

Many-Body Quantum Electrodynamics Networks: Non-Equilibrium Condensed Matter Physics with Light

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Abstract

We review recent developments concerning non-equilibrium quantum dynamics and many-body physics with light, in superconducting circuits and Josephson analogues. We start with quantum impurity models summarizing the effect of dissipation and of driving the system. We mention theoretical and experimental efforts to characterize these non-equilibrium quantum systems. We show how Josephson junction systems can implement the equivalent of the Kondo effect with microwave photons. The Kondo effect is characterized by a renormalized light-frequency and a peak in the Rayleigh elastic transmission of a photon. We also address the physics of hybrid systems comprising mesoscopic quantum dot devices coupled to an electromagnetic resonator. Then, we discuss extensions to Quantum Electrodynamics (QED) Networks allowing to engineer the Jaynes-Cummings lattice and Rabi lattice models. This opens the door to novel many-body physics with light out of equilibrium, in relation with the Mott-superfluid transition observed with ultra-cold atoms in optical lattices. Then, we summarize recent theoretical predictions for realizing topological phases with light. Synthetic gauge fields and spin-orbit couplings have been successfully implemented with ultra-cold atoms in optical lattices — using time-dependent Floquet perturbations periodic in time, for example — as well as in photonic lattice systems. Finally, we discuss the Josephson effect related to Bose-Hubbard models in ladder and two-dimensional geometries. The Bose-Hubbard model is related to the Jaynes-Cummings lattice model in the large detuning limit between light and matter. In the presence of synthetic gauge fields, we show that Meissner currents subsist in an insulating Mott phase.

Keywords: Condensed-Matter Physics with Light, Superconducting Circuit Quantum Electrodynamics Networks, Josephson Effect and NanoScience, Dissipative and Driven quantum impurity Models, Jaynes-Cummings and Rabi lattices, Topological Phases and Synthetic Gauge Fields, Floquet theory

1. Introduction

The field of quantum electrodynamics [1] has attracted some attention with novel and modern applications in non-equilibrium quantum optics and quantum information [2]. In fact, many experiments have accomplished to couple light and matter in a controlled manner using cavity quantum electrodynamics. Quantum matter here can be either Rydberg atoms (cold atoms) or trapped ions [3, 4] for example. A step towards the realization of many-body physics has also been made through the realization of model Hamiltonians such as the Dicke Hamiltonian [5],

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where the associated super-radiant quantum phase transition has been observed in non-equilibrium conditions [6]. A solid-state version of cavity quantum electrodynamics, related to circuit quantum electrodynamics, built with superconducting quantum circuits [7, 8], is also a very active field both from experimental and theoretical points of view. Theorists have predicted novel emergent quantum phenomena either in relation with strong light-matter coupling [9] or non-equilibrium quantum physics [10, 11].

The goal here is to review developments, both theoretical and experimental, towards realizing many-body physics and quantum simulation in circuit QED starting from small networks to larger ensembles of superconducting elements in the microwave limit. An experimental endeavor has been accomplished towards the realization of larger arrays in circuit QED [12, 13, 14, 15] and towards controlling trajectories in small systems [16, 17]. This research is complementary to existing efforts on collective effects in ultra-cold atoms [18, 19, 20, 21, 22], Nitrogen-Vacancy centers [23] and opto-mechanics for example [24, 25]. We shall also summarize (our) progress related to quantum impurity physics with light, such as Kondo physics [26, 27] of photons [28, 29, 30, 31] and non-equilibrium quantum dynamics [32, 33, 34, 35, 36], as well as theoretical predictions concerning many-body physics in larger network circuits described by the Jaynes-Cummings or Rabi lattice [37, 38, 39, 40, 41, 42, 43, 44]. Some progress has been done towards the implementation of an effective chemical potential for photons [45]. The effect of dissipation will be discussed for small and larger networks. In Ref. [46], we have discussed other theoretical aspects in relation with these quantum impurity models; see also Refs. [47, 48]. It is also important to mention recent theoretical developments regarding the strong-coupling light-matter coupling, both at equilibrium [9, 43, 49, 50, 51, 52, 53, 54] and in non-equilibrium [32, 55]. Artificial gauge fields for photons can be realized through nano-circulators, consisting of superconducting Josephson-junction rings penetrated by magnetic fluxes for example [56, 57, 58], allowing the realization of topological phases [59]. The stability towards disorder effects was also studied [59]. An experimental effort to build such nano-circulators has been realized in Ref. [60]. In fact, inspired by the discovery of quantum Hall physics [61, 62, 63] and topological insulators [64, 65, 66], experimentalists have already succeeded in realizing topological phases of light and spin-orbit type coupling [67, 68, 69, 70, 71] (but not yet in microwave superconducting circuits), for example through Floquet time-periodic drive (for recent developments, see Refs. [72, 73]); some aspects of topological phases with light have been discussed in the recent reviews [74, 75]. It is also important to underline the experimental progress in the field of ultra-cold atoms to realize synthetic gauge fields [76, 77, 78, 79, 80, 81]. Haldane and Raghu made a pioneering contribution in this field [82] that was soon confirmed experimentally [67]; they have shown that one can realize quantum Hall phases of light through magneto-optical coupling or Faraday effect. The (chiral) transport of photons at the edges in these systems has been measured [74] by analogy to quantum Hall systems [83]. This opens the door to novel applications in quantum information and quantum computing by analogy to electron systems [84]. The direct measurement of topological invariants [85] in photon lattices is in principle also feasible [59, 86, 87] in particular, using the concept of anomalous quantum velocity [88]. A recent work by the MIT group on photonic gyromagnetic crystals with ferrite rods report the measurement of a Chern number with photon systems [89]. Some aspects of the Berry phase [90] and Chern number [91] are accessible in small superconducting circuits [92, 93, 94, 95]. Artificial graphene with Dirac photons [96] and three-dimensional Weyl systems [97], as well as analogues of flat band models [98], have been realized recently.

Other photonic arrays, including quasi-periodic arrays [99] and one-dimensional geometries [100, 101], have been envisioned and some have been realized experimentally [102].

2. Small Circuits

This Section is organized as follows. First, we will introduce the physics of small superconducting circuits. Then, we will describe progress on quantum impurity systems, at and out of equilibrium. We will also account for dissipation effects and show that it can produce novel many-body physics, for example, based on vacuum quantum fluctuations associated with long transmission lines. We will describe the Kondo effect of light [28, 29] and address recent developments in hybrid systems [103], combining circuit QED and mesoscopic electron quantum dot systems [104, 105, 106, 107, 108]. Note that in relation with quantum vacuum fluctuations, the dynamical Casimir effect was observed in superconducting circuits [109].

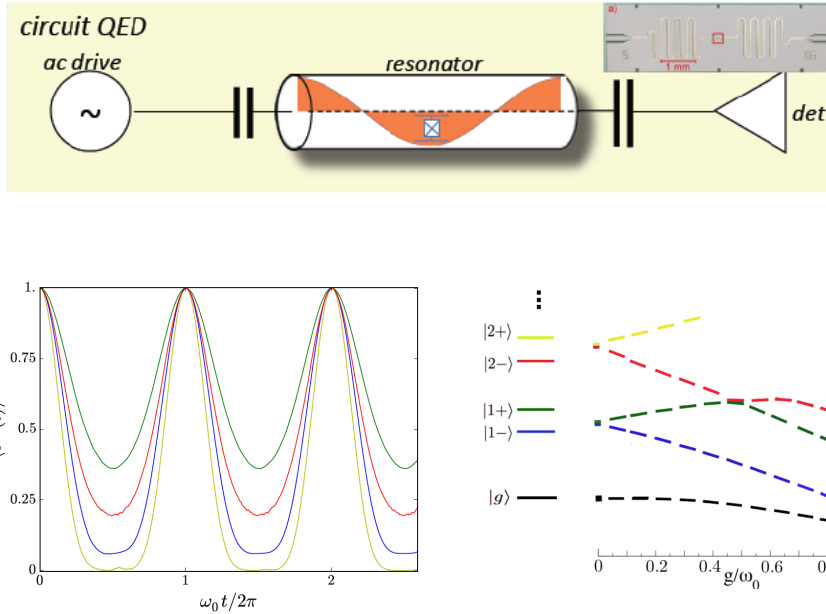


FIGURE 1 : (Color online) Example of Superconducting Circuit Quantum Electrodynamics, with an artificial atom with two levels (spin-1/2 or qubit). The electromagnetic resonator (cavity $\sim 1\text{cm}$) is described through a one-mode transmission line and is capacitively coupled to an AC drive (input signal). We show an example of spin dynamics in the strong-coupling light-matter regime. We apply the methodology described in Ref. [32] and consider the far detuned or adiabatic limit of the Rabi model : g/ω_0 from 0.7 (green) to 1.5 (yellow) and $\Delta/\omega_0 = 0.05$. The system is prepared in the initial state $|+\rangle \otimes |0\rangle = |0, +z\rangle$ and the revival probability of the the initial state can be computed exactly in this limit [133]. The positions of the maxima only depend on the cavity frequency ω_0 . We also show the Jaynes-Cummings polariton ladder for $g/\omega_0 = 0.2$, as well as the evolution of the energy eigenstates of the Rabi model when approaching the strong coupling limit with $g \sim \omega_0$.

2.1. Superconducting Elements

These superconducting circuits embody simple building blocks for quantum simulation, based on harmonic oscillators and artificial atoms - here, two-level mesoscopic systems - such as charge, flux or phase qubits. Nowadays, qubits in superconducting circuits yield a long decoherence time that can reach a few micro-seconds. For a recent progress on superconducting qubits, see for example Refs. [110, 111, 112, 113, 114, 115, 116, 117]. Microwave photons can propagate in on-chip resonators which can be identified to Fabry-Perot type cavities made from finite sections of transmission lines. The physics of such transmission lines is reminiscent of an LC (harmonic) oscillator described by the raising and lowering operator a^\dagger and a . The single-mode approximation is usually appropriate for such cavities of typical size $\sim 1\text{cm}$ (operating in the microwave domain) [118, 119]. At a simple level, $(a+a^\dagger)$ represents the voltage fluctuations in the cavity whereas $(a-a^\dagger)/i$ mimics the conjugate flux variable [118, 120]. Superconducting materials used in such devices are generally Niobium or Aluminum. The propagation of micro-wave photons in the system will be described below by coupling the cavity to transmission lines (see Sec. 2.3). The input AC signal $V_0 \cdot \cos(\omega t)$ will be related to the time-averaged input power $P_{in} = V_0^2/(2R)$ where R is the resistance of the (left) transmission line. This will produce a mean number of photons N such that $P_{in} = \dot{N}\hbar\omega$ and $\dot{N} = dN/dt$. The light-matter coupling will produce non-linearities in the harmonic oscillator spectrum. The superconducting circuits allow a great flexibility afforded by the nature of the nano-fabrication system ; parameters in the Hamiltonian can be widely tuned with conventional lithography. Qubit frequencies can be set from typically 2 to 15 GHz and couplings can range from a few KiloHertz to nearly 1GHz. As the lateral dimension of the transmission lines is typically a few micrometers (which is much smaller than the wave-length), the electric field in the cavity is relatively large and a strong dipolar coupling can be achieved. Capacitors at the ends of the cavity control the photon leakage rates and in multi-cavity systems, the hopping rates between nearest neighbor cavities [10].

A circuit containing the cavity, artificial qubit, dissipation and AC drive effects is drawn in Fig. 1.

2.2. Rabi, Dicke and Jaynes-Cummings Models

The model describing the strong-coupling between a qubit and the electro-magnetic field of the cavity corresponds to the Rabi model (1936) [121] :

$$H = \frac{\Delta}{2}\sigma_z + \omega_0 \left(a^\dagger a + \frac{1}{2} \right) + \frac{\sigma_x}{2} g(a + a^\dagger). \quad (1)$$

Here, ω_0 represents the frequency of the quantized harmonic oscillator (quantized electro-magnetic field), Δ is the resonant frequency between the two levels forming the qubit and σ_i where $i = x, y, z$ are the Pauli spin operators. Furthermore, g represents the dipole coupling between qubit and photon. If σ_x represents the two charge states of a Cooper pair box [122, 123, 124] then the capacitive coupling between a qubit and a resonator realizes a quantum impurity model of Rabi type [125]. An analogous model is obtained in the case of an inductive coupling between the qubit and the cavity.

Although the Rabi model represents one of the simplest interacting quantum models, it does not correspond to a simple theoretical problem. Nevertheless, recently analytical solutions have been pushed forward in connection with the experimental possibilities in superconducting circuits, and more generally in nano-circuits. In particular, the limit $g/\omega_0 \sim 0.1$ is commonly reached experimentally [126, 127]. The discrete \mathbb{Z}_2 symmetry associated with the Hamiltonian ($a \rightarrow -a$ and $\sigma_x \rightarrow -\sigma_x$ ($\sigma_y \rightarrow -\sigma_y$)) cannot be ignored in this regime making analytical solutions more difficult to find. In this respect, it is important to emphasize the theoretical progress first realized by D. Braak in the strong-coupling limit [9, 49, 50, 51, 52, 53, 43, 54], concerning the solvability of the Rabi model. It is also relevant to underline other theoretical progress on the integrability of related quantum impurity models with light [128, 129, 130]. One relatively ‘simple’ limit of the Rabi model in the strong-coupling limit is the so-called adiabatic limit [131, 132], corresponding to the highly detuned limit $\Delta \rightarrow 0$, which has been addressed for example in Ref. [52]. In Refs. [33, 34], in collaboration with P. P. Orth and A. Imambekov, one of the authors has developed a relatively simple stochastic scheme to address certain non-equilibrium properties and quenches on dissipative spin systems. This approach is based on a numerically exact Feynman-Vernon path integral approach [47, 48] (see also Sec. 2.4). In Ref. [32], two of the authors together with Z. Ristivojevic and P. P. Orth have extended this approach to dissipative light-matter systems beyond the weak-coupling limit. We show extra results in the strong-coupling limit (Fig. 1), in the highly detuned limit $\Delta/\omega_0 \ll 1$, in agreement with Ref. [133]. The revival probability of the initial state $|+z\rangle \otimes |0\rangle = |0, +z\rangle$ (with zero photon in the cavity) can be computed exactly in the limit $\Delta \rightarrow 0$, using the shifted $b = \sigma_x a$ basis [133]. More precisely, this probability is of the form $\exp -|Z(t)|^2$ with $Z(t) = (g/\omega_0)(e^{-i\omega_0 t} - 1)$ [133], leading to periodic collapses and revivals with frequency $\omega_0/(2\pi)$. The ratio g/ω_0 controls the sharpness of these revivals peaks.

The limit of large spin then referring to the Dicke Hamiltonian (1954) [134] is also solvable through the Holstein-Primakoff transformation, exemplifying the possibility of superradiant effects for strong light-matter couplings [135]. As mentioned in the introduction, such a superradiant quantum phase transition has been recently observed with ultra-cold atoms in an optical cavity, in a non equilibrium setting [5, 6]. The possibility of a Dicke quantum spin glass has also been discussed [136]. Some efforts have also questioned the possibility of superradiant quantum phase transitions in mesoscopic circuits with Cooper pair boxes [137, 138]. Coupling an ensemble of atoms or artificial qubits to light is now commonly realized with various signatures of collective effects [22, 23]. We have also shown that entanglement between light and matter in the Dicke model can be probed through the properties of the light field [139, 140] or through bipartite fluctuations [141, 142].

In the 1960s, Jaynes and Cummings proposed an approximation to the Rabi model [143], referred to as the rotating-wave approximation, where the interaction part is simplified into $\frac{g}{2}(\sigma^+ a + \sigma^- a^\dagger)$. This approximation is usually valid when $g/\omega_0 \ll 1$ and at small detuning $\delta = |\omega_0 - \Delta| \ll \Delta$. The number of excitations in the system (hybrid excitations of light and matter or polaritons), the number of polaritons $N = a^\dagger a + \frac{1}{2}(\sigma_z + 1)$, is then conserved as a result of the emergent U(1) symmetry ($a \rightarrow ae^{-i\varphi}$ and $\sigma^+ \rightarrow \sigma^+ e^{i\varphi}$). We can diagonalize the model in the basis $|n, +z\rangle$ and $|n+1, -z\rangle$ where n refers to the number of photons in the system and $|\pm z\rangle$ refers to the qubit polarization along the z axis. We have the following eigen-energies and eigenstates for $N \geq 1$ (the ground state has energy $E_0 = \delta/2$) :

$$E_{N\pm} = N\omega_0 \pm \frac{1}{2} \sqrt{\delta^2 + Ng^2}, \quad (2)$$

and

$$\begin{aligned} |N+\rangle &= \alpha_N |N-1, +z\rangle + \beta_N |N, -z\rangle \\ |N-\rangle &= -\beta_N |N-1, +z\rangle + \alpha_N |N, -z\rangle, \end{aligned} \quad (3)$$

where $\beta_N = \cos(1/2 \tan^{-1} \frac{g\sqrt{N}}{\delta})$ and $\alpha_N = \sin(1/2 \tan^{-1} \frac{g\sqrt{N}}{\delta})$. We recover the usual Jaynes-Cummings (JC) ladder (see Fig. 1). The anharmonicity of the Jaynes-Cummings ladder produces exotic phenomena such as photon blockade, preventing a second photon to enter in the cavity when a first one is also present [144, 145]. The \sqrt{N} spacing of the JC ladder can be used to generate nonclassical states of light. Photon-photon interactions occur both in the resonance condition between cavity and qubit and in the dispersive limit, where the light frequency and the qubit level separation Δ are well distinct such that the spin remains in the ground state and the dynamics of the photons is described by an effective Hubbard interaction $U(a^\dagger a)^2$. More precisely, defining the small parameter $\lambda = g/|\omega_0 - \Delta|$ and performing an exact unitary transformation on the Hamiltonian results in an effective photon-photon interaction $U = g\lambda^3$ [146]. This also produces a shift in the resonator frequency. The photon-photon interaction certainly modifies the energy required to add a photon to the cavity depending upon the number of photons occupying the cavity. Although this is a higher order effect, the key point is that the interaction energy or Kerr energy can be made larger than the cavity line width κ and the transport of photons through the cavity is blocked. This photon interaction effect has been observed in various experiments (transmission spectrum as a function of frequency or intensity correlation function), for example, in cavity [147] and circuit-QED [148, 149, 150, 151].

By increasing the light-matter coupling and treating the counter-rotating wave terms perturbatively, one identifies a Bloch-Siegert shift [152] making the two polariton levels $|1-\rangle$ and $|1+\rangle$ repelling each other : when entering the strong-coupling limit the level $|1-\rangle$ becomes closer and closer to the ground state (see Fig. 1). In the strong-coupling limit, the structure of levels becomes more complicated [51]. In the Rabi lattice situation, features reminiscent of the Dicke model appear due to the presence of the counter-rotating wave terms [43]. To get further insights, one can project the rabi ground state in the polariton basis [43]. Due to the parity symmetry, only even polariton states contribute with different weights, which give the probability distribution of having N polaritons in the ground state. This distribution is peaked around some finite value as a result of the counter-rotating terms which mix sectors at different polariton numbers, thus acting effectively as a sort of chemical potential.

In Ref. [32], we used a relatively simple stochastic protocol to address certain non-equilibrium properties and quenches on the qubit strongly-coupled to light (see also Sec. 2.4). In particular, we show how to reach an almost pure state with one polariton in the branch $|1-\rangle$ by driving the cavity, with an almost zero detuning between light and matter. Note that in the limit of large detuning between light and matter, this protocol corresponds to make a π -pulse on the qubit [153]. Below, we shall discuss further applications.

2.3. AC driving, Dissipation and Spin-Bath Models

Let us now discuss dissipation and driving effects in the cavity. First, the cavity line width and driving effects from an AC source can be described in a microscopic manner by coupling the cavity to very long transmission lines [154, 120] which correspond to a bath of harmonic oscillators [155, 156].

Let us envision a configuration where a single cavity is capacitively coupled to a left and a right transmission lines (serving as photon baths). The open cavity system coupled to the transmission lines is described through the Hamiltonian :

$$H_{lines} = \sum_{j=l,r} \sum_k v|k| \left(b_{jk}^\dagger b_{jk} + \frac{1}{2} \right) + \sum_k \alpha_k \left(\gamma_l (b_{lk} + b_{lk}^\dagger) + \gamma_r (b_{rk} + b_{rk}^\dagger) \right) (a + a^\dagger). \quad (4)$$

The transmission lines with dispersion $\omega_k = v|k|$ (the Planck constant \hbar is normalized to unity, for simplicity) are then described by the spectral function

$$J(\omega) = \pi \sum_k \lambda_k^2 \delta(\omega - \omega_k) = 2\pi\alpha\omega \exp(-\omega/\omega_c), \quad (5)$$

and ω_c is an ultraviolet cutoff for photon excitations in the environment (this corresponds to Ohmic dissipation). More precisely, defining the symmetric and anti-symmetric combinations of the two transmission lines, only the symmetric mode couples to the cavity, and the effective coupling constant α_k turns into $\lambda_k = \alpha_k \sqrt{\gamma_l^2 + \gamma_r^2}$. Here, α_k depends on the properties of the transmission lines, which are supposed to be identical, and $\gamma_{l,r}$ encode the capacitive couplings between the cavity and the transmission lines [28, 103]. We relate the bosonic modes in the left line b_{lk} with the input signal and the input power P_{in} in a usual way, and the output signal is expressed in terms of the boson b_{rk} [120]. Using exact equations of motion and the input-output theory [120], we can obtain an expression for the transmission and reflection coefficients of a photon in the system [103] :

$$\begin{aligned} r(\omega) &= 1 - 2i \frac{\gamma_l^2}{\gamma_l^2 + \gamma_r^2} J(\omega) \chi_{xx}^R(\omega) \\ t(\omega) &= |t| \exp(i\varphi) = \frac{2i\gamma_r\gamma_l}{\gamma_l^2 + \gamma_r^2} J(\omega) \chi_{xx}^R(\omega). \end{aligned} \quad (6)$$

In this expression, we introduce the exact retarded photon Green's function $\chi_{xx}^R(t) = -i\theta(t)\langle [x(t), x(0)] \rangle_{sys}$ describing the response of the photon displacement $x = (a + a^\dagger)/\sqrt{2}$. The coupling with the environment then produces a damping of Ohmic type for the photon propagator [103] :

$$\chi(\omega) = \frac{\omega_0}{\omega^2 - \omega_0^2 + i\omega_0\kappa}, \quad (7)$$

where $\kappa = 2\pi\alpha\omega$ is the effective dissipation induced by a coupling α to the modes of the transmission lines. Notice that close to resonance, the dissipation can be described through a complex frequency $\omega_0 \rightarrow \omega_0 - i\kappa/2$. A damping rate linear with frequency is usually typical of long transmission lines which embody a Ohmic environment. Note that by treating these dissipation effects in a Hamiltonian manner through a set of bosonic variables already constitutes a first many-body effect in these systems. Some dissipation effects can be also described through the Lindblad [157] or Bloch-Redfield [158, 159] type equations. Below, we show how many-body physics can be engineered from these Ohmic environments.

Dissipation effects on the spin or the artificial two-level system already results in rich many-body quantum impurity models, such as the spin-boson Hamiltonian [47, 48, 160, 161, 162] :

$$H_{SB} = \sum_k v|k| \left(b_k^\dagger b_k + \frac{1}{2} \right) - \frac{\Delta}{2} \sigma_x + \frac{\hbar}{2} \sigma_z + \frac{1}{2} \sigma_z \sum_k \lambda_k (b_k + b_k^\dagger). \quad (8)$$

In the case of Ohmic dissipation (case of Eq. (5)), the physics of this model is very rich showing some exact mapping to the anisotropic Kondo model and the classical Ising model with long-range forces [163, 164]; these models are now standard models in many-body physics. In particular, this Ohmic spin-boson model yields a quantum phase transition for the dissipative parameter $\alpha_c \sim 1$ by analogy to the ferromagnetic - antiferromagnetic quantum phase transition in the Kondo model [165, 166]; here, we use the definition of α given in Eq. (5). It is important to mention recent progress in nano circuits [167, 168] towards observing such dissipative quantum phase transitions [169, 170, 171, 172, 173]. Rabi oscillations of the qubit also show some incoherent crossover for the Toulouse limit corresponding to a dissipative parameter $\alpha = 1/2$, where the problem becomes formally equivalent to a non-interacting resonant level model [174]. Similar quantum phase transitions have been predicted for a two-spin system [175, 176, 177] and could be observed in light-matter systems [178].

2.4. Dissipative Quantum Trajectories on the Bloch Sphere and Stochastic Dynamics

Various methods have been devised to describe quantitatively the dynamics of these spin-boson systems, from master equations [179], the non-interacting blip approximation (NIBA) and its extensions [47, 48, 180], stochastic wave function approaches [181, 182] and stochastic differential equations [183], time-dependent Numerical Renormalization Group approaches [184, 185, 177], multi-polaron expansion [186], Density Matrix Renormalization Group [187] and Matrix Product States [188, 189, 44] for example. These efforts complement other methods applied for non-equilibrium dissipative quantum systems based on Monte-Carlo techniques on the

Keldysh contour [190, 191, 192]. These theoretical developments are very useful owing to important progress on the experimental side both in quantum optics, ultra-cold atoms and superconducting qubits, for example, by following the real-time dynamics on the Bloch sphere. In the context of superconducting qubits, tracking single quantum trajectories has been explored experimentally with a particular focus on the competition between continuous weak measurement and driven unitary evolution [16, 17]. Some connections also occur with the concept of open quantum walkers [193].

At a classical level, the dynamics of dissipative spins in a macroscopic and dissipative environment can be described through Bloch equations $\dot{\vec{S}}(t) = \vec{H} \times \vec{S}$ where the dissipation is here included in the noisy random magnetic field $\vec{H} = h(\cos \phi(t), \sin \phi(t), 0)$. Classically, the spin then obeys the differential equation $\dot{S}_z(t) = -h^2 \int_0^t ds \cos(\phi(t) - \phi(s)) S_z(s)$. At a quantum level, as a result of the Ehrenfest theorem, $\langle \sigma_z \rangle$ obeys a similar equation, where we also average over different magnetic field configurations :

$$\frac{d}{dt} \langle \sigma_z \rangle = -h^2 \int_0^t ds \langle \cos(\phi(t) - \phi(s)) \sigma_z(s) \rangle. \quad (9)$$

A usual approximation consists to decouple the dynamics of the stochastic field and the spin dynamics, neglecting completely the back-action of the spin dynamics on the stochastic field evolution. This leads to the well-known NIBA approximation [47] (where the noisy phases are then derived from the spin-boson model in Eq. (8) and are formally operators [46]). Nevertheless, it is important to mention recent theoretical progress both analytical and numerical trying to go beyond these type of approximations [33, 34, 194]. These developments also include applications on the driven and dissipative Jaynes-Cummings and Rabi models [32] and networks of spin-boson models, leading to many-body stochastic non-equilibrium (mean-field) theories [195, 196]. Lindblad and Bloch-Redfield approaches give a complementary point of view [197, 198] and can even include non-Markovian effects [183]. Similar approaches have been developed to address heat transfer [180].

In Fig. 1, we show one result obtained for the driven Rabi model, by developing a stochastic Schrödinger equation approach. The formalism is derived on the path-integral real-time formalism [155] and allows to treat Eq. (9) in an exact manner [33, 34]. This formalism has been developed based on previous works in the field [35, 36, 199]. In Ref. [32], the formalism was developed further in the context of the driven and dissipative Rabi model. In Fig. 1, we compute the spin dynamics exactly in the strong-coupling limit, by using (coupled) stochastic Gaussian variables to decouple long-range interactions in time induced by the environment (here, the cavity). The formalism has also been developed to incorporate dissipation effects, and is also inspired from Ref. [199]. Below, we show one application in the driven Jaynes-Cummings lattice including dissipation in the cavities.

2.5. Exploration for Josephson-Kondo Circuits : Rayleigh Resonance of Light and Many-Body Kondo shift

Josephson physics [200, 201] has been explored with superconductors [202, 203] and quantum circuits [204, 122]. Analogous Josephson effects have also been explored with superfluids, for example, in ultra-cold atoms [205, 206], in superfluid He [207] and with polariton condensates [208].

Here, we review small superconducting circuits in the microwave domain allowing to realize Kondo physics with one-dimensional Josephson junction arrays (mimicking long transmission lines) [28, 29] instead of electron reservoirs, as in the original Kondo Hamiltonian [26, 27]. Excitations in the long transmission lines at equilibrium are zero-point fluctuations (bosons instead of electrons in the original Kondo model). Similar geometries have been envisioned in mesoscopic rings [172], with Luttinger liquids [173, 28] or one-dimensional BECs (Bose-Einstein condensates) [209, 210]. In Fig. 2A, we introduce a typical Josephson-Kondo circuit where micro-wave photons are sent across the system [28, 29]. This geometry is also inspired from Refs. [211, 212]. Related circuits with photons have been addressed theoretically [30, 31, 213, 214]. Some (fermonic) Kondo type impurity models have also been explored in ultra-cold atoms [215, 216].

The dissipative parameter in Eq. (5) representing the one-dimensional transmission lines here takes the form $\alpha \sim 2R/R_q$, where R is the resistance of each (superconducting) transmission line and $R_q = h/(2e)^2$ (where $h = 2\pi$ within our units) is the quantum of resistance. With the current efforts in nanotechnology one can build transmission lines with quite large impedances [167, 168, 217]. In particular, realizing highly dissipative transmission lines with Josephson junction arrays is also possible thanks to experimental progress [115]. The

artificial atom was originally considered to be a double-dot charge qubit at resonance allowing for the coherent propagation of Cooper pairs across the system in the micro-wave domain. One could use superconducting qubits with long decoherence times [110, 111, 112, 113, 114, 115]. The Hamiltonian representing the pseudo-spin (representing the double-island charge qubit [218, 219]) coupled to the left and right transmission lines takes the form [28] :

$$H = -\frac{E_J}{2}\sigma^+ e^{i(\Phi_l - \Phi_r)} + h.c. + \sum_{j=l,r} \sum_k v|k| \left(b_{jk}^\dagger b_{jk} + \frac{1}{2} \right). \quad (10)$$

The Josephson phases depict the (generalized) superfluid phases in each transmission line in the vicinity of the artificial qubit. Note that in the geometry of Fig. 2 (left), only the antisymmetric mode ($\Phi_l - \Phi_r$) couples to the transmission line. In the underdamped limit ($0 \ll \alpha \ll 0.3$) of the spin-boson model introduced earlier, the Kondo physics here corresponds to the *perfect* Rayleigh transmission of a micro-wave photon at the frequency corresponding to the Kondo frequency of the system [28]. Such a concept of resonance fluorescence has been observed with artificial superconducting qubits [220]. Note that this frequency is distinct from the bare frequency of the artificial atom (qubit) as a result of the quantum point-fluctuations in the long transmission lines. More precisely, from the two-level perspective, one can define an effective transverse field

$$E_J \langle \cos(\Phi_l - \Phi_r) \rangle. \quad (11)$$

Close to equilibrium, when the input power can be treated in linear response theory, the effective transverse field results from known results on the Ohmic spin-boson model [47, 48] :

$$\omega_R = E_J \left(\frac{E_J}{\omega_c} \right)^{\frac{\alpha}{1-\alpha}} \sim E_J \left(1 + \frac{\alpha}{1-\alpha} \ln \left(\frac{E_J}{\omega_c} \right) \right). \quad (12)$$

The \ln -corrections which stem from the many-body excitations in the long transmission lines can be understood as renormalization effects. This scenario assumes that the high-frequency cutoff ω_c of the order of the plasma frequency in the transmission lines is much larger than the Josephson frequency. The key point is that the spin susceptibility $\chi_{spin}^R(t) = -i\theta(t)\langle [\sigma_z(t), \sigma_z(0)] \rangle$ takes a form identical to Eq. (6) but with the frequency ω_0 being replaced by the Kondo frequency ω_R and a damping rate $\gamma(\omega) = \omega_R J(\omega)$ which is reminiscent of the Korringa-Shiba relation in the spin-boson model [221] and in the Kondo model [27, 222, 223, 224, 225]. Note that this damping rate can be measured directly in quantum RC circuits through the concept of quantum of resistance [226, 227, 228, 229].

To evaluate the elastic transport of a photon through the system, we can still use the input-output theory [120] and relate the reflection and transmission amplitudes of a photon [28]. When the averaged input power tends to zero, in the underdamped limit, we find that the scattering matrix is unitary and at the Kondo frequency the transmission is maximum as a result of $J(\omega_R) \text{Im} \chi_{spin}^R(\omega_R) = -1$ (using the conventions of Eq. (6)). The phase of the reflected signal also changes from zero to π by varying the asymmetry between the left and right couplings with the transmission lines ; see Fig. 2. Note that inelastic effects nevertheless become important when increasing the drive amplitude [28] or by reaching the ultra-strong light-matter coupling (over damped regime) [29]. Open questions concern, for example, the evolution of the Mollow triplet [220] by increasing the drive amplitude and the dissipation strength α . Simple arguments allow to conclude the disappearance of the elastic Kondo peak when increasing the amplitude of the drive [28]. Similar geometries could be considered to simulate the two-channel Kondo effect with light, as in electron systems [230, 231, 232], with finite-frequency measurements [233, 234].

2.6. Hybrid systems

Coupling qubits (artificial spins) to a circuit quantum electrodynamics environment allow to explore new limits in quantum computation, where entangling distant qubits through light [104] has become possible for example [235, 236, 237]. One can also envision the realization of new quantum gates [238] and very robust Einstein-Poldolsky-Rosen or Bell's pairs [239] by analogy to photon systems [240] and Cooper pairs [241]. Coupling a real spin-1/2 object to light has also been realized experimentally [242]. Extending the concepts and techniques of electrons to quantum optics has also resulted in new developments in quantum opto-electronics [243].

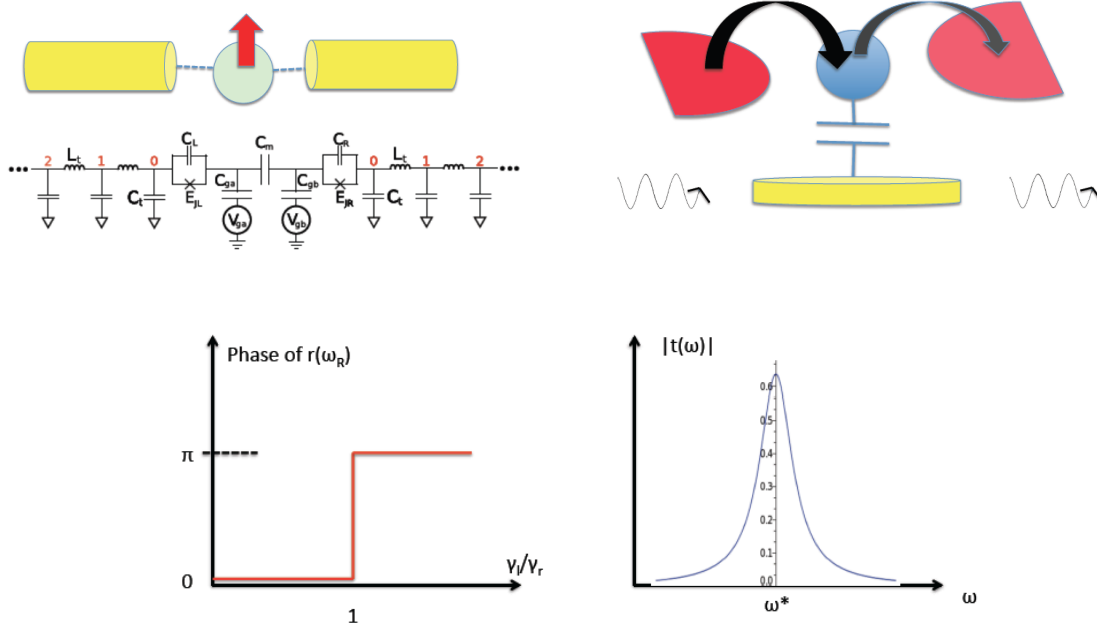


FIGURE 2 : (Left) Typical Josephson-Kondo circuit introduced in Refs. [28, 29]. In the underdamped limit, where dissipation in the long Josephson junction chains is not so important ($0 \ll \alpha \ll 0.3$), one can observe the perfect Rayleigh transmission of a microwave photon through the system at the renormalized Kondo frequency ω_R given in Eq. (12). This many-body Lamb shift exemplifies renormalization effects. Such long transmission lines with tunable resistances can be built with current technology [115]. By adjusting the ratio between the left γ_l and right γ_r couplings between the Josephson junction chains (transmission lines) and the artificial double-dot Cooper-pair box system [218, 219], the phase of the reflected signal at the Kondo frequency passes from 0 to π when changing the ratio γ_l/γ_r . (Right) Cartoon (note the cavity in yellow is much bigger than the mesoscopic (nano-)circuit, as in Fig. 1) : Hybrid system comprising a quantum dot tunnel coupled to two (red) electron reservoir leads and capacitively coupled to a one-mode (yellow) cavity via a capacitive coupling, as realized in Ref. [105]. In Ref. [103], we have shown that at the renormalized frequency ω^* (defined in the text), in the non-linear limit of large bias voltages applied across the quantum dot, the light is also perfectly transmitted for symmetric couplings $\gamma_l = \gamma_r$.

Recently, experimentalists have accomplished another step further by designing hybrid systems, containing electrons with a circuit quantum electrodynamics environment. For example, coupling a quantum dot to an electromagnetic resonator has been achieved experimentally [105, 106, 107, 108]. Similar systems have been built with Nitrogen Vacancy centers [23]. More precisely, the coupling between a quantum dot, a microwave resonator and two biased electronic reservoirs was built on chip and signatures of many-body correlations have been observed, for example, in the phase of the microwave transmitted photon as a function of the bias voltage V applied across the leads. The system can be modeled by an Anderson-Holstein type Hamiltonian [244, 245] :

$$H_{matter} = \sum_{k\sigma\alpha} \epsilon_{k\alpha} c_{k\alpha\sigma}^\dagger c_{k\alpha\sigma} + \sum_{k\alpha\sigma} V_{k\alpha} (c_{k\alpha\sigma}^\dagger d_\sigma + h.c.) + \epsilon \sum_\sigma d_\sigma^\dagger d_\sigma + U n_\uparrow n_\downarrow + \lambda (a + a^\dagger) \sum_\sigma d_\sigma^\dagger d_\sigma + \omega_0 a^\dagger a. \quad (13)$$

Here, $c_{k\alpha\sigma}$ and d_σ are the annihilation operators of the leads and of the dot respectively with spin $\sigma = \uparrow, \downarrow$. In addition, $\epsilon_{k\alpha} = \epsilon_k \pm eV/2$ and $n_\sigma = d_\sigma^\dagger d_\sigma$. The cavity is also coupled to two long transmission lines, similar to Eq. (4), that serve to send and receive the micro-wave signals. A similar Hamiltonian has been addressed to study transport through molecules with particular modes of vibration [246, 247, 248].

One natural question concerns the phase of the transmitted photon signal as a function of the bias voltage V across the two electronic reservoirs [103]. Let us consider the quantum limit in the cavity with a few photon number. The phase of the transmitted (photon) signal can be computed through the relation $\tan \varphi(\omega) =$

$-\text{Re } \chi_{xx}^R(\omega)/\text{Im } \chi_{xx}^R(\omega)$ in Eq. (6). By including the cavity-dot coupling, the (light) susceptibility becomes modified as $\chi_{xx}^R(\omega) = \omega_0/(\omega^2 - \omega_0^2 + i\omega_0\kappa - \omega_0\Pi^R(\omega))$. We have treated the light-matter coupling to second order in perturbation theory. Interestingly, in the limit of large bias voltage applied across the quantum dot, the damping of the light field is only weakly affected by the (fast) motion of the electrons ($\text{Im } \Pi^R(\omega_0) \ll \text{Re } \Pi^R(\omega_0)$), then resulting in a phase $\varphi(\omega_0)$ which tends to a universal value $\pi/2$ in the limit of very small dissipation from the cavity environment ($\kappa \rightarrow 0$) [103]. Note that this phenomenon occurs both when the quantum dot is at resonance and in the Kondo regime. It is also relevant to note that at the renormalized frequency ω^* defined through the equation $\omega^2 - \omega_0^2 - \omega_0 \text{Re } \Pi^R(\omega) = 0$, the transmission coefficient exhibits a pronounced peak (see Fig. 2). At large bias voltages ω^* converges towards ω_0 and the transmission peak approaches unity for symmetric couplings with the left and right transmission lines (in Eq. (6)).

The effect of the electronic non-equilibrium environment on the cavity photon is to induce non linearity, damping/friction term and noise term. These terms in general are independent due to current flow and non-equilibrium effects and can be fully tuned by changing bias and gate voltages. In the limit of large bias, we can formally integrate out all the fermionic modes to obtain an explicit expression for the cavity photon effective action [103]. From this we can see how these hybrid systems can also serve to simulate novel regimes of the Langevin equation and some chaotic properties of the Duffing oscillator [103], which has been realized experimentally [249]. Other recent theoretical developments can be found in Refs. [250, 251, 252, 253, 254, 255]. Similar ideas to build dissipative arrays have been addressed with hybrid system technology [256]. It is also relevant that coupling topological systems such as (one-dimensional) topological superconductors [257, 258, 259] to circuit QED has also attracted some attention, both for simulation and measurement point of view [260, 261, 262, 263, 264]. Implementing Majorana fermions with photons has also engendered some attention in the community [265].

3. Large Circuits

3.1. Circuit QED array : Interaction Effects, Effective Chemical Potential and Driving

In the last few years, some experimental progress were also accomplished towards realizing larger arrays in cQED [10, 12]. Experimental efforts have been realized in various geometries [266]. Two cavities can be coupled through a capacitive or Josephson coupling [178, 267]. This point has also been carefully addressed in Ref. [117]. In particular, a cavity-dimer was realized experimentally [178] allowing to investigate novel non-dynamical steady states and entanglement features [268].

Assuming that the coupling between cavities (usually in the MHz range) is much smaller than the internal frequencies of the cavities, the coupling between cavities turns into an effective hopping term of bosons (photons) from one cavity to the neighboring ones. For a large quality factor, we may restrict our treatment to a single photon mode, similar to the Jaynes-Cummings, Rabi and Dicke Hamiltonians. One model that has been investigated a lot theoretically in the literature is the Jaynes-Cummings lattice model, which takes the form [10] :

$$H = \sum_i H_i^{JC} - J \sum_{\langle i,j \rangle} (a_i^\dagger a_j + a_j^\dagger a_i) - \mu_{eff} \sum_i (a_i^\dagger a_i + \sigma_i^+ \sigma_i^-), \quad (14)$$

where H_i^{JC} describes the Jaynes-Cummings Hamiltonian in each cavity, introduced earlier, and the bracket notation $\langle i,j \rangle$ denotes summation over nearest neighbors (see Fig. 3). One can realize experimentally complex hopping tunneling elements J at each junction, by inserting a Josephson junction ring, which allows to break-time reversal symmetry and to realize topological phases [56, 57, 59, 60], as described below. Similar quantum nano-circulators have also been designed in cQED architectures [58]. This model has attracted some attention, for example, in the light of realizing analogues of Mott insulators with polaritons [37, 40, 41, 42]. More precisely, one observes two competing effects in these arrays. For large values of J , formally the system would tend to form a wave delocalized over the full lattice, by analogy to a polariton superfluid [75], whereas for very small values of J , the photon blockade in each cavity addressed earlier in Sec. 2.2 will play some role. Theoretical works [37, 41, 42], including ours [40, 46], have focused their attentions on solving the phase diagram at equilibrium in the presence of a tunable chemical potential μ_{eff} . For $\mu_{eff} = 0$, the ground state at small J would correspond to the vacuum in each cavity. By driving the system with a quite strong drive amplitude, one can eventually reach for intermediate times, a state with one polariton in average in each cavity [32], by analogy to applying a π pulse on the qubits for

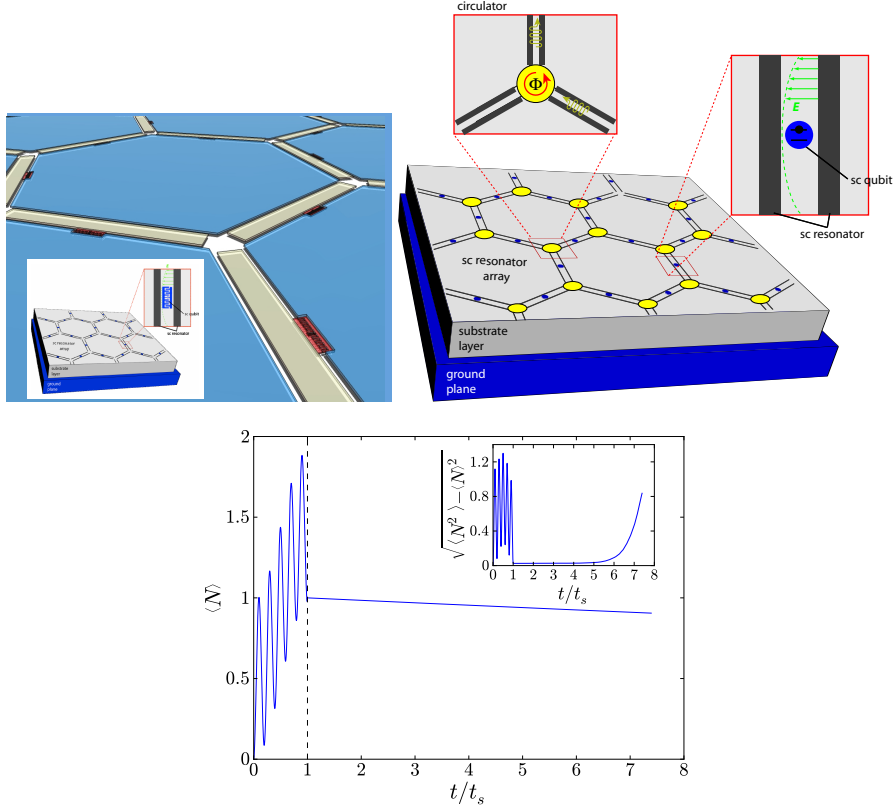


FIGURE 3 : (Color online) (Left) Example of Superconducting Quantum Electrodynamics Networks, from Refs. [10, 40]. Some experimental efforts are currently realized to engineer these artificial photonic lattices operating in the microwave domain [12]. Some other photonic (polaritonic) lattices have been built [98, 96]. (Right) Synthetic gauge fields on photon systems can be generated through nano-circulators or Josephson junction rings at each Y-junction, on the honeycomb lattice [56]. Efforts are currently developed to realize such nano-circulators on-chip [60]. This honeycomb lattice with artificial gauge fields is also dual to a Kagome lattice with particular flux configurations [56, 59], as described below. Below, we also show how these gauge fields can be realized through the Floquet theory. (Bottom) We apply the methodology described in Ref. [32] and include dissipation effects in each cavity ($\omega_0 \rightarrow \omega_0 - i\kappa/2$). The protocol is as follows : we drive the cavities to reach a polariton state $|1-\rangle$ in each cavity at a (sufficiently short) time scale t_s . Then, we increase the coupling between cavities and observe an intermediate state in time, where the standard deviation of the polariton number $N = a^\dagger a + \frac{1}{2}(\sigma_z + 1)$ is negligible (similar to a Mott phase). At long times, we identify a dynamical ‘transition’ towards a delocalized state. The parameters are : $g/\omega_0 = 0.02$, $\Delta/\omega_0 = 0.9$, $V_0/\omega_0 = 0.1$ (V_0 is the amplitude of the AC signal on each cavity), $\kappa/\omega_0 = 10^{-6}$, $J = v_J(t - t_s)$ (for $t \geq t_s$) with $v_J = 10^{-2}$ (in units of $\omega_0 t_s^{-1}$) and $\omega_0 t_s = 311$.

a large detuning between light and matter [153]. By driving cavities individually, an intermediate state analogous to a Mott state, was also identified [32]. In Fig. 3, we apply the methodology of Ref. [32], and apply for the following protocol : we prepare the cavity in a $|1-\rangle$ polariton state by driving each cavity, then we stop the drive and we switch on the coupling between cavities. First, we observe that the effect of dissipation on each cavity described by the modification $\omega_0 \rightarrow \omega_0 - i\kappa/2$ (see Eq. (7)) produces some small leakage on the mean polariton number per cavity $\langle N \rangle = \langle a^\dagger a + \frac{1}{2}(\sigma_z + 1) \rangle$, compared to Ref. [32]. At longer times, the coupling between cavities increases substantially the fluctuations, producing dynamically an extended state for the photons (polaritons). The coupling between cavities here is treated at a mean-field level.

By changing μ_{eff} and keeping J small, one could eventually turn the vacuum in each cavity into a polariton state in the branch $|1-\rangle$ (by analogy to Fig. 1). Simple energetic arguments predict that this change would occur when $E_{1-} - \mu_{eff} = E_0$ (we use the notations of Sec. 2.2). This result can be made more formal by using a mean-field theory and a strong-coupling expansion [37, 40, 41, 42]. In the atomic limit where J is small, one then predicts the analogue of Mott-insulating incompressible phases, as observed in ultra-cold atoms [269], where it costs a finite energy to change the polariton number [37, 39]. This point has been summarized in various recent reviews [10, 11]. By increasing the hopping J , one can build an equivalent of the ψ^4 theory, where $\psi \sim \langle a_i \rangle$,

in order to describe the second-order quantum phase transition between the Mott region of polaritons and the superfluid limit [40]. These results, which make a connection with the Bose-Hubbard model [270, 271], have been confirmed by numerical methods such as Quantum Monte Carlo [272]. The effect of disorder has also been addressed using the Density Matrix Renormalization Group, leading potentially to glassy phases [273]. Disorder can also result in Coddington-Chalker type models for light [274]. Weak-localization effects have been observed in superconducting circuits [116]. These spin-boson networks can also serve to emulate various quantum spin models [40, 38, 275]. It is also relevant to stress the appearance of novel phases in the limit of Rabi lattices, beyond the weak Jaynes-Cummings coupling limit; the presence of the counter-rotating wave terms somehow acts as an effective chemical potential [43]. The limit of strong light-matter coupling has been achieved with two stripline resonators [276]. Dissipation effects in the two resonator problem have been addressed in Ref. [277].

Similar to ultra-cold atoms [18, 19], it seems important to be able to tune the effective chemical potential of photons to develop further quantum simulation proposals and observe these analogues of Mott phases in the long time limit. Recently, an idea to simulate an effective chemical potential for photons was pushed forward in Ref. [45], based on a parametric coupling with a bath of the form $\lambda \cos(\omega_p t) H_{SB}$ where $H_{SB} = \sum_j (a_j + a_j^\dagger) B_j$ where B_j is a bath operator. Superfluid-Mott transitions using this scheme were also addressed in finite lattices. The Bose-Hubbard model was considered which can be achieved for highly detuned cavities [146]. At a more general level, considerable attention has been turned towards describing driven cavity arrays in order to realize novel steady states. In this context, it is certainly important to develop analytical and numerical tools [278, 279, 280], from stochastic non-equilibrium mean-field theories [195, 196] to Matrix Product States for example [44, 281, 282]. We have also done some progress to tackle the real-time dynamics of a spin dimer and the quantum Ising model in a transverse field coupled to a bosonic environment out of equilibrium [195]. See also recent Refs. [283, 284]. Features of the quantum Ising model has been recently observed in cQED [13]. Note that the time-dependent numerical renormalization group is also powerful to describe the dissipative two spin system [177].

Based on all these theoretical developments, we expect more experimental developments in cQED multi-cavity systems [10, 12, 276].

3.2. Artificial Graphene, Kagome lattice and Haldane Model

Some experimental works have also succeeded in realizing topological phases through the Haldane model and analogous ideas. Here, we review these developments starting from the artificial graphene on the honeycomb lattice [98, 285, 286]. The Haldane model takes the form (here, hopping elements are denoted t_1 and t_2 by analogy to condensed-matter systems) [287] :

$$H = -t_1 \sum_{\langle i,j \rangle} c_i^\dagger c_j - t_2 \sum_{\langle\langle i,j \rangle\rangle} e^{i\phi_{ij}} c_i^\dagger c_j + h.c. + \sum_i (-1)^i M c_i^\dagger c_i. \quad (15)$$

We fix the effective chemical potential of photons to be zero here. The presence of the term t_1 will produce the Dirac particles on the graphene lattice and the term t_2 which couples the next nearest neighbor sites corresponds to the Haldane term. We also add the Semenoff mass M which has been introduced in high-energy physics [288]; for cQED arrays, it corresponds to an adjustment of frequencies in each cavity and in ultra-cold atoms it can be engineered through magnetic field gradients, for example. In the continuum limit, we will make an analogy with the equation $(i\gamma^\mu \partial_\mu - M)\psi = 0$ where ψ will be a two-component spinor and the Clifford algebra will be generated from the Pauli matrices. The configuration of fluxes for the Haldane model is shown in Fig. 4 and $\phi_{ij} = \phi$. The key point here is that the t_2 term opens a gap at the Dirac points by breaking the time-reversal symmetry : this produces the emergence of chiral edge modes at a single-particle level. In fact, the system exhibits a non-trivial topology as long as $M < 3\sqrt{3}t_2 \sin \phi$ [287].

The precise analogy with the Dirac equation in two spatial dimensions can be seen by introducing the spinor $\psi_i = (a_i, b_i)$, where a and b correspond to the two sub-lattices [289]. When $t_2 = 0$ two inequivalent Dirac points K and K' naturally emerge in the band structure [290] and are protected by the PT symmetry (Time-reversal and inversion symmetry); for an introduction, see Refs. [289, 291]. The dispersion relation close to the Dirac points in that case obeys $\pm \sqrt{(v_x k_x)^2 + (v_y k_y)^2}$, where v_i are the group velocities (below, $v_i = v$), and one can check that $(PT)H(\mathbf{k})(PT)^{-1} = H(\mathbf{k})^*$. The implementation of Dirac particles has also been discussed in the cQED context [292], in relation with existing experiments. Two linear dispersion bands in the three-dimensional momentum

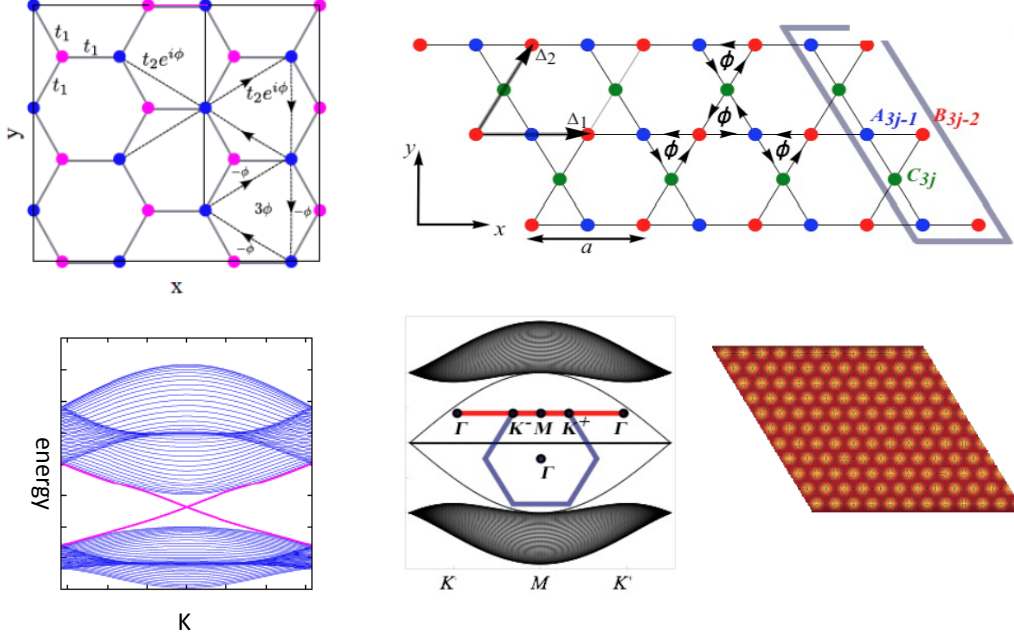


FIGURE 4 : Left : Representation of the Haldane model on the honeycomb lattice [287] (top) and spectrum of the system in the topological phase (bottom). Right : Kagome lattice with artificial fluxes [59], dual to the honeycomb lattice with Josephson junction rings introduced in Ref. [56]. This can be seen as a realization of the Haldane model on the Kagome lattice with no net flux on a parallelogram unit cell. Bottom, middle : spectrum of the Kagome lattice for one particular flux configuration $\phi = \pi/6$ (with the notations of Ref. [59]), showing a flat band in the middle of the energy spectrum in addition to chiral edge modes. Right, the local density of states then also shows states trapped on hexagonal cells [59], as observed in the experimental setup of Ref. [98].

space can also intersect at a single degenerate point – the Weyl point. Recently, these three-dimensional Weyl fermions have also been identified in photonic crystals [97].

The total Hamiltonian can also be written as an effective spin-1/2 particle subject to a magnetic field in the wave-vector space : $H = \int_{BZ} d\mathbf{k} \psi^\dagger(\mathbf{k}) H_H(\mathbf{k}) \psi(\mathbf{k})$ where $H_H = -\mathbf{h}(\mathbf{k}) \cdot \hat{\sigma}$, and in the basis of the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ the effective magnetic field takes the form :

$$\mathbf{h}(\mathbf{k}) = (t_1 \sum_i \cos(\mathbf{k} \cdot \mathbf{a}_i), t_1 \sum_i \sin(\mathbf{k} \cdot \mathbf{a}_i), +M - 2t_2 \sum_i \sin(\mathbf{k} \cdot \mathbf{b}_i)). \quad (16)$$

Here, \mathbf{a}_i and \mathbf{b}_i are the vectors connecting the nearest-neighbors and the next-nearest neighbors respectively and we assume that the phase $\phi = \pi/2$ such that the t_2 term is purely imaginary. The presence of the t_2 opens a gap and produces a quantized Chern number similar to the quantum Hall case [287]. In the presence of P (T broken), the Berry curvature satisfies $\mathcal{F}(\mathbf{k}) = \mathcal{F}(-\mathbf{k})$ and the total Berry flux adds up to 2π and the Chern number equals one. By analogy with a spin-1/2 particle, the Chern number of the lowest band can also be written as : $C_- = (1/4\pi) \int_{BZ} d\mathbf{k} \mathbf{h} \cdot (\partial_x \tilde{\mathbf{h}} \times \partial_y \tilde{\mathbf{h}})$, where $\tilde{\mathbf{h}}$ is the normalized \mathbf{h} vector (here, \mathbf{k} lives on the Brillouin zone torus whereas $\tilde{\mathbf{h}}$ lives on the Bloch sphere). A pedagogical derivation is done in Ref. [291]. An important point is that the t_2 contribution is odd under changing \mathbf{k} in $-\mathbf{k}$. The bulk-edge correspondence ensures the presence of a chiral edge mode in this case similar to the quantum Hall situation [83, 85] (see Fig. 4). A similar quantum Hall state without Landau levels can be generated through (long-range) interaction effects [293, 294].

In fact, Haldane and Raghu have extended this scheme to the field of gyromagnetic photonic crystals with ferrite rods and the magneto-optical Faraday effect then mimics the t_2 term of the Haldane Hamiltonian [82]. This theoretical idea has been first realized at MIT in 2009, where a photonic band gap has been observed as

well as robust chiral edge modes for light in the microwave regime [67]. Note that the experiment has been realized with ferrite rods forming a square lattice instead of the honeycomb lattice (see Fig. 5). Photon systems are particularly appropriate to produce chiral edge modes, where selecting the right-frequency of the incoming wave and observing the local density of states is relatively straightforward [74]. The Haldane Hamiltonian has also been achieved in 2014 at ETH Zuerich [295], with fermions in optical lattices, where the t_2 term has been simulated using the Floquet point of view and the shaking protocol [72, 73]. Similar experiments have also been realized at Hamburg [296]. Preceding experiments on ultra-cold Dirac fermions have managed to move the Dirac points by tuning the hopping amplitudes [297, 298]. The Chern number has also been measured in ultra-cold atoms [299]. The Zak phase has been experimentally observed [300, 301]. A similar quantum Hall state without a uniform magnetic flux was also observed in Bismuth materials [302].

It is also important to note that the Haldane model can be extended to more complicated lattices such as the Kagome lattice, with three sites (atoms) per unit cell. Following the protocol of Refs. [56, 57, 59], summarized in Figs. 3 and 4, one can realize an analogous situation with chiral edge modes of light in cQED. One can map the honeycomb lattice with Josephson rings at each link onto a “dual” Kagome lattice with fluxes in the “bow ties”. The Kagome model contains particles hopping between nearest-neighbor sites with a complex integral $|t|e^{\pm i\phi}$. Photons then acquire a phase 3ϕ around a triangular plaquette and a phase -6ϕ on a honeycomb cell, amounting to a zero net flux in a parallelogram unit cell. By adjusting the flux in each Josephson junction ring, one can realize a quantum Hall effect without Landau levels [59] (see Fig. 4). This model with three sites per unit cell also allows the occurrence of flat bands which corresponds to wave-functions localized on each hexagonal cell (see Fig. 4) [59, 303]. The search of flat bands has a long history in condensed-matter physics [304, 305, 306]. Note that flat bands have been recently observed at LPN Marcoussis in artificial honeycomb systems [98]. Similar lattices (with fluxes) have been widely investigated theoretically and some have been realized in quantum circuits and ultra-cold atoms [307, 308, 309, 310]. The Kagome lattice is also a candidate for the exploration of quantum spin liquids (non-ordered Néel states) [311]. Other correlated states have been predicted on the Kagome lattice [312, 313, 314, 315, 316, 317].

Very recently, another progress was achieved in cQED [93, 94], by observing similar topological effects and simulating directly the Haldane-type Hamiltonian $H(k_x, k_y) = v(k_x\sigma_x + k_y\sigma_y) + (M - m_t)\sigma_z$, where m_t corresponds to the second-neighbor hopping (tunneling) in a local magnetic field. This Hamiltonian has been implemented with one or two superconducting qubits on the Bloch sphere : the BZ on a torus is then simply replaced by an integration over the polar and azimuthal angles on the Bloch sphere. A path from north to south pole on the Bloch sphere was achieved in time, through microwave pulses. Note that these measurements constitute a dynamical (direct) measurement of the Berry phase. These measurements use the general scheme proposed in Ref. [95]. Other measurements of the Berry phase were also done in Ref. [92]. Motion in a curved space will be deflected from a straight trajectory and the state of the system feels a force leading to deviations from the adiabatic path. The experiments have directly measured the Chern number $C = \int_0^\pi F_{\theta\phi} d\theta$ where the Berry curvature component $F_{\theta\phi}$ encodes the information of the force and is related to the spin observable $\langle\sigma_y\rangle$ (the experiments work at constant speed allowing a linear response). The two experiments report a quantized Chern number 1 for $M < m_t$ and a topological phase transition for $M = m_t$ where the Chern number drops to zero.

3.3. Topological Quantum Fluids and Spin-Orbit Coupling

The goal to achieve arbitrary control of photon flows has motivated an increasing recent research on photonic crystals, metamaterials and artificial quantum circuits. Here, we first review the theoretical proposal of Ref. [318] and describe the emergence of protected one-way photon edge states that are robust against disorder, without the use of magneto-optical effects [67, 74] (see also Fig. 5, top, left). Using simple protocols, based on particular choices of spatial distribution and modulation phases, one can produce an effective magnetic field for photons, leading to a Lorentz force. The Hamiltonian takes the form (see Figure 5, bottom left)

$$\begin{aligned}
H &= \omega_A \sum_i a_i^\dagger a_i + \omega_B \sum_i b_i^\dagger b_i \\
&+ \sum_{\langle i,j \rangle} V \cos(\Omega t + \phi_{ij})(a_i^\dagger b_j + h.c.).
\end{aligned} \tag{17}$$

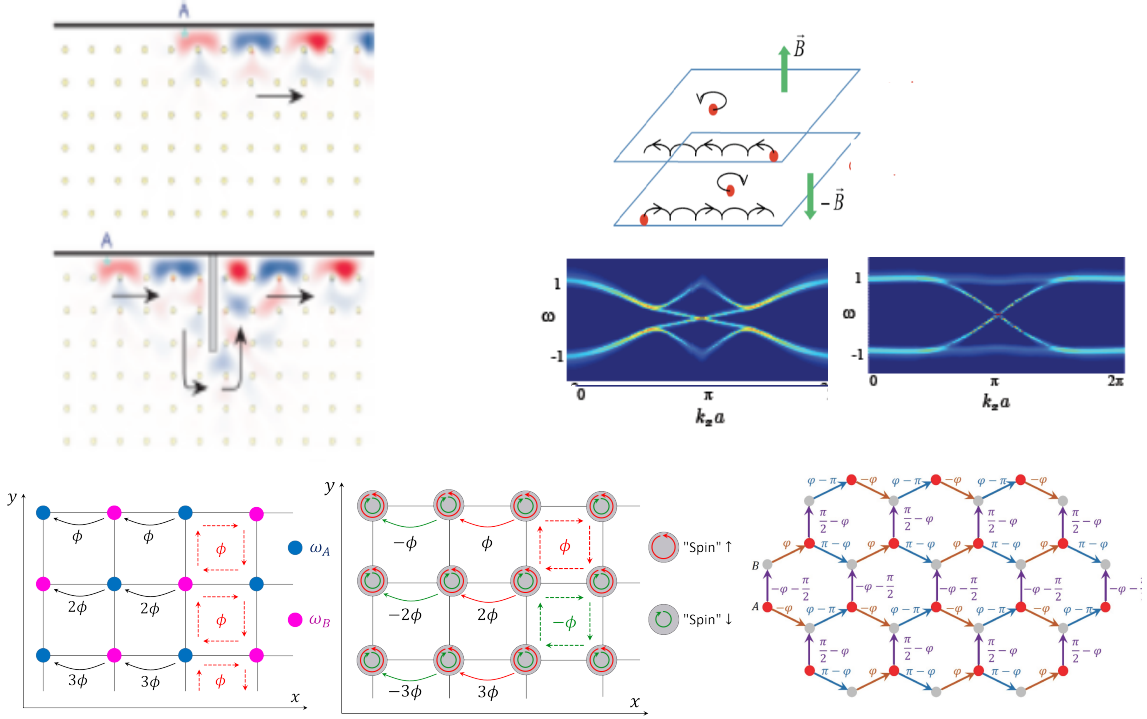


FIGURE 5 : Top : Left, first experimental observation of chiral edge modes and quantum Hall phases of photons in gyromagnetic photonic crystals [67, 74]. Bottom, Left : possible Floquet scheme suggested in Ref. [318] to realize Peierls phases and quantum Hall phases for photon systems without the Faraday effect. Bottom, middle : Generalization to a quantum Spin Hall state, as observed experimentally in silicon photonics at University of Maryland [69, 70]. Top, right : Spectral function from Ref. [332] computed with CDMFT [336, 337] exemplifying that the helical edge modes of the Kane-Mele model are robust towards interaction effects. Bottom, right : Possible phase configuration using the Floquet scheme of Eq. (17) producing an Haldane state with anisotropic couplings (see Eq. (19)).

In this scheme, two neighboring resonators have different frequencies. One can simulate various situations by adjusting the modulation frequency Ω . Let us assume that we are close to resonance $\Omega = \omega_A - \omega_B$, then we can build a simple physical argument based on the rotating-wave approximation. Assuming that $V \ll \Omega$, the evolution in time of the photons is controlled by ω_A and ω_B mostly. It is then justified to define the operators $c_{i(j)} = e^{[i\omega_{A(B)}t]c_{i(j)}^\dagger} a_i (b_j)$.

In the rotating frame, the Hamiltonian is equivalent to charged particles subject to a magnetic field [319, 320] :

$$H_{eff} = \sum_{\langle i;j \rangle} \frac{V}{2} (e^{-i\phi_{ij}} c_i^\dagger c_j + e^{i\phi_{ij}} c_j^\dagger c_i). \quad (18)$$

Using this resonant scheme, then one can produce an equivalent of Peierls phases on a given lattice, for neutral particles ; here, the nearest-neighbor coupling is dominant. In particular, by adjusting experimentally the phases ϕ_{ij} (see Fig. 5, bottom left) one can realize for example a uniform magnetic field related to the effective vector potential \mathbf{A}_{eff} defined as $\int_i^j \mathbf{A}_{eff} \cdot d\mathbf{l} = \phi_{ij}$. An idea to realize the dynamical coupling in photon (quantum circuit) systems has been suggested in Ref. [318]. A similar scheme has been used in ultra-cold atoms, based on resonant laser-assisted tunneling and tilted lattices [79, 80, 81]. In the context of ultra-cold atoms, tilts can be achieved in various ways, from magnetic to gravitational gradients.

In the case of a uniform net flux, the effective Hamiltonian then exhibits the same properties as a two-dimensional gas in the presence of a uniform magnetic field. If the flux is rational, one observes a Hofstadter

butterfly and chiral edge modes, as achieved in ultra-cold atoms for example [79]. To show this, Fang et al. applied a simple Floquet argument [318]. The original Hamiltonian is periodic in time $H(t) = H(t + \frac{2\pi}{\Omega})$ and the eigenstates then take the form $|\psi(t)\rangle = e^{-i\epsilon t}|\chi(t)\rangle$ where ϵ represents the “quasi-energy” which can be found using the equation $(i\partial_t - H(t))|\chi(t)\rangle = -\epsilon|\chi(t)\rangle$. The time-dependent Hamiltonian here takes the particular form : $H(t) = H_0 + H_1 e^{i\Omega t} + H_{-1} e^{-i\Omega t}$, where $H_0 = H_{0,a} + H_{0,b} = \sum_i \omega_i c_i^\dagger c_i$. The problem can be diagonalized using the decomposition $|\chi(t)\rangle = \sum_{n=-\infty}^{+\infty} |\chi_n\rangle e^{in\Omega t}$ and using the (a, b) basis, by writing $|\chi_n\rangle = |\chi_{n,a}\rangle + |\chi_{n,b}\rangle$ [321]. Outside the resonant condition, one can apply the Floquet theory, a multi-phase expansion and a $\mathcal{O}(1/\Omega)$ expansion [72, 73]. This allows to achieve various flux configurations and spin-orbit couplings [322, 323]. In particular, one can realize an effective Haldane-type model, as follows. We expand the effective Floquet Hamiltonian in powers of $1/\Omega$. To first order in $1/\Omega$, away from the resonance condition, one obtains

$$H = H_{\text{graphene}} + \frac{V^2}{\Omega} \left(\sum_{\langle\langle i;k \rangle\rangle} (2ia_i^\dagger a_k \sin(\phi_{ij} + \phi_{jk}) + h.c.) + (\text{same terms on B-sublattice}) \right). \quad (19)$$

We assume that the Hamiltonian also contains a constant nearest-neighbor hopping term (which can be generated through a perturbation of the form $V \cos^2(\Omega t + \phi_{ij})$). Here, j is the unique nearest neighbor of both sites i and k . To realize a t_2 term coupling next-nearest-neighbor sites, we can fix the phases ϕ_{ij} to zero. Another interesting choice is as drawn in Fig. 5 (bottom right). In that case, the generated t_2 terms are anisotropic : the t_2 term parallel to the horizontal axis is formally zero.

A generalization of the Haldane model on the honeycomb lattice to spin-1/2 particles has been made by Kane and Mele in 2005 [324] (see also Refs. [64, 65, 66]) :

$$H = - \sum_{\langle i;j \rangle} t_1 c_{ia}^\dagger c_{ja} + \sum_{\langle\langle i;j \rangle\rangle} i t_2 v_{ij} s_{\alpha\beta}^z c_{ia}^\dagger c_{j\beta} + h.c. \quad (20)$$

The it_2 term (here, purely imaginary) mimics an atomic spin-orbit coupling $s^z L^z$. More precisely, on a given pair i and j of next-nearest-neighbor sites within the same sub-lattice, we must satisfy $v_{ij} = -v_{ji} = \pm 1$. Similar to the Haldane model, the spin-orbit t_2 term opens a gap in the system. The main difference is that now the system respects the T-symmetry and is characterized by a different topological invariant. A simple picture of the edge states is the Kramers pair (Fig. 5, top right) : a particle (electron) with spin-up propagates to the left whereas a particle (electron) with spin-down propagates to the right. This leads to a spin current at the edges and to a \mathbb{Z}_2 type topological invariant [325, 326] and a spin Chern number [327]. The \mathbb{Z}_2 invariant can also be expressed in terms of the four time-reversal invariant momenta (points) in the Brillouin zone [328]. Progress have been accomplished in ultra-cold atoms, with magnetic field gradients, to observe such a quantum spin Hall state [79]. Another important progress from the experimental point of view comes from the discovery of Mercury Telluride materials which have allowed to observe a similar physics in two dimensions, leading to the discovery of two-dimensional topological insulators [329, 330]. In Refs. [331, 332], together with S. Rachel, W. Wu and W.-M. Liu, we have shown that the helical edge modes in the Kane-Mele model are robust towards finite to moderate interactions. For free particles, the spectrum of the edge modes can be computed analytically [333]. For recent reviews on interaction effects in topological spin-orbit systems and iridate materials, consult for example, Refs. [334, 335]. In Fig. 5, we show the spectral functions in the presence of a Hubbard interaction obtained with a real-space extension [332] of the Cluster Dynamical Mean-Field Theory (CDMFT) [336, 337]. We have reached a similar conclusion using the Slave-Rotor approach [331, 338]. It is also interesting to note the possibility of chiral bogoliubons in a BEC system [339].

Recently, a quantum spin Hall state has been realized in photon systems by Hafezi et al. [69, 70] based on a Silicon architecture (see Fig. 5, bottom, middle) : the two spin-polarizations correspond to the two possible circulations (clockwise or anti-clockwise) in a given resonator. The phase between resonators are carefully adjusted to follow the scheme of Fig. 5. The main difference with the usual quantum spin Hall effect is that here the two spin polarizations couple when T is not broken. To decouple the two spin polarizations, one requires to break the T symmetry. The measurement of a topological invariant in these systems has been suggested [86]. Defining (time-dependent) topological invariants in relation with observables is certainly timely for Floquet theories [340, 341]. Measuring topological invariants in quantum fluids can also be done exploiting the Karplus-Luttinger anomalous velocity [88, 87, 59]. Another important progress concerns the first realization of a photonic

topological Floquet insulator [68]. Rechtsman et al. translated the modulation from the time domain to the spatial domain, leading to the experimental demonstration of the photonic analogue of the quantum Hall effect at optical frequencies (633nm). Similar developments have been done in micro-cavities where a spin-orbit coupling for polaritons has been engineered [71]. Concerning quantum circuits, a topological insulator has been realized based on meta-materials, composed of capacitively coupled high-Q inductors [342], and other suggestions have been addressed [343]. Other quantum circuit architectures have been proposed [344]. The effect of dissipation in driven lattices has also been considered theoretically, in particular, on honeycomb lattice systems [345]. In fact, these developments are not restricted to two dimensions in these quantum fluids by analogy to the topological properties of Bismuth materials [64, 65, 66].

Let us also emphasize that a plethora of theoretical arguments have been proposed to realize strongly-correlated quantum Hall type phases in photon fluids, such as fractional quantum Hall phases, in particular, in the presence of out of equilibrium driving terms [346, 347, 348, 349].

3.4. Josephson Effect and Chiral Bosonic Phases

The effect of interactions on topological phases have been addressed in various contexts [350, 351, 352]. The stability of topological insulators, for example, towards disorder has been thoroughly analyzed from a theoretical point of view [353, 354, 59]. For problems such as the quantum spin Hall state, one can apply various analytical arguments to answer these types of questions [324, 331]. The stability of topological insulators and superconductors is in general ensured perturbatively due to the gap protection, for fermion systems. Extending the notion of topological invariants to interacting fermion systems has also attracted some attention in the community in relation with numerical developments [355, 356, 357, 358].

For boson systems, the situation is different since the bosons can condense in a Bose-Einstein type manner and occupy macroscopically the bottom of the lowest Bloch band. Defining topology in these many-body boson systems has attracted attention recently. Here, we review two examples of non-trivial effects in chiral bosonic systems, in relation with current experiments [295, 359]. The breaking of time-reversal symmetry via the spontaneous formation of chiral order has been observed in Ref. [360]. The physics here is associated with Meissner physics. A connection to quantum Hall phases in ladder systems will be addressed.

The first effect concerns a situation related to Eqs. (17) and (18), which is obtained from the time-dependent Hamiltonian at the resonance condition. Here we follow the recent experiment realized at Muenich on a ladder-type geometry [359], and focus on the effective (time-independent) Hamiltonian :

$$\begin{aligned}
H &= -t \sum_{\langle i,j \rangle} e^{iLA_{ij}^\alpha} a_{i\alpha}^\dagger a_{j\alpha} - t_\perp \sum_{i\alpha} e^{-iLA_{\perp i}} a_{i2}^\dagger a_{i1} + h.c. \\
&+ \frac{U}{2} \sum_{i\alpha} n_{i\alpha}(n_{i\alpha} - 1) + V_\perp \sum_i n_{i1} n_{i2} - \mu \sum_{i\alpha} n_{i\alpha}.
\end{aligned} \tag{21}$$

$\alpha = 1, 2$ denotes the two chain index, l is the lattice spacing in both directions (parallel and perpendicular to the chains) and we have considered a general gauge field acting on the boson operators of each chain. We have also added a chemical potential which is justified in ultra-cold atoms, Josephson junction arrays and can be simulated in photon networks, as discussed in Sec. 3.1. Here, we also consider the effect of on-site repulsion and repulsion between chains, $n_{i\alpha} = a_{i\alpha}^\dagger a_{i\alpha}$. The t_\perp term represents a Josephson coupling between chains favoring (superfluid) phase-locking between the chains and the occurrence of a superfluid. The artificial flux phases can be explicitly derived from a Floquet approach [361]. The application of a small (synthetic) magnetic field perpendicular to the ladder plane then will result in the Meissner effect as in superconductors. We can describe this Meissner effect as follows [362, 363]. Let us introduce the explicit bosonic representation $a_{i\alpha} = \sqrt{n} e^{i\theta_{i\alpha}}$, where n represents the mean density in each chain, describing the (quasi-)superfluid in each chain. The Josephson term can be re-written as $-t_\perp n \cos(LA_{\perp i} + \theta_{1i} - \theta_{2i})$. The current (density) operators at point i take the form : $j_\parallel = it(-e^{iLA_{ij}^1} a_{i1}^\dagger a_{j1} + e^{iLA_{ij}^2} a_{i2}^\dagger a_{j2}) + h.c.$ and $j_\perp = -it_\perp a_{i1}^\dagger a_{i2} e^{iLA_{\perp i}} + h.c..$ The key point is that for strong values of t_\perp , the superfluid phases are pinned such that $\theta_{1i} - \theta_{2i} + LA_{\perp i} = 0$ and therefore we obtain a Meissner effect, meaning that $\langle j_\perp \rangle = 0$ in the bulk (see Fig. 6). Furthermore, in the small field limit, we may expand to get

$$\langle j_\parallel \rangle = \langle j_1 \rangle - \langle j_2 \rangle = -2tn \text{ phase}_{ij}. \tag{22}$$

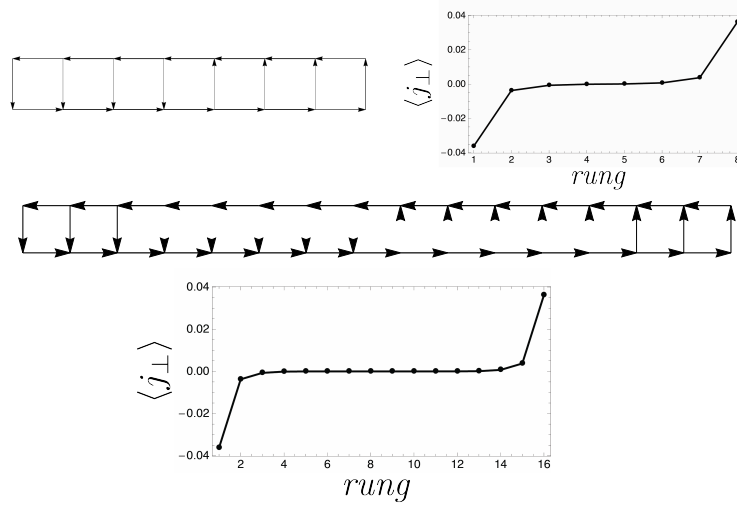


FIGURE 6 : DMRG and Exact Diagonalization results on small ladders with bosons, with 8 and 16 rungs, showing that the Meissner currents subsist in a Mott phase with total density 1, as predicted by the theory [365, 366]. Here, the parameters are fixed to : formally $U \rightarrow +\infty$ (hard-core bosons), $V_{\perp} = 4t$, $t = 1$, $t_{\perp} = 1$ and the flux is fixed to $\chi = \pi/25$ using the notations of Ref. [366]. Other careful DMRG analyses of this model have been performed in Refs. [367, 368, 369].

Here, we have defined the phase around a plaquette, $phase_{ij} = (A_{ij}^2 - A_{ij}^1)l + (A_{\perp i} - A_{\perp j})l$. Then, this corresponds to the usual Meissner current, here defined on a lattice : a chiral current moves in opposite direction than the applied external flux. This is an example of chiral superfluid phase in low dimensions which has been observed experimentally [359]. This argument is in fact general since the t_{\perp} hopping term is a relevant perturbation under Renormalization Group arguments in the superfluid phase ; one can also apply bosonization techniques explicitly [364]. By increasing the magnetic flux, one sees the emergence of vortices ; similarly to type II superconductors, there is a critical field strength. Another nice phenomenon is that these Meissner currents can subsist when entering into a Mott phase. For example, for one boson per rung in average, as shown in Refs. [365, 366], by adjusting the parameters U and V_{\perp} , one can enter a Mott state where the total density in each rung is fixed but still Meissner currents develop : this is an example of chiral Mott phase which breaks time-reversal symmetry. In Fig. 6, we show that the Meissner currents in the Mott phase can be already observed for relatively small systems (with 8 or 16 rungs for example ; these results are obtained with the Density matrix Renormalization Group (DMRG) and Exact Diagonalization). More results have been confirmed independently based on the DMRG [367, 368]. Other chiral phases have been predicted for other filling factors [369, 370, 371]. Note that Laughlin type phases in ladder systems [372, 373, 374] characterized by chiral edge states are also possible by adjusting densities and the magnetic flux, as shown in Ref. [366]. These correlated phases can also be realized in quantum circuits such as Josephson junction ladders [366].

Another example of chiral bosonic phases has been identified through the Haldane model on the honeycomb lattice discussed in Sec. 3.2, with bosons. Extensive results have been obtained through a combination of bosonic DMFT, Exact Diagonalization and analytical arguments [375]. Here, we review simple physical results. One can easily identify the presence of two superfluid phases in the phase diagram. For $t_2 = 0$ (here, we follow the notations of Sec. 3.2) all the bosons condense at the center of the Brillouin zone, at zero momentum. On the other hand, for $t_1 = 0$, the model turns into two decoupled triangular lattices, and the bosons condense at the two inequivalent corners of the Brillouin zone K and K' . Whereas the superfluid phase at $t_2 = 0$ will exhibit Meissner physics in response to the Haldane phases $\phi_{ij} = \pi/2$ between the next-nearest-neighbor bonds, the phase at large t_2 constitutes a novel chiral superfluid phase. More precisely, as an order parameter, one can define the average current $\langle j_{ij} \rangle$ coupling a pair of next-nearest-neighbors which belong to the same sub-lattice. In the superfluid phase with $t_2 = 0$, the Meissner response to the phase $\phi_{ji} = -\phi_{ij} = -\pi/2$ is $\langle j_{ij} \rangle = -2nt_2 \text{Im}(\exp(-i\pi/2)) = 2nt_2$. In the chiral superfluid phase, one rather observes a sign change of the response $\langle j_{ij} \rangle = -nt_2$. This implies that

the quantum phase transition between the two phases is a first-order-phase transition. A thorough analysis has been performed to build the complete phase diagram [375]. The properties of the Mott phase, emerging for one particle per site, are also interesting; the Mott phase exhibits at high energy topological quasiparticle and quasi hole excitations visible in the spectral function. Similar chiral superfluid phases have been predicted on the square lattice [376].

We note that the emergence of chiral currents have been predicted in other Hamiltonians, for example, in doped Mott insulators [377, 378], high-Tc superconductors [379] and other ladder models [380, 381, 382].

3.5. Quantum Link Model in superconducting circuits

Substantial theoretical effort has gone into implementing gauge theories in ultra cold atoms and quantum circuits [383]. Superconducting circuits have also attracted some attention in the context of topological quantum computing [384]. One can provide a useful scheme to think about these gauge fields with potential realizations in superconducting circuits, using loops of Josephson junctions as described in Ref. [385]. One can build a Quantum link model. Let us consider a square lattice and two nearest neighboring sites connected with the phase $\varphi_{ij} = \int_i^j d\mathbf{l} \cdot \mathbf{A}$. The flux variable is then defined as $U_{ij} = \exp(i\varphi_{ij}) \in U(1)$, and the electric field (conjugate) variable connecting this link is then defined as $E_{ij} = -i\partial/\partial\varphi_{ij}$. These variables are operators satisfying the angular momentum type algebra $[E_{ij}, U_{ij}] = U_{ij}$ and $[E_{ij}, U_{ij}^\dagger] = -U_{ij}^\dagger$; U_{ij} and U_{ij}^\dagger then act as lowering and raising operators of the electric field e_{ij} , such that

$$E_{ij}|e_{ij}\rangle = e_{ij}|e_{ij}\rangle. \quad (23)$$

Here, eigenvalues of the electric field are integer or half-integer. One can write down the Hamiltonian, by introducing the flux per plaquette. For example, for a square type geometry, the flux per plaquette reads $U_\square = (U_{ij}U_{jk}U_{kl}U_{li}) = \exp(i\Phi)$ where the magnetic flux per plaquette is $\Phi = \varphi_{ij} + \varphi_{jk} + \varphi_{kl} + \varphi_{li}$. The dynamics of the gauge fields here is described by the Hamiltonian [385] :

$$H = \frac{g^2}{2} \sum_{\langle i;j \rangle} E_{ij}^2 - \frac{1}{4g^2} \sum_{\square} (U_\square + U_\square^\dagger). \quad (24)$$

This Hamiltonian corresponds to a Maxwell-type Hamiltonian on a lattice (the second term in $\cos\Phi$ is related to Josephson physics). Lattice gauge transformations keeping the Hamiltonian invariant have been discussed explicitly in Ref. [385]. One can envision different Quantum link Models. A class of models making links with spin-1/2 Hamiltonians and dimer models is defined as follows. If $e_{ij} = \pm 1/2$, then one can define spin-up and spin-down states through $(E_{ij} = S_{ij}^z, U_{ij} = S_{ij}^-)$ and $(E_{ij} = -S_{ij}^z, U_{ij} = S_{ij}^+)$, where $E_{ij}^2 = (S_{ij}^z)^2 = 1/4$. In this language, the U_\square represent ring-exchange terms $S_{ij}^+ S_{jk}^- S_{kl}^+ S_{li}^-$. In a neutral sub-space of the Hilbert space we have the same number of spin-up and spin-down states. Then, one can also engineer a Rokhsar-Kivelson type interaction between dimers [386]

$$H = -J \sum_{\square} (U_\square + U_\square^\dagger - \lambda(U_\square + U_\square^\dagger)^2). \quad (25)$$

This model has attracted some attention in condensed-matter physics, in relation with confined-deconfined phase transitions. Theoretical questions about gauge theories are also relevant [387]. Generalizations to non-Abelian situations might lead to more complex gauge theories. Some applications of quantum link models to quantum chromodynamics have been suggested [388]. Other gauge theories may have applications for quantum computing and the toric code [389]. In fact, theorists have recently shown that linear circuits can already mimic non-trivial hopping matrix elements, in relation with current technology, allowing potentially to realize non-Abelian gauge fields. In particular, loops of “triangle-like capacitors” have been considered as a potential simulator of a spin-doubled Azbel-Hofstadter [320, 390] model, with 1/3 magnetic flux per plaquette [344]. This model is related to the time-reversal invariant Hofstadter model [391, 392, 393, 394]. Non-abelian models have also been considered in ladder systems [395].

4. Conclusion

We have summarized several recent developments, both theoretical and experimental, to realize many-body physics with photons in superconducting Quantum Electrodynamics (QED) Networks. At the theoretical level, we have outlined analytical and numerical approaches to describe quantitatively these open, dissipative and many-body quantum systems beyond the equilibrium limit. We hope that this review will also serve to inspire novel experimental developments, related for example to the realization of Kondo physics with light in Josephson junction arrays [28, 29] or to the realization of the Jaynes-Cummings lattice [10, 37, 38, 39, 40, 41, 42] or Rabi model [43, 44] with superconducting circuits. Some experimental efforts already exist for a few cavities [178, 266, 267]. We have presented results on the driven Jaynes-Cummings lattice from a stochastic Schrödinger equation approach [32, 33, 34, 35, 36] including dissipation effects in the cavities. The coupling between cavities is treated at a mean-field level. A more rigorous dynamical-mean field theory approach could be, in principle, built [336] by analogy with recent results on dissipative spin chains with long-range forces [195]. We have also extended stochastic approaches for two-spin dimer problems [195], which have attracted some attention both theoretically [268] and experimentally [178]. On the experimental side, hybrid mesoscopic circuits [104, 105, 106, 107, 108] comprising quantum dots and photons will hopefully confirm our theoretical predictions [103]. Regarding the implementation of synthetic gauge fields and spin-orbit couplings in photon systems, experimentalists have already managed to observe topological phases of light [67, 68, 69, 70, 71, 74], in synchronization with the developments in ultra-cold systems [76, 77, 78, 79, 80, 81]. Using the Floquet approach, we have shown that the Haldane model on the honeycomb lattice can be engineered from a perturbation periodic in time. The Haldane model has been recently implemented with ultra-cold fermions [295]. A similar scheme has been considered to simulate a Kane-Mele model for photons [69, 70]. Josephson junction loops can serve to realize artificial gauge fields and break time-reversal symmetry [56, 57, 59, 58, 60]. Applying the Density Matrix Renormalization Group and Exact Diagonalization (on small systems), we have checked that Meissner currents can develop in a Mott insulating phase, in a ladder geometry with synthetic gauge fields and Josephson physics [365, 366, 371]. Such ladder geometries have been built with ultra-cold bosonic atoms [359] based on the theoretical work of Ref. [362]. Experimental accomplishments with lumped elements [342] and small superconducting circuits [93, 94] pave the way towards observing topological phases in superconducting Quantum Electrodynamics (QED) networks. Entanglement properties in such quantum Hall phases could be probed through charge noise [141, 142, 396, 397, 398].

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