

# Correlation functions of the Lieb-Liniger gas and the LeClair-Mussardo formula

Garry Goldstein and Natan Andrei  
*Department of Physics, Rutgers University and  
Piscataway, New Jersey 08854*

In this letter we derive formulas for multi point correlation functions, in the thermodynamic limit, for the Lieb Liniger gas taken with respect to arbitrary eigenstates. These results apply for the ground state, thermal states and GGE states. We obtain these correlation functions as a series of multiple integrals of progressively higher dimensions. These integrals converge rapidly for short distance correlation functions and low densities of particles. The series derived matches exactly the LeClair Mussardo formula for correlation functions of relativistic integrable models.

*Introduction.* The field of integrable models has reached a certain level of maturity. For many integrable models in 1-D the exact eigenstates and eigen energies are known for arbitrary size systems and their form in the thermodynamic limit is well known through the string hypothesis [1–4]. The thermodynamics of many integrable models is well understood through the thermodynamic Bethe ansatz which is also based on the string hypothesis [1–4]. One of the key remaining challenges is the calculation of correlation functions for local observables [2]. This task is very important as the theoretical description of a many body system is usually given in terms of correlation functions of local observables. In particular the results of any experiments that may be carried out on the system can be computed in terms of sufficiently complicated multi point correlation functions,  $G(x_1 \cdots x_n) = \langle O_1(x_1) \cdots O_n(x_n) \rangle$ , with the expectation value is taken with respect to the initial state - typically the ground state or a thermal ensemble (which is known to be equivalent to an eigenstate calculation [2]). Many correlation functions are directly measurable. For example, in the cold atoms context, correlation functions are measurable through time of flight interferometry [5–14]. The calculation of these correlation functions has been a major challenge over the years and many approximation methods were developed [2, 3]. Even when the system is described by an integrable Hamiltonian with all its eigenstates given by the Bethe Ansatz [4] or by equivalently by the Quantum Inverse Scattering method [2], the correlation functions are known for only few cases and are implicitly expressed in terms of complicated infinite determinants.

The study of correlation functions for arbitrary states, rather than with respect to the ground state, has recently been motivated by the study of quench dynamics in integrable systems. In some cases - when no bound states occur and when the initial state is translationally invariant or close to it - the system is described in the long time limit by a generalized Gibbs ensemble (GGE) [15–24]. Furthermore the equilibrium state which corresponds to a GGE ensemble is equivalent to a specially chosen eigenstate of the integrable model [25]. By judicious choice of the initial state virtually any eigenstate may be obtained at long times. It is of great theoretical interest to compute correlation functions for such eigenstates (or

equivalently understand quenches at long times).

The model we will study the Lieb-Liniger Hamiltonian describes bosons moving on the continuous line and interacting via a short range potential [26]. Imposing periodic boundary condition with periodicity  $L$  the Hamiltonian is given by,  $H_{LL} = \int_{-L/2}^{L/2} dx \left\{ \partial_x b^\dagger(x) \partial_x b(x) + c (b^\dagger(x) b(x))^2 \right\}$ , with  $b^\dagger(x)$  being the creation operator of the bosons at point  $x$ . The  $N$ -bosons eigenstates of the model,  $|k_1 \dots k_N\rangle$ , labeled by rapidities  $\{k_1 \dots k_N\} \equiv \{k\}$ , are explicitly given by:

$$\int_{-L/2}^{L/2} dx_N \dots \int_{-L/2}^{L/2} dx_1 \times \prod_{i < j} Z_{x_i - x_j}(k_i - k_j) \prod e^{ik_i x_i} \prod b^\dagger(x_i) |0\rangle \quad (1)$$

with energy  $E = \sum k_i^2$  and momentum  $P = \sum k_i$ . The scattering factor  $Z_x(k) \equiv \frac{k+ic(1-2\theta(x))}{k+ic}$  incorporates the  $S$ -matrix of the Lieb-Liniger model,  $S_{LL}^{ij} = \frac{k_i - k_j - ic}{k_i - k_j + ic}$ . The  $S$ -matrix describes collisions between two particles in the model and the integrability of the model hinges on the fact that products multi-particle collisions can be consistently described in terms of 2-particles collisions [1–4]. The correlation functions in the Lieb-Liniger model take the form  $\langle \{k\} | O_1(x_1) \cdots O_n(x_n) | \{k\} \rangle$  where the states  $|\{k\}\rangle$  will be specified by the density  $\rho_p(k)$  of the Bethe-Ansatz momenta  $k$  and the density of the holes  $\rho_h(k)$ . Both densities are obtained from solving the Bethe-Ansatz equations that follow from imposing periodic boundary conditions,  $\exp(iLk_j) \prod_{k \neq j} S_{LL}(k_k - k_j) = 1$ . In particular all correlation functions will depend explicitly on the occupation probability,  $f(k) = \frac{\rho_p(k)}{\rho_t(k)}$ , with  $\rho_t(k) = \rho_p(k) + \rho_h(k)$  being the total quasiparticle density. Such states  $|\{k\}\rangle$  capture, for appropriately chosen states both thermal averages or their GGE generalizations which are the long time limit of the Lieb-Liniger system quenched from some given initial state. Thus the ability to compute expectation values is very useful in understanding the not only the thermodynamics of this model (and other integrable models) but also the non equilibrium evolution dynamics of integrable many body quantum systems.

In this letter we present a series expansion formula for arbitrary multipoint correlation functions for arbitrary eigenstates of the Lieb Liniger gas. This expansion is given in terms of a series of finite dimensional inte-

grals where all the information about the eigenstate being given in terms of the occupation probability for the state. The series converges efficiently for a small density of particles (or equivalently a large coupling constant) and for short distances between the points of the correlation function. The series presented is identical in form to the LeClair Mussardo formula [27] for correlation functions of relativistically invariant integrable models. More precisely we will prove that for a generic local operator  $O$  (which can be a multipoint functions) for the Lieb liniger gas we have that:

$$\langle \{k\} | O | \{k\} \rangle = \sum_n \frac{1}{n!} \int \prod_{j=1}^n f(k_j) \frac{dk_j}{2\pi} F_{2n}^O(k_1, \dots, k_n) \quad (2)$$

Here  $F_{2n}^O(k_1, \dots, k_n)$  is appropriately chosen and depends only on  $k_1 \dots k_n$  (in particular there are no dressing equations [29–32]). We will derive explicit formulas for  $F_{2n}^O(k_1, \dots, k_n)$  for short distance expansions of the field field correlation function, for the density-density correlation function and for the emptiness probability all of which are directly measurable in cold atoms experiments.

*Algebraic Bethe ansatz.* The correlation functions are conveniently expressed in the equivalent language of the Algebraic Bethe Ansatz (ABA) as it applies to the Lieb Liniger model. The main object used in the ABA is the so called transfer matrix [2]:

$$T(k) = \begin{pmatrix} A(k) & B(k) \\ C(k) & D(k) \end{pmatrix} \quad (3)$$

with  $A(k)$ ,  $B(k)$ ,  $C(k)$  and  $D(k)$  are operators on the space of the bosons, e.g.  $A(k) = A(k, \{b(x)\}, \{b^\dagger(y)\})$  etc. Also  $C(k) = B^\dagger(k)$ . The vacuum eigenvalues of the operators  $A(k)$ ,  $D(k)$  and  $C(k)$  are given by:  $C(k)|0\rangle = 0$ ,  $A(k)|0\rangle = a(k)|0\rangle$ ,  $D(k) = d(k)|0\rangle$ , with  $a(k) = \exp(-i\frac{k}{2}k)$ ,  $d(k) = \exp(i\frac{k}{2}k)$ . It is possible to show that the state  $B(k_1) \dots B(k_N)|0\rangle$  is an eigenstate of the Lieb-Liniger Hamiltonian if the rapidities  $\{k_i\}$  satisfy the Bethe Ansatz equations. It is convenient to normalize our states and write:

$$|k_1 \dots k_N\rangle = (-ic)^{-N/2} \prod_{j < k} \frac{1}{f(k_j, k_k)} B(k_1) \dots B(k_N) |0\rangle$$

where with  $f(k, q) = \frac{k-q+ic}{k-q}$ . With this normalization it is possible to show [28] that the wavefunction of the state  $|k_1, \dots, k_N\rangle$  is given by Eq. (1) above. Furthermore the states are normalized such that:

$$\langle k_1 \dots k_N | k_1 \dots k_N \rangle = \det(M_{jk}) \quad (4)$$

with  $M_{jk} = \delta_{jk} \left( L + \sum_{l=1}^N \frac{2c}{c^2 + (k_j - k_l)^2} \right) - \frac{2c}{c^2 + (k_j - k_k)^2}$ , see [2].

*Field field correlation functions.* As an example of the general formalism we proceed now to calculate the field field correlation functions. Its Fourier transform yields

the velocity distribution and is directly measurable in experiment. It is known that the correlation function for  $b^\dagger(x)b(y)$  with respect to a state  $|\{k\}\rangle$  with occupation density  $f(k) = \frac{\rho_p(k)}{\rho_t(k)}$  is given by [2]:

$$\frac{\langle \{k\} | b^\dagger(x)b(y) | \{k\} \rangle}{\langle \{k\} | \{k\} \rangle} = \frac{\frac{\partial}{\partial \alpha} (\tilde{0} | \det(1 + \frac{1}{2\pi} W_T) | 0)_{\alpha=0}}{\det(1 - \frac{1}{2\pi} K_T)} \quad (5)$$

with the determinant extending over all the rapidities  $k_m$  entering the state  $|\{k\}\rangle$  and where the operator valued matrix is given by:

$$\begin{aligned} W_T(k, q) = & \frac{f(q)}{c} [t(k, q) + t(q, k) \exp(i(y-x)(q-k) + \\ & + \Phi_{A_2}(q) - \Phi_{D_2}(q) + \Phi_{D_2}(k) - \Phi_{A_2}(k)) - \\ & - [t(k, q) \exp(i(y-x)(q-k) + \Phi_{A_1}(k) + \\ & + \Phi_{D_1}(q)) + t(q, k) \exp(\Phi_{A_1}(q) + \Phi_{D_1}(k))] \\ & \times \exp(\psi_{D_2}(k) - \psi_{A_1}(k) - \Phi_{A_1}(k) + \\ & + \psi_{A_2}(q) + \psi_{D_3}(q) - \psi_{D_1}(q) - \Phi_{D_2}(q)) + \\ & + \alpha c \exp(-i(y-x)k + \psi_{D_2}(k) - \Phi_{A_2}(k) + \\ & + \psi_{A_3}(q) + \psi_{A_2}(q) - \psi_{D_1}(q) - \Phi_{D_2}(q))] \end{aligned} \quad (6)$$

Here  $f(k, q) = \frac{k-q+ic}{k-q}$ ,  $g(k, q) = \frac{ic}{k-q}$ ,  $h(k, q) = \frac{k-q+ic}{ic}$  and  $t(k, q) = \frac{(ic)^2}{(k-q)(k-q+ic)}$ . The defining relations for these fields are given by:  $\Phi_{A_k} = P_{A_k} + Q_{A_k}$ ,  $\Phi_{D_k} = P_{D_k} + Q_{D_k}$  and  $\psi_{A_k} = p_{A_k} + q_{A_k}$ ,  $\psi_{D_k} = p_{D_k} + q_{D_k}$ . The various expectation values are  $P_a|0\rangle = p_a|0\rangle = (\tilde{0}|q_a = (\tilde{0}|Q_a = 0, (\tilde{0}|0) = 1$ . Here  $|0\rangle$  is not related to any states of the boson system but is a fictitious auxiliary ground state. The only nonzero commutation relations are given by:  $[P_{A_i}, Q_{A_k}] = \delta_{ik} \ln(h(q, k))$ ,  $[P_{D_i}, Q_{D_k}] = \delta_{ik} \ln(h(k, q))$ ,  $[p_{A_i}, q_{A_k}] = \delta_{ik} \ln(h(q, k))$  and  $[p_{D_i}, q_{D_k}] = \delta_{ik} \ln(h(k, q))$ . Furthermore the determinant  $W_T(k, q)$  in Eq. (6) is well defined since  $[\Phi_a(k), \Phi_b(q)] = [\psi_a(k), \psi_b(q)] = [\Phi_a(k), \psi_b(q)] = 0$ . Using the Taylor series formula for Fredholm determinants [33] it is possible to obtain a series solution for the determinant:

$$\begin{aligned} & \frac{\partial}{\partial \alpha} (\tilde{0} | \det(1 + \frac{1}{2\pi} W_T(k, q)) | 0) = \\ & = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{dk_1}{2\pi} \dots \int \frac{dk_n}{2\pi} \prod f(k_j) \frac{\partial}{\partial \alpha} \det(\tilde{0} | (\hat{W}_T(k_j, k_k))_{n \times n} | 0) \end{aligned} \quad (7)$$

Here  $(\hat{W}_T(k_j, k_k))_{n \times n}$  is the  $n \times n$  matrix whose  $j, k'$ th entry is  $\hat{W}_T(k_j, k_k)$ , where  $\hat{W}_T(k, q) = \frac{W_T(k, q)}{f(q)}$ . Similarly we have that

$$\begin{aligned} & \det(1 - \frac{1}{2\pi} K_T) = \\ & = \sum_{n=0}^{\infty} \frac{-1^n}{n!} \int \frac{dk_j}{2\pi} \prod f(k_j) \det\left(\frac{2c}{(k_j - k_k) + c^2}\right) \end{aligned} \quad (8)$$

Hence we obtain a LeClair Mussardo like formula for the correlation function by Taylor expanding the ratio of these determinants,

$$\frac{(\tilde{0} | \det (1 + \frac{1}{2\pi} W_T) | 0)}{\det (1 - \frac{1}{2\pi} K_T)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod f(k_j) \frac{dk_j}{2\pi} F_{2n}^{b^\dagger(x)b(y)}(k_1, \dots, k_n)$$

where the  $F_{2n}^{b^\dagger(x)b(y)}(k_1, \dots, k_n)$  are given by

$$F_2^{b^\dagger(x)b(y)} = \frac{\partial}{\partial \alpha} \det (\tilde{0} | (\hat{W}_T(k_j, k_j))_{1 \times 1} | 0)$$

$$F_4^{b^\dagger(x)b(y)} = \frac{\partial}{\partial \alpha} \det (\tilde{0} | (\hat{W}_T(k_j, k_k))_{2 \times 2} | 0) + \frac{\partial}{\partial \alpha} \det (\tilde{0} | (\hat{W}_T(k_j, k_j))_{1 \times 1} | 0) \cdot (\hat{K}_T(k_k, k_k))_{1 \times 1}$$

etc and where  $(\hat{K}_T(k_j, k_k))_{n \times n}$  is the  $n \times n$  matrix whose  $j, k$ 'th entry is  $\hat{K}_T(k_j, k_k) = \frac{2c}{(k_j - k_k) + c^2}$ . We can calculate the first few terms in the expansion for the field field correlation function, it is given by:

$$\begin{aligned} \langle b^\dagger(x)b(y) \rangle_2 &= \int dk \frac{f(k)}{2\pi} \exp[-i(y-x)k] dk + \int dk \frac{f(k)}{2\pi} \int dq \frac{f(q)}{2\pi} \frac{2}{c} \exp[-i(y-x)k] + \\ &- \frac{1}{c} \int dk \frac{f(k)}{2\pi} \int dq \frac{f(q)}{2\pi} \exp[-i(y-x)q] (g(k, q) (h(q, k) h^{-1}(k, q) + h^{-1}(k, q) h^{-1}(k, q) + 2h^{-1}(q, k))) + \\ &+ i \int dk \frac{f(k)}{2\pi} \int dq \frac{f(q)}{2\pi} \exp[-i(y-x)q] \left( -2i(x-y) f^{-1}(k, q) + \frac{2}{ic} h^{-1}(q, k) + \frac{4i}{c} f^{-1}(k, q) \right) + \dots \end{aligned} \quad (9)$$

From this expression we see that the LeClair Mussardo formula is a short distance expansion (in particular the term  $\sim x - y$  diverges at large distances). This is a general feature of the expansion. This expression is valid for  $x > y$ , while for  $x < y$  one needs to replace the expression by its complex conjugate. In the supplement we give some examples of how to use these formulas for various initial states.

*Density density correlation functions.* We now proceed to consider density density correlation functions, this would be another completely explicit example. Actually it is easier to consider the generating function of density density correlation  $\exp(\alpha Q_{xy})$ , where  $Q_{xy} = \int_x^y b^\dagger(z)b(z) dz$ . It is known that the correlation function for  $\exp(\alpha Q_{xy})$  with respect to a state  $|\{k\}\rangle$  with occupation density  $f(k) = \frac{\rho_p(k)}{\rho_i(k)}$  is given by [2]:

$$\frac{\langle \{k\} | \exp(\alpha Q_{xy}) | \{k\} \rangle}{\langle \{k\} | \{k\} \rangle} = \frac{(\tilde{0} | \det (1 + \frac{1}{2\pi} V_T) | 0)}{\det (1 - \frac{1}{2\pi} K_T)} \quad (10)$$

Here the we have the ratio of two Fredholm determinants.  $K_T(k, q) = \frac{2c}{(k-q)^2 + c^2} f(q)$ , and

$$\begin{aligned} V_T(k, q) &= \frac{f(q)}{c} [t(k, q) + t(q, k) \exp(-i(y-x)(k-q)) \times \\ &\times \exp\{\varphi_1(q) - \varphi_1(k)\} + e^\alpha \exp\{\varphi_3(k) + \\ &+ \varphi_4(q)\} \times (t(q, k) + t(k, q) \times \\ &\exp(-i(y-x)(k-q)) \exp\{\varphi_2(k) - \varphi_2(q)\})] \end{aligned} \quad (11)$$

The various fields are defined as  $\varphi_i(k) = q_i(k) + p_i(k)$ ,  $i = 1, 2, 3, 4$ . The commutation relations for these fields are given by  $[p_a(k), p_b(q)] = [q_a(k), q_b(q)] = 0$  and  $[p_a(k), q_b(q)] = H_{a,b}(k, q)$ . With  $H_{a,b}(k, q) = (\ln(h(k, q))) A + (\ln(h(q, k))) A^T$ ,

$$A = \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}.$$

The vacuum states  $|0\rangle$  and  $(\tilde{0} |$  are defined as  $p_a(k) |0\rangle = (\tilde{0} | q_a(k) = 0$ ,  $(\tilde{0} | 0) = 1$ . Furthermore this determinant is well defined since  $[\varphi_i(k), \varphi_j(q)] = 0$ ,  $i, j = 1, 2, 3, 4$ . Proceeding with a Taylor expansion much like in the field field case we obtain that the density density correlation function is given by:

$$\begin{aligned}
\langle \rho(x) \rho(y) \rangle &= -\frac{1}{2} \frac{d^2}{d\alpha^2} \frac{d^2}{dx dy} \langle \exp(\alpha Q_{xy}) \rangle = \\
&= \rho^2 + \frac{1}{c^2} \int dk \int dq \frac{f(k)}{2\pi} \frac{f(q)}{2\pi} \frac{d^2}{d\alpha^2} \frac{d^2}{dx dy} (\tilde{0} | \{ 2e^{\alpha t} t(q, k) t(k, q) \exp(-i(y-x)(k-q)) \times \\
&\times \exp(\varphi_1(k) - \varphi_1(q)) \exp(\varphi_3(q) + \varphi_4(k)) + e^{2\alpha t^2}(k, q) \exp(-i(y-x)(k-q)) \times \\
&\times \exp(\varphi_3(k) + \varphi_4(q)) \exp(\varphi_2(k) - \varphi_2(q)) \exp(\varphi_3(q) + \varphi_4(k)) \} | 0) + \dots = \\
&= \rho^2 - \int dk \int dq \frac{f(k)}{2\pi} \frac{f(q)}{2\pi} \exp(-i(y-x)(k-q)) + \dots
\end{aligned} \tag{12}$$

We see that to leading order in the small density expansion we have that  $\langle \rho(x) \rho(y) \rangle \cong \rho^2 - \left| \langle b^\dagger(x) b(y) \rangle_1^2 \right|$  [34]. For example this means that at zero temperature:  $\langle \rho(x) \rho(y) \rangle = \rho^2 - \frac{\sin^2((y-x)k_F)}{\pi^2(y-x)^2} + \dots$

*Probability of forming an empty interval.* Of particular interest is the expectation value of the operator  $\langle \exp(\alpha Q_{xy}) \rangle$  in the limit when  $\alpha \rightarrow -\infty$ . In that case the only terms that contribute to the expectation value are those when there are no particles in the interval  $[x, y]$  and  $Q_{xy} = 0$ . We call this expectation value the probability of having an empty interval. In this limit the expression given in Eq. (10) greatly simplifies [2]:

$$P(x, y) \equiv \frac{\langle \{k\} | \exp(\alpha Q_{xy})_{\alpha \rightarrow -\infty} | \{k\} \rangle}{\langle \{k\} | \{k\} \rangle} \tag{13}$$

$$= \frac{\langle 0 | \det \left( 1 + \frac{1}{2\pi} M_T \right) | 0 \rangle}{\det \left( 1 - \frac{1}{2\pi} K_T \right)} \tag{14}$$

Here

$$\begin{aligned}
M_T(k, q) &= -cf(q) \left[ \frac{\exp\left(\frac{i}{2}k(y-x) + \frac{1}{2}\phi(k)\right)}{(k-q)(k-q+ic)} \times \right. \\
&\times \exp\left(-\frac{i}{2}q(y-x) - \frac{1}{2}\phi(q)\right) + \\
&+ \frac{\exp\left(\frac{i}{2}q(y-x) + \frac{1}{2}\phi(q)\right)}{(q-k)(q-k+ic)} \times \\
&\left. \times \exp\left(-\frac{i}{2}k(y-x) - \frac{1}{2}\phi(k)\right) \right]
\end{aligned} \tag{15}$$

Here  $\phi(k) = P(k) + Q(k)$  with  $P(k)|0\rangle = \langle 0|Q(k) = 0 = [P(k), P(q)] = [Q(k), Q(q)]$  and  $[P(k), Q(q)] = \ln\left(\frac{c^2}{(k-q)^2 + c^2}\right)$ . The leading order expression for this correlation function (keeping only terms with  $2 \times 2$  matrices or less and noticing that  $P(x=y) = 1$ ) is given by:

$$\begin{aligned}
P(x, y) &= 1 - \int \frac{f(k)}{2\pi} (y-x) + \frac{1}{2} \int \frac{f(k)}{2\pi} \frac{f(q)}{2\pi} (y-x)^2 \\
&- \frac{1}{2} \int \frac{f(k)}{2\pi} \frac{f(q)}{2\pi} \left\{ \frac{(k-q)^2 + c^2}{(k-q)^2} \left( \frac{\exp(i(y-x)(q-k)) - 1}{(q-k+ic)^2} + \frac{\exp(i(y-x)(k-q)) - 1}{(k-q+ic)^2} \right) \right\}
\end{aligned}$$

For the correlation functions with respect to the ground state,  $f(k) = \theta[-k_F, k_F]$ , and we obtain that:

$$\begin{aligned}
P(x, y) &= 1 - \frac{k_F}{\pi} (y-x) + \frac{k_F^2}{\pi^2} (y-x)^2 - \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \frac{1}{4\pi^2} \left\{ \frac{(k-q)^2 + \left(\frac{c}{k_F}\right)^2}{(k-q)^2} \times \right. \\
&\times \left. \left( \frac{\exp(i[k_F(y-x)](q-k)) - 1}{\left(q-k + i\left(\frac{c}{k_F}\right)\right)^2} + \frac{\exp(i[k_F(y-x)](k-q)) - 1}{\left(k-q + i\left(\frac{c}{k_F}\right)\right)^2} \right) \right\}
\end{aligned}$$

For small  $(y-x)$  this series gives  $P(x, y) = 1 - \frac{k_F}{\pi} (y-x) + \frac{k_F^2}{\pi^2} (y-x)^2 \left\{ \left( \frac{3}{2} - \frac{c}{k_F} \arctan\left(\frac{2k_F}{c}\right) + \left(\frac{c^2}{k_F^2}\right) \ln\left(1 + \frac{4k^2}{c^2}\right) \right) \right\}$ .

*General correlation functions.* We would like to extend our results to general multipoint correlation functions.

The most general correlation function can be written as:

$$O = \prod_{j=0}^{n-1} \phi_{0,\pm 1}(x_j) \exp(\theta_j Q_{x_j, x_{j+1}}) \quad (16)$$

Here  $\phi_0(x_j) = 1$ ,  $\phi_1(x_j) = b^\dagger(x_j)$ ,  $\phi_{-1}(x_j) = b(x_j)$  and  $Q_{x_j, x_{j+1}} = \lim_{\epsilon \downarrow 0} \int_{x_j + \epsilon}^{x_{j+1} + \epsilon} b^\dagger(y) b(y) dy$ . Furthermore let us denote by  $\alpha_0$  the set of  $x_j$  where we have a  $\phi_0(x_j)$ ;  $\alpha_1$  the set of  $x_j$  where we have  $b^\dagger(x_j)$  and  $\alpha_{-1}$  the set of  $x_j$  where we have  $\phi_{-1}(x_j)$ . Let the cardinality of  $\alpha_{\pm 1}$  be equal to  $m$ . In the supplementary online information we show that:

$$\frac{\langle \{k\} | O | \{k\} \rangle}{\langle \{k\} | \{k\} \rangle} = \frac{(\tilde{0} | \frac{\partial}{\partial \beta_1} \dots \frac{\partial}{\partial \beta_m} \det \left( 1 + \frac{1}{2\pi} \hat{U}_T \right)_{\beta_1 = \dots = \beta_m = 0} | 0)}{\det \left( 1 - \frac{1}{2\pi} K_T \right)} \quad (17)$$

Where  $\hat{U}_T(k, q) \sim f(k)$ . Arguing again exactly in the same way as following Eq. (7) we can obtain a series expansion for a general correlation function thereby providing efficient methods to evaluate them a deriving the LeClair mussardo formula.

*Conclusions.* Here we have presented a LeClair Mussardo like formula for generic correlations of the Lieb-Liniger gas. We have proven that for a generic local op-

erator  $O$  we have the result that is given in Eq. (2). We have presented a useful expansion for correlation functions of the Lieb liniger gas at short distances and low densities. This result can be used in the future to compute correlation functions for quench problems. We hypothesize that a similar relation is valid for arbitrary integrable models. In these models there there are many particle types. We conjecture the following formula for the correlation functions

$$\langle \{k\} | O | \{k\} \rangle = \sum_n \frac{1}{n!} \int \frac{dk_1}{2\pi} \dots \int \frac{dk_n}{2\pi} \times \sum_{p_i} \left( \prod_{j=1}^n f^{p_j}(k_j) \right) F_{2n}^{O, p_1 \dots p_n}(k_1, \dots, k_n) \quad (18)$$

Here  $\sum_{p_i}$  is a sum over particle types. It is of importance to derive this relation for correlation functions for a general integrable models. The authors are also currently working on extending the formula to multi time correlation functions.

**Acknowledgments:** This research was supported by NSF grant DMR 1410583 and Rutgers CMT fellowship.

- 
- [1] M. Takahashi, *Thermodynamics of one-dimensional solvable models*, (Cambridge University Press, 1999).
- [2] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum inverse scattering method and correlation functions*, (Cambridge university Press 1993).
- [3] G. Mussardo, *Statistical field theory: an introduction to exactly solved models in statistical physics*, (Oxford university press 2010).
- [4] B. Sutherland, *Beautiful models: 70 years of exactly solved quantum many-body problems*, (Word Scientific Publishing, 2004).
- [5] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature (London)* **440**, 900 (2006)
- [6] M. Greiner, O. Mandel, T. W. Hansch and I. Bloch, *Nature* **419**, 51, (2002).
- [7] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm and J. Schniedmayer, *Nature* **449**, 324 (2007).
- [8] E. Haller, M. Gusatvsson, M. J. Mark, J. G. Danzl, R. Hart, G. Pupillo and H.-C. Nagerl, *Science* **325**, 1224 (2009).
- [9] S. Trotsky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert and I. Bloch, *Nature Phys.* **8**, 325 (2012).
- [10] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauss, T. Fukuhara, C. Gross, I Bloch, C. Kollath and S. Kuhr, *Nature* **481**, 484 (2012).
- [11] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, *Science* **337**, 1318 (2012).
- [12] U. Schneider, L. Hakermuller, J. P. Ronzheimer, S. Will, S. Braun, T. Best, I Bloch, E. Demler, S. Mandt, D. Rasch, and A. Rosch, *Nature Phys.* **8**, 213 (2012).
- [13] J. P. Ronzheimer, M. Schreiber, S. Braun, S. S. Hodgman, S. Langer, I. P. McCulloch, F. Heidrich-Meisner, I. Bloch and U. Schneider, *Phys. Rev. Lett.* **110**, 205301 (2013).
- [14] I. Bloch, J. Dalibard and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
- [15] M. Rigol, V. Dunjko, and M. Olshanii, *Nature* **452**, 854, (2008).
- [16] P. Calabrese and J. Cardy, *Phys. Rev. Lett.* **96**, 136801 (2006).
- [17] M. Eckstein, M. Kollar, and P. Werner, *Phys. Rev. Lett.* **103**, 056403 (2009).
- [18] A. Faribault, P. Calabrese and J. S. Caux *J. Stat. Mech.* P03018 (2009).
- [19] S. Sotiriadis, D. Fioretto, and G. Mussardo, *J. Stat. Mech.* P02017 (2012).
- [20] J. Mossel and J. S. Caux, *J. Phys. A* **45**, 255001 (2012).
- [21] T. Barthel, and U. Schollwöck, *Phys. Rev. Lett.* **100**, 100601 (2008).
- [22] M. Rigol and M. Fitzpatrick, *Phys. Rev. A* **84**, 033640 (2011).
- [23] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, *Phys. Rev. Lett.* **98**, 050405 (2007).
- [24] M. Kormos, G. Mussardo and A. Tronchetti, *Phys. Rev. A* **81**, 043606 (2010).
- [25] J. Mossel, J.-S. Caux, *J. Phys. A: Math. Theor.* **45**, 255001, (2012).
- [26] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
- [27] A. LeClair and G. Mussardo, *Nucl. Phys.* **B552**, 624 (1999).

- [28] G. Goldstein and N. Andrei arXiv 1309.3471.  
 [29] A. G. Izergin, V. E. Korepin and N. Yu Reshetikhin, J. Phys. A: Math Gen. **20**, 4799 (1987).  
 [30] A. G. Izergin, and V. E. Korepin, Comm. Math. Phys. **94**, 67 (1984).  
 [31] V. E. Korepin, Comm. Math. Phys. **94**, 93 (1984).  
 [32] N. M. Bogoliubov and V. E. Korepin, Nuclear Physics **B257**, 766 (1985).  
 [33] M. Stone and P. Goldbart, *Mathematics for physics: a guided tour for graduate students*, (Cambridge University press, Cambridge, 2009).  
 [34] See supplementary online information.

## Supplementary online information

### I. LECLAIR MUSSARDO FORMULA

We begin by reviewing some properties of relativistically invariant integrable one dimensional systems (for simplicity we focus on the single species of particle case- generalizations to multiple species of particles is straightforward). Eigenstates of integrable models are parametrized by sets of rapidities  $|\{\theta_i\}\rangle$ . For finite sized systems with periodic boundary conditions these rapidities satisfy bethe ansatz equations

$$e^{ip(\theta_j)L} \prod_{k \neq j} S(\theta_j - \theta_k) = 1 \quad (19)$$

Here  $L$  is the system size,  $p(\theta) = m \sinh(\theta)$  is the momentum of the particle (here  $m$  is the mass of the particle) and  $S(\theta_j - \theta_k)$  is the scattering matrix between particles  $j$  and  $k$ . In integrable models the two particle scattering matrix determines all multiparticle scattering. In the thermodynamic limit when both the particle number and the system size is large it is possible to introduce quasiparticle densities. To do so let us denote by  $L\rho_p(\theta) d\theta$  as the number of particles in the interval  $[\theta, \theta + d\theta]$ ,  $L\rho_h(\theta) d\theta$  as the number of holes in the interval  $[\theta, \theta + d\theta]$  and  $L\rho_t(\theta) d\theta$  as the number of states in the interval  $[\theta, \theta + d\theta]$  so that  $\rho_t(\theta) = \rho_p(\theta) + \rho_h(\theta)$ . It is also convenient to introduce  $f(\theta) \equiv \frac{\rho_p(\theta)}{\rho_t(\theta)}$ . The quasiparticle density satisfies the so called thermodynamic Bethe ansatz equations:

$$\rho_t(\theta) = \frac{1}{2\pi} p'(\theta) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \rho_p(\theta') \quad (20)$$

Here  $\varphi(\theta) \equiv -i \frac{d}{d\theta} \log(S(\theta))$ . This is the continuum version of Eq. (19). Using these quantities various thermodynamic quantities like the Yang-Yang entropy may be defined. The Yang-Yang entropy associated with the densities,  $\{\rho_p(\theta), \rho_h(\theta)\}$ , measures the number of states

$|\{k\}\rangle$  consistent with the densities. It is given by:

$$S(\{\rho^n\}) = \int_{-\infty}^{\infty} d\theta \left( \rho_h(\theta) \ln \left( \frac{\rho_t(\theta)}{\rho_h(\theta)} \right) + \rho_p(\theta) \ln \left( \frac{\rho_t(\theta)}{\rho_p(\theta)} \right) \right). \quad (21)$$

With these definitions the LeClair Mussardo formula for the expectation of an operator with respect to an eigenstate of the theory may be written as

$$\langle \{k\} | O | \{k\} \rangle = \sum_n \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \dots \int \frac{d\theta_N}{2\pi} \times \left( \prod_{j=1}^n f(\theta_j) \right) F_{2n,c}^O(\theta_1, \dots, \theta_n) \quad (22)$$

Here  $f(\theta) = \frac{\rho_p(\theta)}{\rho_t(\theta)}$  corresponds to the state  $|\{\theta\}\rangle$  and  $F_{2n,c}^O(\theta_1, \dots, \theta_n)$  is the so called connected correlation functions for the operator  $O$ . Its definition is given below. To define the connected correlator recall the Kinematic pole axiom for correlation functions of integrable field theories [3] (on the infinite interval) which in the simplest case says:

$$-i \lim_{\tilde{\theta} \rightarrow \theta} (\tilde{\theta} - \theta) \langle \tilde{\theta}, \theta'_1, \dots, \theta'_N | O | \theta, \theta_1, \dots, \theta_N \rangle \quad (23)$$

$$= \left( \prod S(\theta'_i - \theta) - \prod S(\theta - \theta_i) \right) \times \langle \theta'_1, \dots, \theta'_N | O | \theta_1, \dots, \theta_N \rangle$$

The pole when  $\tilde{\theta} \rightarrow \theta$  is universal but the residue may be more complex than shown in Eq. (23) above. As a result of this axiom, the expectation value of  $\langle \theta_1 + \epsilon_1, \dots, \theta_n + \epsilon_n | O | \theta_1, \dots, \theta_n \rangle$  has poles when  $\epsilon_i \downarrow 0$ . The connected part of a correlation function is defined by  $F_{2n,c}^O(\theta_1, \dots, \theta_n) =$

$$\text{Finite Part} \left( \lim_{\epsilon_i \downarrow 0} \langle \theta_1 + \epsilon_1, \dots, \theta_n + \epsilon_n | O | \theta_1, \dots, \theta_n \rangle \right) \quad (24)$$

In particular it does not have any factors of the form  $\frac{\epsilon_i}{\epsilon_j}$ .

### II. LIEB-LINIGER GAS AS A NON-RELATIVISTIC LIMIT OF THE SINH-GORDON MODEL

The LeClair Mussardo formula has been defined for relativistic field theories. We would like to explain why it should work for the Lieb-Liniger gas. We would like to note that it has never been rigorously proved for general correlation functions. We closely follow the derivations given in [24]. In the work the authors show that the Lieb-Liniger gas (given by the Hamiltonian in the main text) can be obtained as the nonrelativistic limit of the Sinh Gordon model. The Sinh Gordon model consists of a single massive boson governed by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2v^2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{v^2}{4g^2} (\cosh(g\phi) - 1) \quad (25)$$

Here  $v$  is the velocity of light. The particles in this model have mass

$$M^2 = \frac{1}{4} \frac{\sin(\pi\alpha)}{\pi\alpha} \quad (26)$$

Here  $\alpha = \frac{vg^2}{8\pi+cg^2}$ . The non-relativistic limit is obtained by the following procedure

$$v \rightarrow \infty, g \rightarrow 0, vg = \text{const} \quad (27)$$

In this case it is possible to obtain, by ignoring some terms oscillating at a frequency  $\frac{1}{2}v^2$ , the Lieb-Liniger Hamiltonian given in the main text with the coupling constant

$$c = \frac{v^2 g^2}{16} \quad (28)$$

We note that the results in this section are not a proof of the LeClair Mussardo formula which is merely a hypothesis for the Sinh Gordon model, but are merely motivational for our discussion below.

### III. MULTIPOINT CORRELATION FUNCTION

We would like to extend our results to general correlation functions. The most general correlation function can be written as:

$$O = \prod_{j=0}^{n-1} \phi_{0,\pm 1}(x_j) \exp(\theta_j Q_{x_j, x_{j+1}}) \quad (29)$$

Here  $\phi_0(x_j) = 1$ ,  $\phi_1(x_j) = b^\dagger(x_j)$ ,  $\phi_{-1}(x_j) = b(x_j)$  and  $Q_{x_j, x_{j+1}} = \lim_{\epsilon \downarrow 0} \int_{x_j+\epsilon}^{x_{j+1}+\epsilon} b^\dagger(y) b(y) dy$ . Because of translation invariance we may as well assume  $x_0 = 0$ . Furthermore let us denote by  $\alpha_0$  the set of  $x_j$  where we have a  $\phi_0(x_j)$ ;  $\alpha_1$  the set of  $x_j$  where we have  $b^\dagger(x_j)$  and  $\alpha_{-1}$  the set of  $x_j$  where we have  $b(x_j)$ . Let the cardinality of  $\alpha_{\pm 1}$  be equal to  $m$ . Now we want to calculate

$$\langle 0 | C(k_1^C) \dots C(k_N^C) O B(k_1^B) \dots B(k_N^B) | 0 \rangle \quad (30)$$

Take the limit  $k_j^C = k_j^B$  and express everything in terms of a determinant given in Eq. (17) in the main text. Now we use the notation:

$$T_{(0,L)}(k) (0, L) = \prod_{j=0}^n T_{(x_j, x_{j+1})}(k) \quad (31)$$

$$= \prod_{j=0}^n \begin{pmatrix} A_j(k) & B_j(k) \\ C_j(k) & D_j(k) \end{pmatrix} \quad (32)$$

Here  $x_{n+1} = L$  and  $T_{(a,b)}(k)$  is the transfer matrix for the interval  $(a, b)$ . We now use the relationship [2]:

$$\begin{aligned} B(k_1^B) \dots B(k_N^B) | 0 \rangle &= \sum_{\{k^B\} = \cup_{j=0}^n \{k_j^B\}} \prod_{j=0}^n B_j(k_j^B) | 0_j \rangle \times \\ &\times \prod_{0 \leq j < k \leq n} a_k(k_j^B) d_j(k_k^B) f(k_j^B, k_k^B) \\ \langle 0 | C(k_1^C) \dots C(k_N^C) &= \sum_{\{k^B\} = \cup_{j=0}^n \{k_j^B\}} \langle 0_j | \prod_{j=0}^n C_j(k_j^C) \times \\ &\times \prod_{0 \leq j < k \leq n} d_k(k_j^C) a_j(k_k^C) f(k_k^C, k_j^C) \end{aligned} \quad (33)$$

Here  $|0_j\rangle$  is the state with no bosons on the interval  $(x_j, x_{j+1})$ . Next we use the identity [2]:

$$\begin{aligned} b(x_j) \prod_{l=1}^{n_j} B(k_l^B) | 0_j \rangle &= -i\sqrt{c} \sum_{\{k_{0,j}^B\}} a_j(k_{0,j}^B) \\ &\times \prod_{k_l^B \neq k_{0,j}^B} f(k_{0,j}^B, k_l^B) \prod_{k_l^B \neq k_{0,j}^B} B_j(k_l^B) | 0_j \rangle \\ \langle 0_j | \prod_{l=0}^{n_j} C_j(k_l^C) b^\dagger(x_j) &= i\sqrt{c} \sum_{\{k_{0,j}^C\}} d_j(k_{0,j}^C) \\ &\times \prod_{k_l^C \neq k_{0,j}^C} f(k_l^C, k_{0,j}^C) \prod_{k_l^C \neq k_{0,j}^C} \langle 0 | C_j(k_l^C) \end{aligned} \quad (34)$$

Next we use the relationship [2]:

$$\begin{aligned} \langle 0_j | \prod_{l=0}^{n_j} C_j(k_l^C) \prod_{l=1}^{n_j} B(k_l^B) | 0_j \rangle &= \\ = \prod_{j>k} g(k_j^C, k_k^C) g(k_k^B, k_j^B) \langle 0 | \det_{n_j}(S^j) | 0 \rangle \end{aligned} \quad (35)$$

Here

$$\begin{aligned} S_{lm}^j &= t(k_l^C, k_m^B) a_j(k_l^C) d_j(k_m^B) \exp(\Phi_{A_j}(k_l^C) + \Phi_{D_j}(k_m^B)) + \\ &+ t(k_m^B, k_l^C) a_j(k_m^B) d_j(k_l^C) \exp(\Phi_{A_j}(k_m^B) + \Phi_{D_j}(k_l^C)) \end{aligned} \quad (36)$$

Here

$$\Phi_{A_j}(k) = Q_{A_j}(k) + P_{D_j}(k) \quad \Phi_{D_j} = Q_{D_j}(k) + P_{A_j}(k) \quad (37)$$

The only nonzero commutation relations are given by:

$$\begin{aligned} [P_{D_j}(k), Q_{D_l}(q)] &= \delta_{jl} \ln(h(k, q)) \\ [P_{A_j}(k), Q_{A_l}(q)] &= \delta_{jl} \ln(h(q, k)) \end{aligned} \quad (38)$$

The relevant expectation values are given by:

$$P_a | 0 \rangle = \langle 0 | Q_a = 0 \quad (39)$$

We note that the state  $|0\rangle$  is not related to the state with no bosons in any way but is merely an auxiliary vacuum used for the purpose of calculating correlation functions. Once again

$$[\Phi_a(k), \Phi_b(q)] = 0 \quad (40)$$

With these results we can see that:

$$\begin{aligned}
\langle 0|C(k_1^C) \dots C(k_N^C)OB(k_1^B) \dots B(k_N^B)|0\rangle &= (-1)^{[P_B]+[P_C]} \prod_{j>k} g(k_j^C, k_k^C) g(k_k^B, k_j^B) \sum_{\{k^B\}=\cup_{j=0}^n \{k_j^B\} \cup_{\{\alpha_{-1}\}} \{k_{j_0}^B\}} \times \\
&\times \sum_{\{k^C\}=\cup_{j=0}^n \{k_j^C\} \cup_{\{\alpha_{-1}\}} \{k_{j_0}^C\}} \times \prod_{0 \leq j < k \leq n} a_k(k_j^B) d_j(k_k^B) h(k_j^B, k_k^B) \times \\
&\times \prod_{0 \leq j < k \leq n} d_k(k_j^C) a_j(k_k^C) h(k_k^C, k_j^C) \times \prod_j (0|\det_{n_j}(\tilde{S}^j)|0) \times \\
&\times \prod_{\{\alpha_{-1}\}} \sqrt{c} a_j(k_{0,j}^B) \prod_{k_l^B \neq k_{0,j}^B} h(k_{0,j}^B, k_l^B) \times \prod_{\{\alpha_{-1}\}} \sqrt{c} d_j(k_{0,j}^C) \prod_{k_l^C \neq k_{0,j}^C} h(k_l^C, k_{0,j}^C)
\end{aligned}$$

where  $[P_B]$  is the parity of the permutation  $\{1, 2, 3, \dots, N\} \rightarrow \cup_{j=0}^n (k_{j_0}^B \cup \{k_j^B\})$  and  $[P_C]$  is the parity of the permutation  $\{1, 2, 3, \dots, N\} \rightarrow \cup_{j=0}^n (k_{j_0}^C \cup \{k_j^C\})$ , and  $\tilde{S}_{lm}^j = e^{\theta_j} S_{lm}^j$ . Now introducing  $\Gamma_j = \sum_{l \in \{\alpha_{-1}\}, l \leq j} 1 - \sum_{l \in \{\alpha_{-1}\}, l \leq j} 1$ . We have that

$$\begin{aligned}
\langle 0|C(k_1^C) \dots C(k_N^C)OB(k_1^B) \dots B(k_N^B)|0\rangle &= \\
\prod_{j>k} g(k_j^C, k_k^C) g(k_k^B, k_j^B) \frac{\partial}{\partial \beta_1} \dots \frac{\partial}{\partial \beta_m} (0|\det S|0)_{\beta_1=\dots=\beta_m=0} & \quad (41)
\end{aligned}$$

To define  $S$  we need the following functional fields

$$\begin{aligned}
\Phi_{A_j,k}(k) &= Q_{A_j,k}(k) + P_{D_j,k}(k) \quad \Phi_{D_j,k} = Q_{D_j,k}(k) + P_{A_j,k}(k) \\
\Psi_{A_j,k}(k) &= \tilde{Q}_{A_j,k}(k) + \tilde{P}_{D_j,k}(k) \quad \Psi_{D_j,k} = \tilde{Q}_{D_j,k}(k) + \tilde{P}_{A_j,k}(k)
\end{aligned} \quad (42)$$

Here  $0 \leq j < k \leq n$ . The nonzero commutation relations are given by:

$$\begin{aligned}
[P_{D_j,k}(k), Q_{D_{l,m}}(q)] &= \delta_{(j,k),(l,m)} \ln(h(k, q)) \quad (43) \\
[P_{A_j,k}(k), Q_{A_{l,m}}(q)] &= \delta_{(j,k),(l,m)} \ln(h(q, k)) \\
[\tilde{P}_{D_j,k}(k), \tilde{Q}_{D_{l,m}}(q)] &= \delta_{(j,k),(l,m)} \ln(h(k, q)) \\
[\tilde{P}_{A_j,k}(k), \tilde{Q}_{A_{l,m}}(q)] &= \delta_{(j,k),(l,m)} \ln(h(q, k))
\end{aligned}$$

The relevant expectation values are given by:

$$P_a|0\rangle = (0|Q_a = \tilde{P}_a|0\rangle = (0|\tilde{Q}_a = 0 \quad (44)$$

Furthermore we will need the functional fields:

$$\begin{aligned}
\psi_{A_j}(k) &= q_{A_j}(k) + p_{D_j}(k) \quad \psi_{D_j} = q_{D_j}(k) + p_{A_j}(k) \\
\varphi_{A_k}(k) &= r_{A_k}(k) + s_{D_k}(k) \quad \varphi_{D_k} = r_{D_k}(k) + s_{A_k}(k)
\end{aligned} \quad (45)$$

Here  $j \in \alpha_{-1}$ ,  $k \in \alpha_1$ . The non-zero commutation relations are given by:

$$\begin{aligned}
[p_{D_j}(k), q_{D_l}(q)] &= \delta_{jl} \ln(h(k, q)) \quad (46) \\
[p_{A_j}(k), q_{A_l}(q)] &= \delta_{jl} \ln(h(q, k)) \\
[s_{D_j}(k), r_{D_l}(q)] &= \delta_{jl} \ln(h(k, q)) \\
[s_{A_j}(k), r_{A_l}(q)] &= \delta_{jl} \ln(h(q, k))
\end{aligned}$$

The relevant expectation values are given by:

$$p_a|0\rangle = (0|q_a = s_a|0\rangle = (0|r_a = 0 \quad (47)$$

In terms of these variables

$$S = \sum_{j \in \alpha_0} S^{j,0} + \sum_{j \in \alpha_1} S^{j,1} + \sum_{j \in \alpha_{-1}} S^{j,-1} + \sum_{l=1}^m \beta_l \Lambda^l \quad (48)$$

Here:

$$\begin{aligned}
S_{k,l}^{j,0} &= -1^{\Gamma_j} \tilde{S}_{k,l}^j \prod_{o>j} a_o(k_l^B) \prod_{o<j} d_o(k_l^B) \prod_{o>j} d_o(k_k^C) \prod_{o<j} a_o(k_k^C) \times \\
&\times \exp \left( \sum_{o>j} \Phi_{A_{j,o}}(k_l^B) + \sum_{o<j} \Phi_{D_{o,j}}(k_l^B) + \sum_{o>j} \Psi_{D_{j,o}}(k_l^C) + \sum_{o<j} \Psi_{A_{o,j}}(k_l^C) \right)
\end{aligned} \quad (49)$$

Furthermore:

$$\begin{aligned}
S_{k,l}^{j,1} &= S_{k,l}^{j,0} \times \exp(\psi_{A_j}(k_k^C)) \\
S_{k,l}^{j,-1} &= S_{k,l}^{j,0} \times \exp(\varphi_{D_j}(k_l^B))
\end{aligned} \quad (50)$$



We now note that:

$$\begin{aligned} \Lambda_{k,l}^d &= c \prod_{o>l} a_o(k_l^B) \prod_{o<i} d_o(k_l^B) \prod_{o>i} d_o(k_k^C) \prod_{o<j} a_o(k_k^C) \times d_j(k_k^C) a_i(k_l^B) \times \exp(\psi_{D_j}(k_k^C) + \varphi_{A_i}(k_l^B)) \\ &\times \exp\left(\sum_{o>i} \Phi_{A_{i,o}}(k_l^B) + \sum_{o<i} \Phi_{D_{o,i}}(k_l^B) + \sum_{o>j} \Psi_{D_{j,o}}(k_l^C) + \sum_{o<j} \Psi_{A_{o,j}}(k_l^C)\right) \end{aligned} \quad (51)$$

Where  $j$  is the  $d$ 'th entry of  $\alpha_1$  and  $i$  is the  $d$ 'th entry of  $\alpha_{-1}$ . Next we need to manipulate these formulas a little bit, we start with:

$$\begin{aligned} \langle 0 | \det S | 0 \rangle &= \prod_{j=1}^N a(k_j^C) d(k_j^B) \langle 0 | \prod_{j=1}^N \exp(\Phi_{A_n}(k_j^C) + \Phi_{D_n}(k_j^B)) \prod_{j<n} \exp(\Phi_{D_{j,n}}(k_m^B) + \Psi_{A_{j,n}}(k_l^B)) \det \tilde{S} | 0 \rangle \\ &= \prod_{j=1}^N a(k_j^C) d(k_j^B) \prod_{j,k=1}^N h(k_j^C, k_k^B) \langle \tilde{0} | \det \tilde{S} | 0 \rangle \end{aligned} \quad (52)$$

Where

$$\langle \tilde{0} | = \langle 0 | \prod_{j=1}^N \exp(P_{D_n}(k_j^C) + P_{A_n}(k_j^B)) \prod_{j<n} \exp(\Phi_{D_{j,n}}(k_m^B) + \Psi_{A_{j,n}}(k_l^B)) \quad (53)$$

and  $\tilde{S}_{l,m} = S_{l,m} d(k_l^C) a(k_m^B) \exp(-\Phi_{A_n}(k_l^C) - \Phi_{D_n}(k_m^B))$ . Furthermore  $\langle \tilde{0} | 0 \rangle = 1$ . Now introducing  $\tilde{\Phi}_{A_n} = \Phi_{A_n} - \langle \tilde{0} | \Phi_{A_n} | 0 \rangle$ ,  $\tilde{\Phi}_{D_n} = \Phi_{D_n} - \langle \tilde{0} | \Phi_{D_n} | 0 \rangle$ ,  $\tilde{\Phi}_{A_{j,n}} = \Phi_{A_{j,n}} - \langle \tilde{0} | \Phi_{A_{j,n}} | 0 \rangle$ ,  $\tilde{\Psi}_{D_{j,n}} = \Psi_{D_{j,n}} - \langle \tilde{0} | \Psi_{D_{j,n}} | 0 \rangle$  and specializing to the case when  $k_j^B = k_j^C$  we obtain that [2]:

$$\tilde{S}_{lm} = \left( \sum_{j \in \alpha_0, j \neq n} S_{lm}^{j,0} + \sum_{j \in \alpha_1} S_{lm}^{j,1} + \sum_{j \in \alpha_{-1}} S_{lm}^{j,-1} + \sum_{l=1}^m \beta_l \Lambda^l \right) \exp\left(-\tilde{\Phi}_{A_n}(k_l^B) - \tilde{\Phi}_{D_n}(k_m^B)\right) a(k_l^B) d(k_m^B) \times \quad (54)$$

$$\times \prod_{j<n} \exp\left(-\Phi_{D_{j,n}}(k_m^B) - \Psi_{A_{j,n}}(k_l^B)\right) + \delta_{lm} \left( L + \sum_{o=1}^N K(k_m^B, k_o^B) \right) + \hat{S}_{lm}^n \quad (55)$$

Where the expressions for  $S_{lm}^{j,0} + S_{lm}^{j,1} + S_{lm}^{j,-1} + \Lambda^l$  we have  $\Phi_{A_{j,n}} \rightarrow \tilde{\Phi}_{A_{j,n}}$  and  $\Psi_{D_{j,n}} \rightarrow \tilde{\Psi}_{D_{j,n}}$ . Here

$$\begin{aligned} \hat{S}_{lm}^n &= (d(k_l^B) a(k_m^B) t(k_l^C, k_m^B) a_n(k_l^C) d_n(k_m^B) + a(k_l^B) d(k_m^B) t(k_m^B, k_l^C) a_n(k_m^B) d_n(k_l^C) \times \\ &\times \exp\left(\tilde{\Phi}_{A_n}(k_m^B) + \tilde{\Phi}_{D_n}(k_l^C) - \tilde{\Phi}_{D_n}(k_m^B) - \tilde{\Phi}_{A_n}(k_l^C)\right)) \prod_{j<n} a_j(k_l^B) d_j(k_m^B) \end{aligned} \quad (56)$$

Now we factorize out the part  $\propto \delta_{lm} \left( L + \sum_{o=1}^N K(k_m^B, k_o^B) \right)$ . First we note that in the thermodynamic limit  $L + \sum_{o=1}^N K(k_m^B, k_o^B) = 2\pi L \rho_t(k_m^B)$ . Now introduce the matrix  $\Theta_{lm} = \delta_{lm} 2\pi L \rho_t(k_m^B)$ . Using this matrix we can see that

$$\langle \tilde{0} | \det \tilde{S} | 0 \rangle = \det \Theta_{lm} \langle \tilde{0} | \det \left( 1 + \frac{1}{2\pi} U_T(k, q) \right) | 0 \rangle \quad (57)$$

Where

$$U_T(k, q) = f(q) \tilde{S}(k, q) - \delta_{lm} \left( L + \sum_{o=1}^N K(k_m^B, k_o^B) \right) \quad (58)$$

Now using the relationship [2]:

$$\begin{aligned} \langle 0 | C(k_1^B) \dots C(k_N^B) B(k_1^B) \dots B(k_N^B) | 0 \rangle \\ = \prod_{j=1}^N 2\pi L \rho_t(k_j^B) \times \prod_{j,k} f(k_j^B, k_k^B) \det \left( 1 - \frac{1}{2\pi} K_T \right) \end{aligned} \quad (59)$$

We obtain that

$$\frac{\langle \{k\} | O | \{k\} \rangle}{\langle \{k\} | \{k\} \rangle} = \frac{\langle \tilde{0} | \frac{\partial}{\partial \beta_1} \dots \frac{\partial}{\partial \beta_m} \det \left( 1 + \frac{1}{2\pi} \hat{U}_T \right)_{\beta_1 = \dots = \beta_m = 0} | 0 \rangle}{\det \left( 1 - \frac{1}{2\pi} K_T \right)} \quad (60)$$

Arguing again exactly as in the main text we can obtain a LeClair Mussardo like formula.

#### IV. FIELD FIELD CORRELATION FUNCTIONS (EXAMPLES)

##### A. Field Field correlations

We would like to give an example of how the various formulas to the field field correlation functions work in practice. These formulas are useful for calculation the short distance correlation functions for various initial states. For example keeping the leading order term in Eq. (9) we obtain that:

$$\langle b^\dagger(x) b(y) \rangle_1 = \int dk \frac{f(k)}{2\pi} \exp[-i(y-x)k] dk + \dots \quad (61)$$

We would like to evaluate this expression for various states  $f(k)$ . We will concentrate on thermal states. In the limit of zero temperature we have that  $f(k) = \theta[-k_F, k_F]$ . From this we obtain that

$$\langle b^\dagger(x) b(y) \rangle_1 \cong \frac{1}{\pi} \frac{\sin((x-y)k_F)}{x-y} \quad (62)$$

This is a good approximation for short distances and low densities. For the case of a finite temperature state we have that  $f(k) = \frac{1}{\exp(\beta(k^2 - \tilde{\mu})) + 1} + O\left(\frac{1}{c^3}\right)$  where  $\tilde{\mu} = \mu + \frac{T}{\pi} \int dq \ln\left(1 + e^{-\beta(k^2 - \tilde{\mu})}\right) [2]$ . From this we obtain that

$$\langle b^\dagger(x) b(y) \rangle_1 = \int \frac{dk}{2\pi} \frac{\exp[-i(y-x)k]}{\exp(\beta(k^2 - \tilde{\mu})) + 1} + O\left(\frac{1}{c^3}\right) \quad (63)$$

We can consider two cases when  $\tilde{\mu} < 0$  and when  $\tilde{\mu} > 0$ . In the case when  $\tilde{\mu} < 0$  we obtain that:

$$\langle b^\dagger(x) b(y) \rangle_1 = \sum_{n=1}^{\infty} (-1)^{n-1} e^{n\beta\tilde{\mu}} \frac{\pi}{\sqrt{n}\beta} \exp\left(-\frac{1}{4} \frac{(x-y)^2}{n\beta}\right) \quad (64)$$

In the case when  $\tilde{\mu} > 0$  we may approximate  $k^2 - \tilde{\mu} \cong \theta(k)(k - \sqrt{\tilde{\mu}})2\sqrt{\tilde{\mu}} - \theta(-k)(k + \sqrt{\tilde{\mu}})2\sqrt{\tilde{\mu}}$ . Using this expression we obtain that:

$$\langle b^\dagger(x) b(y) \rangle_1 = 2Re \left[ \frac{1}{i|x-y| + \beta\sqrt{\tilde{\mu}}} \times \right. \quad (65) \\ \left. \times {}_2F_1 \left\{ 1, 1 - i \frac{|x-y|}{\beta\sqrt{\tilde{\mu}}}, 2 + i \frac{|x-y|}{\beta\sqrt{\tilde{\mu}}}, -e^{-\beta\tilde{\mu}} \right\} \right] \quad (66)$$

where  ${}_2F_1$  is the hypergeometric function.

##### B. Velocity probability distribution

The velocity probability density is an easily measurable experimentally relevant quantity [5]. It is given by:

$$P(v) \sim \int dx \int dy \exp\left(i\frac{v}{2}(y-x)\right) \langle b^\dagger(x) b(y) \rangle \quad (67)$$

The factor of  $\frac{1}{2}$  in the above formula comes from the fact that the mass of the Lieb Liniger gas is normalized to  $\frac{1}{2}$ . For the simpler case where  $f(k) = f(-k)$  Eq. (9) simplifies for both  $x > y$  and  $x < y$  giving:

$$\langle b^\dagger(x) b(y) \rangle_2 = \int dk \frac{f(k)}{2\pi} \exp[-i(y-x)k] dk + \int dk \frac{f(k)}{2\pi} \int dq \frac{f(q)}{2\pi} \frac{2}{c} \exp[-i(y-x)k] + \\ - \frac{1}{c} \int dk \frac{f(k)}{2\pi} \int dq \frac{f(q)}{2\pi} \exp[-i|y-x|q] (g(k, q) (h(q, k) h^{-1}(k, q) + h^{-1}(k, q) h^{-1}(k, q) + 2h^{-1}(q, k))) + \\ + i \int dk \frac{f(k)}{2\pi} \int dq \frac{f(q)}{2\pi} \exp[-i|y-x|q] \left( -2i|y-x| f^{-1}(k, q) + \frac{2}{ic} h^{-1}(q, k) + \frac{4i}{c} f^{-1}(k, q) \right) + \dots \quad (68)$$

Now using the relations:

$$\int dx \exp\left(-ik|x| + i\frac{v}{2}x\right) = -P.V. \frac{i}{k - \frac{v}{2}} - P.V. \frac{i}{k + \frac{v}{2}} + \pi\delta\left(\frac{v}{2} - k\right) + \pi\delta\left(\frac{v}{2} + k\right) \\ \int dx |x| \exp\left(-ik|x| + i\frac{v}{2}x\right) = P.V. \frac{1}{\left(k + \frac{v}{2}\right)^2} - P.V. \frac{1}{\left(k - \frac{v}{2}\right)^2} - i \frac{d}{dk} \delta\left(\frac{v}{2} - k\right) + i \frac{d}{dk} \delta\left(\frac{v}{2} + k\right)$$

We get

$$\begin{aligned}
P(v)_2 \sim & \frac{1}{2}f\left(\frac{v}{2}\right) + f\left(\frac{v}{2}\right) \cdot \int dk \frac{1}{c} \cdot \frac{f(k)}{2\pi} + \\
& - \frac{1}{2c} \int dk \frac{f(k)}{2\pi} \cdot f\left(\frac{v}{2}\right) \left( g\left(k, \frac{v}{2}\right) \left( h\left(\frac{v}{2}, k\right) h^{-1}\left(k, \frac{v}{2}\right) + h^{-1}\left(k, \frac{v}{2}\right) h^{-1}\left(k, \frac{v}{2}\right) + 2h^{-1}\left(\frac{v}{2}, k\right) \right) \right) + \\
& + i \int dk \frac{f(k)}{2\pi} \left( 2 \frac{d}{dv} \left( f\left(\frac{v}{2}\right) f^{-1}\left(k, \frac{v}{2}\right) \right) + f\left(\frac{v}{2}\right) \frac{1}{ic} h^{-1}\left(\frac{v}{2}, k\right) + \frac{2i}{c} f^{-1}\left(k, \frac{v}{2}\right) \right) + c.c. + P.V. \dots \quad (69)
\end{aligned}$$

Where *c.c.* means the complex conjugate of the entire expression and *P.V.* is the principle value part. This formula becomes very accurate for large *v* where the principle value part becomes subleading. For this case (if we denote by  $N = \int dk \frac{f(k)}{\pi}$ ) we have that

$$\begin{aligned}
P(v)_2 \sim & \frac{1}{2}f\left(\frac{v}{2}\right) \left\{ 1 + N \left\{ \frac{2}{c} - \frac{1}{c}g\left(0, \frac{v}{2}\right) \left( h\left(\frac{v}{2}, 0\right) h^{-1}\left(0, \frac{v}{2}\right) + h^{-1}\left(0, \frac{v}{2}\right) h^{-1}\left(0, \frac{v}{2}\right) + 2h^{-1}\left(\frac{v}{2}, 0\right) \right) \right\} + \right. \\
& \left. \frac{1}{c} \left( 2h^{-1}\left(\frac{v}{2}, 0\right) - 4f^{-1}\left(0, \frac{v}{2}\right) \right) \right\} + 2iN \frac{d}{dv} \left( f\left(\frac{v}{2}\right) f^{-1}\left(0, \frac{v}{2}\right) \right) + c.c.
\end{aligned}$$


---