [-\nabla \lambda \hat{D} \nabla \rho + (1/2) \nabla \lambda \hat{F} \nabla \lambda], \quad (86)$$

with $\hat{D} = D(\rho) \hat{T}$, $\hat{F} = F(\rho) \hat{T}$, and \hat{T} being an arbitrary constant symmetric tensor,⁵² bear the same kind of the universality as the shot noise in diffusive conductors discussed above (see Refs. 49 and 50). The reason is that the field theory with the action (86) can be mapped on

the 1D theory with the action (76) by making use of the parameterization

$$\rho(\mathbf{r}) = \rho[\varphi(\mathbf{r})], \quad \lambda(\mathbf{r}) = \lambda[\varphi(\mathbf{r})], \quad (87)$$

where the function $\varphi(\mathbf{r})$ satisfies the equation

$$\nabla \cdot [\hat{T}\nabla\varphi(\mathbf{r})] = 0. \quad (88)$$

Using Eqs. (77) for ρ and λ as functions of φ , it is straightforward to verify that the fields $\rho(\mathbf{r})$ and $\lambda(\mathbf{r})$ given by (87) and (88) satisfy the saddle-point equations for the action (86). One of the equations is the conservation of current:

$$\mathbf{j} = -\hat{D}\nabla\rho + \hat{F}\nabla\lambda = \mathcal{I}\hat{T}\nabla\varphi. \quad (89)$$

Since the 1D Hamiltonian density is conserved, the action takes the following form

$$S = t\mathcal{G}\mathcal{H}, \quad (90)$$

where the constant \mathcal{G} depends only on the geometry of the boundary $\partial\Omega$ by Eq. (88):

$$\mathcal{G} = \int_{\Omega} d\mathbf{r} \nabla\varphi \hat{T} \nabla\varphi = \int_{\partial\Omega} ds \cdot \varphi \hat{T} \nabla\varphi. \quad (91)$$

Consider now a two-terminal diffusive wire, so that the surface $\partial\Omega$ consists of the left $\partial\Omega_L$ and right $\partial\Omega_R$ contact surfaces, and the open surface $\partial\Omega_0$ with no current through it. We choose the boundary conditions for φ to be

$$\varphi(\mathbf{r})|_{\partial\Omega_L} = 0, \quad \varphi(\mathbf{r})|_{\partial\Omega_R} = 1, \quad ds \cdot \hat{T}\nabla\varphi(\mathbf{r})|_{\partial\Omega_0} = 0, \quad (92)$$

so that $\rho(\mathbf{r})|_{\partial\Omega_L} = \rho_L$, $\rho(\mathbf{r})|_{\partial\Omega_R} = \rho_R$, $\lambda(\mathbf{r})|_{\partial\Omega_L} = 0$, $\lambda(\mathbf{r})|_{\partial\Omega_R} = \chi$, and $ds \cdot \mathbf{j}(\mathbf{r})|_{\partial\Omega_0} = 0$. Then \mathcal{H} becomes a function of ρ_L , ρ_R , and χ , and the action (90) is the generator of the cumulants of the transmitted charge. If instead, \mathcal{H} and χ are expressed in terms of \mathcal{I} (as above for the 1D theory), then one obtains the logarithm of the distribution of the current, $\ln[P(I)] = S(I) - tI\chi(I)$, where $I = \mathcal{G}\mathcal{I}$, according to equations (89), (91), and (92). The constant \mathcal{G} may be interpreted as a ‘‘geometrical conductance’’ of a wire. In particular, in the ‘‘ohmic’’ regime, i.e. when D is independent of ρ , we have $I_0 = \mathcal{I}(\chi)|_{\chi=0} = D(\rho_L - \rho_R)$, and therefore $\mathcal{G} = I_0/[D(\rho_L - \rho_R)]$. In this case, the ratio S/I_0 does not contain \mathcal{G} and becomes fully universal, proving also the universality of the result (81) as a special case.

To summarize, we have proven the universality of the FCS of the transmitted charge for a two-terminal multi-dimensional generalized wire described by the action (86) with the noise tensor $F(\rho)\hat{T}$, being an arbitrary function of the charge density ρ , and with the constant diffusion tensor $D\hat{T}$. The universality means that the FCS depends neither on the shape of the conductor, nor on its dimensionality.⁵³ The FCS of a mesoscopic wire given by Eq. (81) is a particular example of universal FCS. In the more general case, when \hat{D} is a function of ρ , the FCS depends on the geometry through only one parameter \mathcal{G} , the geometrical conductance given by Eq. (91).

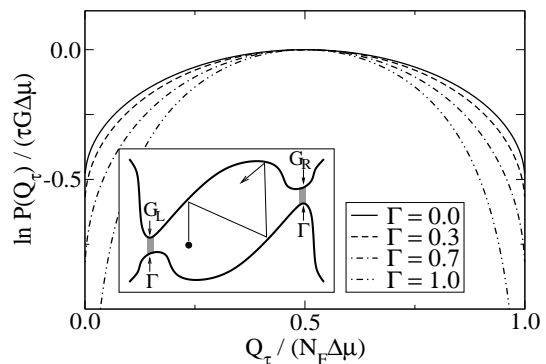


FIG. 8: The logarithm of the distribution of charge Q_τ in a symmetric cavity, $G_L = G_R$, averaged over measurement time τ in the long time limit $\tau \gg \tau_D$ and at zero temperature. The results are presented for several transparencies Γ of the point contacts. It is clearly seen that the tails of the distribution grow in the tunneling limit $\Gamma \ll 1$. The distribution is symmetric, i.e. odd cumulants vanish.

B. Charge fluctuations in a chaotic cavity.

As another example of the applicability of the stochastic path integral approach, we now consider transport through a chaotic cavity. This problem is often investigated in mesoscopic physics because of its simplicity and conceptual clarity. A cavity consists of a large conducting island of irregular shape that is connected to two metallic leads through quantum point contacts (see inset of Fig. 8). The distinctive property of the chaotic cavity separating it from diffusive conductors is that the conductance is determined solely by the ballistic point contacts. The chaotic cavity itself may be either disordered or ballistic. Chaotic cavities can be described by a semiclassical theory if the point contacts have conductances much larger than e^2/h . The statistics of current flow through the cavity have been addressed using various methods. The zero-frequency noise power has been calculated using random matrix theory⁵⁴ and the minimal correlation principle.¹⁸ The higher order current cumulants have been obtained in Refs. 7 and 20. The results are in complete agreement with random matrix theory.

In this section we will address another type of statistics. In a typical experimental setup, the cavity is connected to the electrical circuit not only through the leads, but also through nearby metallic gates via the electrostatic interaction. Observing potential fluctuations at these additional gates gives direct insight into the statistics of charge on the cavity. The noise power of the charge fluctuations in this system has been calculated in Ref. 55. The full statistics have been recently addressed using a random matrix theory.⁵⁶ Here, we rederive these results using the stochastic path integral, show new results on the temperature dependence of these statistics, and also investigate the instantaneous fluctuation statistics.

In a semiclassical approach, both leads L, R and the cavity are described by electron distribution functions

f_L, f_R , and f . The Fermi functions in the leads $f_\alpha = f_F(E - \mu_\alpha)$ are characterized by their chemical potential μ_α and their temperature T . The chaotic electron motion inside the cavity makes the cavity distribution function $f(E, t)$ isotropic and position independent. Only its energy dependence must be retained. From now on we set the electron charge to one, $e = 1$. Then the charge Q in the cavity is given in terms of the electron distribution function and density of states N_F as $Q = N_F \int dE f$. The average value of charge is determined by the low-energy cut-off of the integral and is not relevant for the present discussion. The charge and electrostatic potential of the cavity are related by a geometrical capacitance C_g . In the following, we restrict ourselves to the case $C_g \gg e^2 N_F$ which describes complete screening of the charge in the cavity. A more general discussion can be found in Ref. 56. To analyze the time evolution of the charge, we note that if the size of the cavity is smaller than the electron-electron and electron-phonon scattering length, every electron entering the cavity at a certain energy leaves it at the same energy. The single electron energy is thus conserved and we can formulate a current conservation law separately for each energy interval dE ,

$$N_F \dot{f}(E, t) = J_L(E, t) + J_R(E, t), \quad (93)$$

where J_α denote ingoing particle currents per energy interval dE in the left and right contacts. These currents are described by binomial processes with the cumulant generating function given by⁵

$$H_\alpha(f, i\lambda_\alpha) dE = \Gamma^{-1} G_\alpha dE \ln [1 + \Gamma f_\alpha (1 - f) (e^{i\lambda_\alpha} - 1) + \Gamma f (1 - f_\alpha) (e^{-i\lambda_\alpha} - 1)], \quad (94)$$

where we have introduced the conductances of the point contacts G_α , $\alpha = L, R$, and their transparency Γ .

The quantity of interest is the total number of electrons in the cavity averaged over the measurement time τ ,

$$Q_\tau = (N_F/\tau) \int_0^\tau dt \int dE f(E, t). \quad (95)$$

We first consider the long time limit, $\tau \gg \tau_D$, where $\tau_D = N_F/(G_L + G_R)$ is the average dwell time of an electron in the cavity. In this limit, the action is stationary with respect to the variables f and λ ,

$$S = \tau \int dE [H(f, i\lambda) + i(N_F/\tau)\chi f], \quad H = H_L + H_R, \quad (96)$$

where the external variable χ generates the statistics of the desired quantity Q_τ .

At zero temperature $T = 0$, the variables λ and f are independent of the energy E , and the integration in Eq. (96) amounts to a multiplication by $\Delta\mu = \mu_L - \mu_R$. Evaluating the Fourier transform of the characteristic function $Z(i\chi)$,

$$Z(i\chi) = (2\pi)^{-1} \int dQ d\lambda \exp(S), \quad (97)$$

we express the full probability distribution $P(Q_\tau)$ of charge on the cavity as an integral

$$P(Q_\tau) = (2\pi)^{-1} \int d\lambda \exp[\tau \Delta\mu H(f, i\lambda)], \quad f = Q_\tau / (N_F \Delta\mu). \quad (98)$$

This integral will be calculated in the saddle-point approximation. For the tunneling limit $\Gamma \ll 1$ and for open point contacts $\Gamma = 1$ we obtain:

$$\ln P(Q_\tau) = \tau G \Delta\mu K(f), \quad (99a)$$

$$K(f)|_{\Gamma \ll 1} = - \left[\sqrt{f(1-f_0)} - \sqrt{f_0(1-f)} \right]^2, \quad (99b)$$

$$K(f)|_{\Gamma=1} = f_0 \ln \left(\frac{f}{f_0} \right) + (1-f_0) \ln \left(\frac{1-f}{1-f_0} \right), \quad (99c)$$

where $G = G_L + G_R$, and where we have introduced the average distribution function $f_0 = G_L/(G_L + G_R)$ in the cavity. We summarize that the results (99) have been obtained under the conditions $T = 0$, $\tau \gg \tau_D$, and for $\Gamma \ll 1$ and $\Gamma = 1$. These results can be easily generalized to the case of a multi-terminal cavity.

Although the general case of an arbitrary transparency Γ has been also solved analytically, the final expression for the charge distribution is too lengthy to be presented here. The Fig. 8 shows the distribution $P(Q_\tau)$ at zero temperature for various transparencies Γ of the point contacts. The cavity is taken to be symmetric $G_L = G_R$. It is clearly seen that the tails of the distribution grow towards the tunneling limit.

At finite temperature, further analytical progress can be made by considering the first few cumulants of the charge Q_τ . The integral (96) for the cumulant generating function has to be evaluated at the saddle point. For $\chi = 0$ the solution of the saddle-point equations $\partial S/\partial \lambda = 0$ and $\partial S/\partial f = 0$ are simply given by $\lambda = 0$ and $f = f_0$, where $f_0 = (G_L f_L + G_R f_R)/(G_L + G_R)$ is the average electron distribution function in the cavity. From the diagrammatic technique discussed in Sec. IV we derive analytical expressions for the first few cumulants. The second cumulant has been obtained in Ref. 55. As an example, we present here the result for the third cumulant for the case of open point contacts, $\Gamma = 1$:

$$\langle\langle Q_\tau^3 \rangle\rangle = -\frac{2\tau_D^3}{\tau^2} \frac{G_L G_R (G_L - G_R)}{(G_L + G_R)^2} F(\Delta\mu, T), \quad (100a)$$

$$F(\Delta\mu, T) = \Delta\mu + 3 \frac{\Delta\mu - k_B T \sinh(\Delta\mu/k_B T)}{\cosh(\Delta\mu/k_B T) - 1}, \quad (100b)$$

where the function $F(\Delta\mu, T)$ is always positive for $\Delta\mu > 0$. The first few cumulants are plotted in Fig. 9 as a function of the dimensionless bias $\Delta\mu/k_B T$. Note that the fourth cumulant may change its sign as one goes from a symmetric cavity ($\beta = 0$) to an asymmetric cavity ($\beta = 0.9$).

So far we have considered the time of measurement τ longer than the dwell time τ_D . Next we consider the opposite limit $\tau \ll \tau_D$ (but still larger than $\tau_0 = \hbar/\Delta\mu$) and

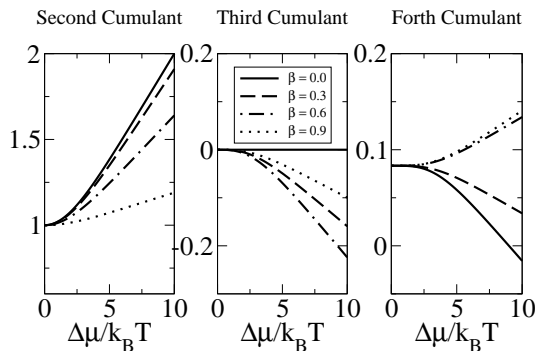


FIG. 9: Cumulants of the charge inside a chaotic cavity, $\langle\langle Q^n \rangle\rangle$, $n = 2, 3, 4$ (in arbitrary units) as functions of the dimensionless potential difference $\Delta\mu/k_B T$. The parameter $\beta = (G_L - G_R)/(G_L + G_R)$ characterizes the asymmetry of the cavity.

study the instantaneous fluctuations of the charge Q in the cavity at zero temperature, $T = 0$. For this purpose we will use the stochastic path integral (3) for the propagator $U(Q_f, Q_i, t)$ of the cavity charge. The distribution $P(Q)$ of instantaneous fluctuations can be obtained by taking the $t \rightarrow \infty$ limit of the propagator $U(Q_f, Q_i, t)$ and setting $Q_f = Q$. We note that in the long time limit, $t \gg \tau_D$, the initial state Q_i relaxes to the stationary state \bar{Q} , and as a result the saddle-point expression of the propagator $U = \exp(S_{sp})$ factorizes according to $S_{sp} = S_0(Q) + S_i(Q_i) + S_f(Q_f)$. Here the stationary contribution to the action is zero, $S_0 = 0$, since there is no charge accumulation on a long time scale. We will show that the initial state contribution vanishes, $S_i = 0$, so the system loses its memory about the initial state. Thus we obtain $\ln P(Q) = S_f(Q)$.

We now focus on the case of a cavity with two tunneling contacts ($\Gamma \ll 1$). Using the Hamiltonians in Eq. (94), and replacing the counting variable $\lambda \rightarrow i\lambda$, we write the action as

$$S = G\Delta\mu \int dt [\tau_D \lambda \dot{f} + h_s(\lambda, f)], \quad (101a)$$

$$h_s = (1 - f_0)f(e^\lambda - 1) + f_0(1 - f)(e^{-\lambda} - 1), \quad (101b)$$

where h_s is the scaled Hamiltonian. The saddle point equations take the following form

$$\tau_D \dot{f} = -(1 - f_0)f e^\lambda + f_0(1 - f)e^{-\lambda}, \quad (102a)$$

$$\tau_D \dot{\lambda} = \sinh(\lambda) + (1 - 2f_0)[\cosh(\lambda) - 1]. \quad (102b)$$

The solution of the Eq. (102b) for λ reads

$$\lambda(t) = \ln \left[\frac{1 + Af_0 \exp(t/\tau_D)}{1 - A(1 - f_0) \exp(t/\tau_D)} \right], \quad (103)$$

where A is the integration constant.

To show that the initial contribution to the action S_i is zero, we note that independent of the constant A , the absolute value of λ is a growing function with the stationary state given by $\bar{\lambda} = 0$ at $t = -\infty$. This means

that starting from early times $t_0 \rightarrow -\infty$, the solutions are $\lambda(t) = 0$ and $f(t) - f_0 = [f(t_0) - f_0] \exp[-(t - t_0)/\tau_D]$. They describe the relaxation of the initial state $f(t_0)$ to the stationary state $\bar{f} = f_0$. Substituting these solutions to Eqs. (101) we immediately find that $S_i = 0$.

After making this point we skip the rest of the details and present the final result for $\ln P(Q) = S_f(Q)$:

$$\ln P(Q)|_{\Gamma \ll 1} = -\tau_D G \Delta\mu \times \left[f \ln \left(\frac{f}{f_0} \right) + (1 - f) \ln \left(\frac{1 - f}{1 - f_0} \right) \right], \quad (104)$$

which can now be compared to the results (99). The cumulant generating function for the distribution (104) is given by $S(\chi) = \tau_D G \Delta\mu \ln[1 + f_0(e^\chi - 1)]$. Note that $\tau_D G \Delta\mu = N_F \Delta\mu$ is the total number of the semi-classical states in the cavity which participate in transport. Therefore the distribution (104) can be interpreted as being a result of uncorrelated binomial fluctuations of the Fermi occupations of each semi-classical state. We would like to mention that the same result can be obtained by solving the stationary master equation.

VI. CONCLUSIONS

We have put forth a stochastic path integral formulation of fluctuation statistics in networks. The mathematical building blocks of the theory are 1) the probability distributions of transport processes through the connectors, 2) a continuity equation linking the connector currents to the charge accumulation in nodes (charge conservation), and 3) a separation of time scales between nodal dynamics and connector fluctuations. The relevant action of the path integral is derived from these considerations and is related to the probability of (charge conserving) paths in phase space. The dominant contribution to the statistics comes from the saddle point approximation to the path integral, and the generating function for the interacting system is simply the action at the saddle point. Fluctuations are suppressed by the number of transporting elementary charges in the network. We have considered the continuum limit to obtain a field theory, and mapped it onto a Langevin equation with Gaussian noise. Cascade diagrammatic rules were found in agreement with Nagaev for the one node case, and extended to general current correlation functions in an arbitrary network. Applications to the current statistics of the diffusive wire and fluctuation statistics of the charge inside a mesoscopic cavity were also discussed. As the building blocks of the theory are classical probability theory, the potential application of this formalism is very broad and applicable to any field where fluctuations are important, including mesoscopics, biology, economics, fluid and chemical dynamics.

Note added in proof.—After this paper was submitted

for publication, the authors learned of previous related work by Bertini *et al.*⁵⁷ Although they did not consider transport statistics, they did consider the probability to manifest a given macroscopic fluctuation of the particle density in diffusive lattice gas models and arrive at the action Eq. (33). However, the Gaussian nature of the local fluctuations was assumed *a priori*. We thank B. Derrida for bringing these papers to our attention.

Acknowledgments

The authors thank M. Kindermann for helpful discussions, and M. Büttiker who collaborated in the initial stages of this work. This work was supported by the Swiss National Science Foundation, and by INTAS (project 0014, open call 2001).

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ization. If this is not the case, one may often use the freedom of stochastic quantization to render it a constant.²⁹

To systemically investigate fluctuations, the determinant may be written as a fermionic functional integral.³²

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