

Quantum dynamics for a Cantor spectrum, tight-binding model - log-periodic behaviour as a consequence of time-averaging

This note is about the signature of the fractal structure of the spectrum in time-averaged dynamical quantities. The model studied is the same as in the notes of June 1 (tight-binding model and Cantor spectrum).

As showed in the notes of 1 June, the wave function is given by :

$$\begin{aligned}\psi(k, t) &= \int_K \phi_\epsilon(1)^* \phi_\epsilon(k) e^{-i\epsilon t} d\mu \\ &= \int_K |\phi_1(\epsilon)|^2 p_k(\epsilon) e^{-i\epsilon t} d\mu\end{aligned}$$

We define $|\phi_1(\epsilon)|^2 d\mu$ as the local measure associated to ψ ; for simplicity we write now $|\phi_1(\epsilon)|^2 d\mu = d\mu$. Using : $p_k(x) = \sum_{0 \leq j_k \leq k} c_{j_k} x^{j_k}$, we have :

$$\psi(k, t) = \sum_{j_k \leq k} c_{j_k} \int_K \epsilon^{j_k} e^{-i\epsilon t} d\mu$$

Since the RMS (and many other dynamical quantities) can be expressed in terms of the amplitude of ψ at the sites of the lattice, namely $|\psi(k, t)|^2$, the dynamics of the system depends essentially on the cumulants :

$$h_k(t) = \int_K e^{-i\epsilon t} \epsilon^k d\mu(\epsilon)$$

The $h_k(t)$ are not scaling functions, and therefore neither is the RMS. However, we showed that there is a structure and some invariant properties, with dynamical implications which were discussed in the notes of June 1.

We discuss here the influence of time-averaging : instead of studying instantaneous dynamical quantities, we are interested in their time average, given by the transformation $\frac{1}{t} \int_0^t$. Thus, we study $\frac{1}{t} \int_0^t |\psi(k, t')|^2 dt'$, which narrows down to the study of the quantities :

$$\frac{1}{t} \int_0^t h_k^*(t') h_l(t') dt'$$

Since $|\psi(k, t)|^2$ is real, we need only to consider the real part of $\frac{1}{t} \int_0^t h_k^*(t') h_l(t') dt'$, which we note $f_{k,l}(t)$. A standard calculation leads to :

$$\begin{aligned}f_{k,l}(t) &= \Re \left(\frac{1}{t} \int_0^t h_k^*(t') h_l(t') dt' \right) \\ &= \iint_{K \times K} x^k x^l \text{sinc}((x' - x)t) d\mu(x) d\mu(x')\end{aligned}$$

Note that $f_{0,0}(t)$ is the time-averaged return probability $C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$, which we calculated exactly and which we know to have the following asymptotic expression :

$$C(t) \sim_{t \rightarrow \infty} \frac{t^{-d_K}}{\ln(3)} \sum_n \gamma_n t^{-\frac{2i\pi n}{\ln(3)}} \sin \left(\frac{\pi}{2} (d_{K,n} - 1) \right) \Gamma(d_{K,n} - 1)$$

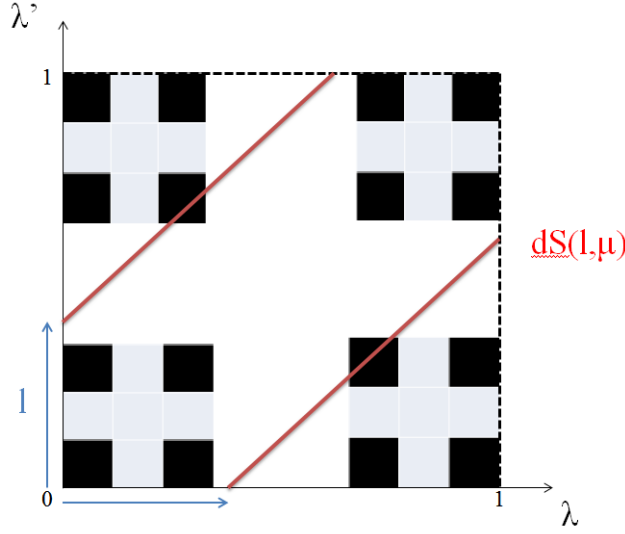


FIGURE 1 – 2 dimensional Cantor dust (represented here at the second step of construction); the infinitesimal surface $dS(l, \mu)$ corresponds to the surface of the Cantor dust, in the sense of $d\mu$, contained in the two red strips.

with $d_K = \frac{\ln 2}{\ln 3}$. The asymptotic behaviour of $C(t)$ can be intuited by the following observation : using the scaling property of the measure, we find that :

$$C(t) = \frac{1}{2}C(t/3) + \frac{1}{2} \iint_{K \times K} \text{sinc}\left(\frac{x - x' + 2}{3}t\right) d\mu(x) d\mu(x')$$

When $t \rightarrow \infty$, the second term of the sum on the right tends to zero, since for all x, x' in K , $1 < x - x' + 2 < 3$. Thus, $C(t)$ is asymptotically a scaling function :

$$C(t) \sim_{t \rightarrow \infty} \frac{1}{2}C(t/3)$$

This is due to the *sinc* in the expression of $C(t)$: $C(t)$ is a wavelet transform of the spectrum. For the same reason, we expect $\iint x^k x'^l \text{sinc}((x' - x)t) d\mu(x) d\mu(x')$ to be, asymptotically, a scaling function.

For the calculation of $C(t)$, we used the change of variable $l = |x - x'|$ and reduced the problem to calculating the area $dS(l, \mu)$ (see fig. 1). The log-periodic behaviour comes from the expression of $dS(l, \mu)$.

If we make the approximation that : $\text{sinc}(xt) = 1$ for $|xt| < \pi$ and $\text{sinc}(xt) = 0$ elsewhere, we get the following approximate expression for $C(t)$:

$$C(t) = \int_{l=0}^{\pi/t} dS(l, \mu) = t^{-\ln 2 / \ln 3} \sum_n d_n t^{-2i\pi n / \ln 3}$$

which accounts for the typical asymptotic behaviour, although the coefficients d_n are a little bit different from the coefficients in the exact expression.

If we apply the same reasoning for $f_{k,l}(t) = \iint x^k x'^l \text{sinc}((x' - x)t) d\mu(x) d\mu(x')$, we find :

$$f_{k,l}(t) \approx \iint_{|x-x'| < \frac{\pi}{t}} x^k x'^l d\mu(x) d\mu(x')$$

Using a similar method as for $C(t)$, we can show that (see detail in the next section) :

$$\iint_{|x-x'| < \frac{\pi}{t}} x^k x^l d\mu(x) d\mu(x') = t^{-\ln 2 / \ln 3} \left[G(\ln t) + \mathcal{O}\left(\frac{1}{t}\right) \right]$$

See a graph of $f_{1,2}(t)$ fig. 2.

Note that this implies that the wave function has the following asymptotic behaviour : $\frac{1}{t} \int_0^t |\psi(k, t')|^2 dt' \sim t^{-\ln 2 / \ln 3}$, which is consistent with the results of Guarneri [1].

Now, since

$$f_{k,l}(t) = \iint x^k x^l \text{sinc}((x' - x)t) d\mu(x) d\mu(x') \sim_{t \rightarrow \infty} t^{-\ln 2 / \ln 3} G(\ln t / \ln 3)$$

we deduce, adapting the argument of Guarneri in [1], the following lower-bound for the time-averaged RMS :

$$\frac{1}{t} \int_0^t \Delta x(t') dt' \geq t^{\frac{\ln 2}{\ln 3}}$$

which is consistent with the numerics (fig. 3).

Calculation of $f_{k,l}(t)$

We give the analytical expression of the approximated function for $f_{k,l}(t)$, which we call

$$r_{k,l}(t) = \iint_{|x-x'| < \frac{\pi}{t}} x^k x^l d\mu(x) d\mu(x')$$

We show that :

$$r_{k,l}(t) = t^{-\ln 2 / \ln 3} \left[G(\ln t) + \mathcal{O}\left(\frac{1}{t}\right) \right]$$

We follow a similar method as for $C(t)$.

First, we calculate the infinitesimal area $dS_{k,l}(u, \mu)$, which is the integral of $x^k y^l$, for the measure $d\mu$, on the two strips $y = x + u$ and $y = x - u$ (red strips on fig. 1) :

$$dS_{k,l}(u, \mu) = \int_K (x^k (x + u)^l \theta(1 - u - x) + x^k (x - u)^l \theta(x - u)) d\mu(x)$$

To derive $dS_{k,l}(u, \mu)$, we study the poles of its Mellin transform and apply the residue theorem. The Mellin transform is :

$$m_{k,l}(s) = \int_0^1 u^{s-1} dS_{k,l}(u, \mu) = \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu$$

We now show by recurrence that the poles of $m_{k,l}(s)$ is the set :

$$S_{k,l} = \left\{ 1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3}; j \in [0, k + l], n \in \mathbb{Z} \right\}$$

Initialization :

$m_{0,0}(s) = \iint_{K \times K} |x - y|^{s-1}$ has been calculated for $C(t)$ and has poles at $S_{0,0} =$

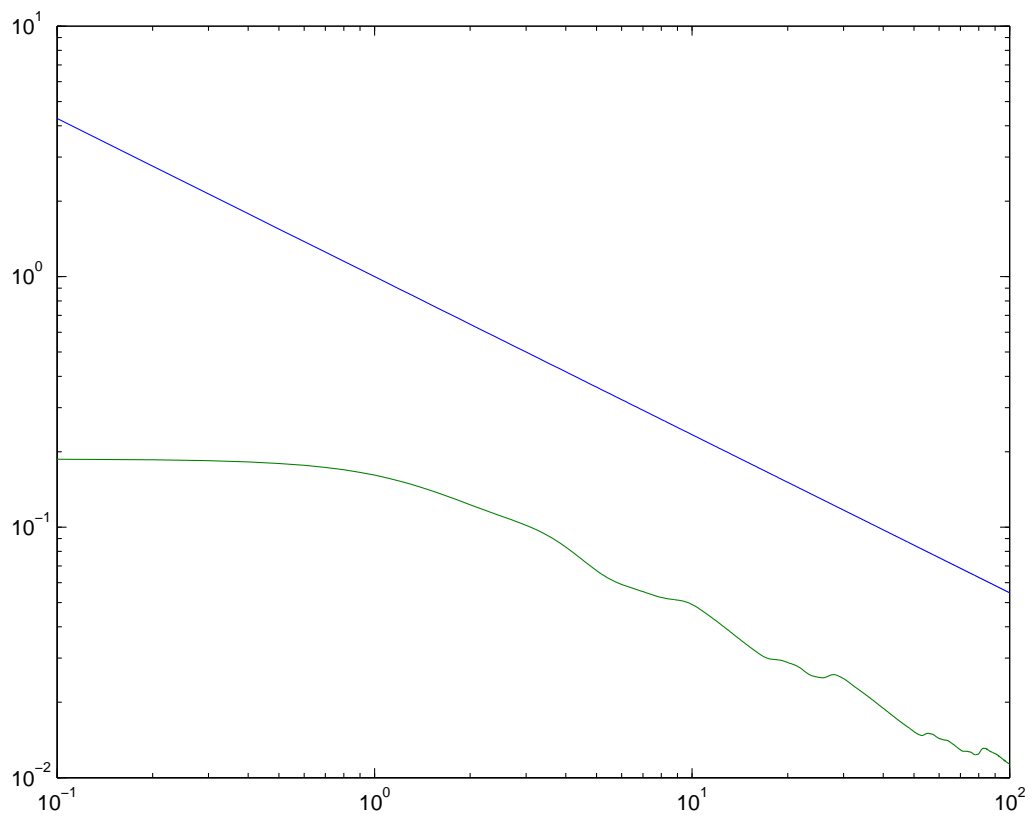


FIGURE 2 – Loglog scale graphs of $f_{1,2}(t)$ (green) and $t^{-\ln 2 / \ln 3}$ (blue)

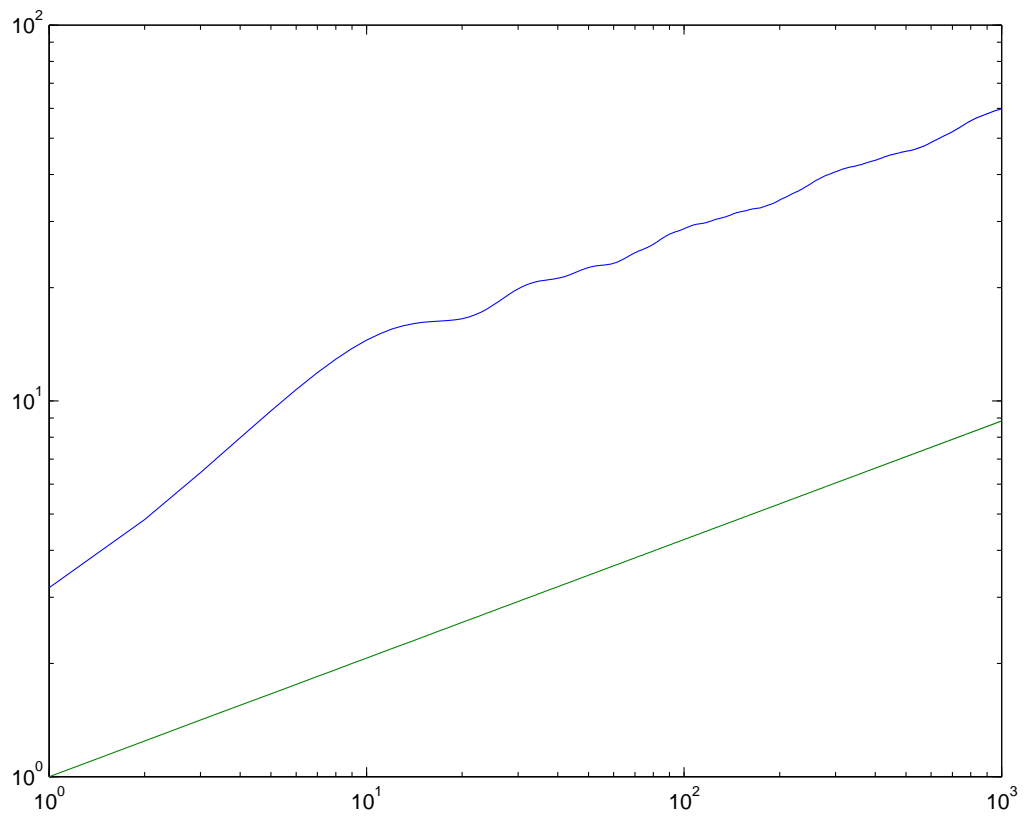


FIGURE 3 – Loglog scale graphs of $\frac{1}{t} \int_0^t \Delta x(t') dt'$ (blue) and $t^{\frac{\ln 2}{2 \ln 3}}$ (green)

$$\left\{1 - \frac{\ln 2}{\ln 3} + \frac{2i\pi n}{\ln 3}; n \in \mathbb{Z}\right\}.$$

Inductive step :
using the symmetry of the measure, we find :

$$\begin{aligned} m_{k,l}(s) &= \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu \\ &= \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu \\ &\quad + \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} (x+2)^k (y+2)^l |x - y|^{s-1} d\mu d\mu + R_{k,l}(s) \end{aligned}$$

with

$$R_{k,l}(s) = \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} (x+2)^k y^l |x-y+2|^{s-1} d\mu d\mu + \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} x^k (y+2)^l |x-y-2|^{s-1} d\mu d\mu$$

has no poles. Expanding the second term :

$$\begin{aligned} m_{k,l}(s) &= \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu \\ &\quad + \frac{3^{1-s-(k+l)}}{4} \sum_{i=0}^k \sum_{j=0}^l \binom{k}{i} \binom{l}{j} 2^{k+l-i-j} \iint_{K \times K} x^i y^j |x - y|^{s-1} d\mu d\mu + R_{k,l}(s) \\ &= \frac{3^{1-s-(k+l)}}{2} m_{k,l}(s) + \frac{3^{1-s-(k+l)}}{4} \sum_{\substack{i=1 \\ (i,j) \neq (k,l)}}^k \sum_{j=1}^l \binom{k}{i} \binom{l}{j} 2^{k+l-i-j} m_{i,j}(s) + R_{k,l}(s) \end{aligned}$$

Thus, the poles of $m_{k,l}(s)$ is the set of the poles of all the $m_{i,j}(s)$ with $i \leq k$ and $j \leq l$, plus the set of complex points $\left\{1 - \frac{\ln 2}{\ln 3} - (k+l) + \frac{2i\pi n}{\ln 3}; n \in \mathbb{Z}\right\}$. Using the induction hypothesis, we conclude that the poles of $m_{k,l}(s)$ is the set :

$$S_{k,l} = \left\{1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3}; j \in [0, k+l], n \in \mathbb{Z}\right\}$$

which completes the proof.

Now, we apply the inverse Mellin transform and use the residue formula to find $dS_{k,l}(u, \mu)$:

$$\begin{aligned} dS_{k,l}(u, \mu) &= \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} u^{s-1} m_{k,l}(s) ds \\ &= \sum_{j=0}^{k+l} \sum_{n \in \mathbb{Z}} c_{j,n} u^{j+\alpha-1-2i\pi n/\ln 3} \end{aligned}$$

with $\alpha = \frac{\ln 2}{\ln 3}$ and

$$c_{j,n} = R_{k,l} \left(1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3}\right) + \sum_{\substack{i=1 \\ i+i' \neq j}}^k \sum_{i'=1}^l \binom{k}{i} \binom{l}{i'} 2^{k+l-i-j} m_{i,i'} \left(1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3}\right)$$

By integrating, we obtain $r_{k,l}(t)$:

$$\begin{aligned}
r_{k,l}(t) &= \int_0^{1/t} dS(u, \mu) \\
&= \sum_{j=0}^{k+l} \sum_{n \in \mathbb{Z}} c_{j,n} \int_0^{1/t} u^{j+\alpha-1-2i\pi n/\ln 3} \\
&= t^{-\alpha} \sum_{j=0}^{k+l} \sum_{n \in \mathbb{Z}} c_{j,n} \frac{t^{-j+2i\pi n/\ln 3}}{\alpha + j - 2i\pi n/\ln 3}
\end{aligned}$$

Thus :

$$r_{k,l}(t) = t^{-\alpha} \left[G(\ln t) + \mathcal{O}\left(\frac{1}{t}\right) \right]$$

with $G(\ln t) = \sum_{n \in \mathbb{Z}} \frac{t^{2i\pi n/\ln 3}}{\alpha - 2i\pi n/\ln 3} c_{0,n}$.

References

[1] Guarneri, "Spectral Properties of Quantum Diffusion on Discrete Lattices", Eurquhys. Lett., 10 (2), pp. 95-100 (1989)