

Summary : Quantum dynamics for a fractal spectrum : Thouless coefficient - Thouless formula - quantum dynamics for a Fibonacci potential and for a Cantor spectrum in the tight-binding model

We note K the triadic Cantor set, embedded in $[0, 1]$, and $d\mu$ the associated measure. $d\mu$ verifies :

$$\int_k f(x)d\mu(x) = \frac{1}{2} \int_k f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_k f\left(\frac{x+2}{3}\right) d\mu(x) \quad (1)$$

Return probability

cf. correlation-fun-cantor.pdf

Thouless coefficient

Thouless formula

Fibonacci potential in the tight-binding model

Quantum dynamics for a triadic Cantor set spectrum in the tight-binding model

Therefore, the wave function $\psi(k, t)$, describing the wave packet, is given by :

Thus, the dynamical behaviour of the system depends essentially on the functions : aka the Fourier transforms on K of the moments x^k .

Study of $h_k(t)$

Using the Mellin transform (see annexe), we may express $h_k(t)$ as a series :

It is not clear, from this expression, whether or not $h_k(t)$ has a log-periodic feature, which is expected as a fingerprint of the scaling symmetry of the spectrum expressed in (1). We shall now show that there is, however, such a behaviour. We will do so by calculating the zeroes of $h_k(t)$ and the maximas of $|h_k(t)|$.

First note that $h_0(t)$ verifies the following functional equation, derived by direct application of (1) :

By repeating this process, and using $h_0(0) = \int_K 1d\mu = 1$, we find the exact expression of $h_0(t)$:

This is consistent with the expression derived in [1]. Furthermore, there is a perfect fit between the graph of the theoretical expression (2) and the one obtained from the direct calculation of $h_0(t)$ using : with K_n the set of the left edges of the remaining segments after n iterations in the construction of the Cantor set from the segment $[0, 1]$.

From (2), we easily deduce the zeroes of $h_0(t)$: it is the set :

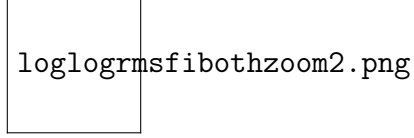


FIGURE 1 – Numerical curves of $h_0(t)$, $h_1(t)$ and $h_{31}(t)$.

We also get the local maxima of $|h_0(t)|$: with the property :

Now, using (1) again, we derive the following recurrence formula for $h_k(t)$:

$$\begin{aligned}
 h_k(t) &= \int_K e^{-i\epsilon t} \epsilon^k d\mu \\
 &= \frac{3^{-k}}{2} \int_K e^{-i\epsilon t/3} \epsilon^k d\mu + \frac{3^{-k} e^{-i2t/3}}{2} \int_K e^{-i\epsilon t/3} (\epsilon + 2)^k d\mu \\
 &= \frac{3^{-k}}{2} h_k(t/3) + \frac{3^{-k}}{2} e^{-i2t/3} \sum_{j=0}^k \binom{k}{j} h_j(t/3) 2^{k-j}
 \end{aligned}$$

We will use this relation to show, by recurrence, that the set Z_k of zeroes of h_k , $k \neq 0$ contains the set :

It seems that there are no other zeroes for $h_k(t)$, e.g. that but it is not rigorously proven ; the numerics however seem to support this hypothesis (see fig. 2).

We therefore deduce that $h_k(t)$ has the same zeroes than $h_{k-1}(t/3)$.

Using a similar reasoning, we can study the points at which the module $|h_k(t)|$ has a local maximum. One can show (see annexe) that these maxima are reached, for $|h_0(t)|$, at points of the form : $t_{m,n} = 3^n m \pi$. It is also easy to show that, for all n, m , one has : $h_0(t_{m,n}) = h_0(t_{m,0})$. Using the recurrence relation between the h_k , one can see that $h_k(t_{m,n}) \approx 3^{-k} h_0(t_{m,n-1})$.

Thus, we may infer, with a good approximation, that :

$$h_k(t) \propto h_{k-1}(t/3)$$

This is confirmed by the numerics.

To summarize, the functions $h_k(t)$ themselves are not log-periodic, and have periodically spaced zeroes of the form $t_{m,n} = \frac{\pi}{2}(2m+1)3^n$. The fingerprint of the fractal spectrum lies in the recurrence relation :

$$h_k(t) = \frac{3^{-k}}{2} h_k(t/3) + \frac{3^{-k}}{2} e^{-i2t/3} \sum_{j=0}^k \binom{k}{j} h_j(t/3) 2^{k-j}$$

which, up to a good approximation, can be simplified in the following renormalization relation :

$$h_k(t) \propto h_{k-1}(t/3)$$

in good approximation (e.g, it is not a rigorous proportionality relation, since the h_k

Application to the RMS and the participation ratio

References

[1] E. HILLE and J. D. TAMARKIN, The American Mathematical Monthly, Vol. 36, No. 5 (May, 1929), pp. 255-264