

Quantum dynamics for a fractal spectrum - Applications to simple processes in electrodynamics and quantum optics

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Introduction

During the past decades, there has been a growing interest for Schroedinger operators $H = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})$ with an associated fractal spectrum. In fact, while the dynamics and the transport properties of quantum systems having a continuous or discrete spectrum are well understood, little is known about singular continuous spectra (like fractals), which are ubiquitous (quasi-crystals, quasi-periodic potentials, some glasses).

Some recent experiments [1], pursued at the Laboratoire de Photonique et Nanostructure (CNRS, Marcoussis, France) in partnership with my group at the Technion, on cavity polaritons placed in a quasi-periodic potential (which we will now refer to as "Fibonacci potential") with an associated Cantor like spectrum, have evidenced a log-periodic behaviour for some dynamical quantities of the system. Although expected theoretically in the case of a fractal spectrum, such log-periodic oscillations had never been observed experimentally before.

Objectives

Our goal is to understand, for various cases, how the fractal feature of the spectrum affects the dynamics and the propagation properties of the system, to determine which relevant physical quantities carry its signature, and to derive the analytical expression of those quantities. This study should then allow us to conduct further experiments. My initial project contains three main topics, but so far I have focused on two of them :

- 1 - Analytical and numerical study of a wave packet for a triadic Cantor spectrum ; this study was initiated by the experimental results obtained with the cavity polaritons, and was the subject of my master's research internship pursued between April and July 2014.
- 2 - Spontaneous emission and superradiance for atoms coupled to a fractal spectrum.

First results

Dynamics of a wave packet for a triadic Cantor spectrum

Consider a 1D quantum system, described by a Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R})$, with associated triadic Cantor set spectrum K . The density of states is degenerate, but can be described by an appropriate measure $d\mu$ on K such that, for any function f defined on K , one has :

$$\int_K f(x)d\mu(x) = \frac{1}{2} \int_K f\left(\frac{x}{3}\right)d\mu(x) + \frac{1}{2} \int_K f\left(\frac{x+2}{3}\right)d\mu(x)$$

This scaling property is very important and has deep consequences in the dynamics of the system, as we saw it through the numerical, experimental and theoretical study of a few relevant physical quantities : the return probability, the root mean square displacement and the participation ratio. It also explains the unusual behaviour of the Thouless coefficient associated to the system.

Return probability

As suggested by a numerical study for the Fibonacci potential, the time-averaged return probability

$$C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$$

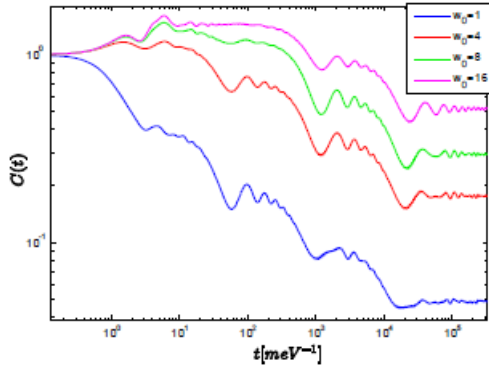


FIGURE 1 – Time-averaged return probability of a wave packet in a Fibonacci potential.

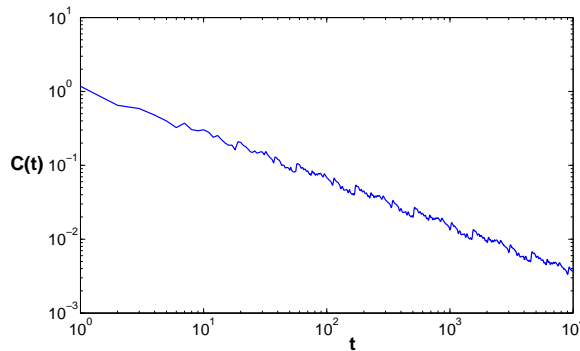


FIGURE 2 – Time-averaged return probability for a triadic Cantor spectrum.

displays a log-periodic feature. The derivation of the analytical expression of $C(t)$ was part of my master's internship project, but it has been completed since and a numerical analysis was pursued. We find that $C(t)$ has a power law behaviour modulated by log-periodic oscillations :

$$C(t) = \frac{t^{-d}}{\ln(3)} \sum_n \epsilon_n t^{-\frac{2i\pi n}{\ln(3)}} \sin\left(\frac{\pi}{2}(d_n - 1)\right) \Gamma(d_n - 1)$$

with $d = \frac{\ln(2)}{\ln(3)}$ (the fractal dimension of the triadic Cantor set), $d_n = d + \frac{2i\pi n}{\ln 3}$, $\epsilon_n = \int_{K \times K} \frac{d\mu(x)d\mu(y)}{|x-y+2|^{d + \frac{2i\pi n}{\ln(3)}}}$

and Γ the Euler gamma function.

Furthermore, the method used to derive this expression shows that such a behaviour (power law and log-periodic oscillation) is in fact the signature of a fractal feature, and should be observed in other dynamical quantities. In fig. 1 and 2, we compare the return probability for the Fibonacci cavity and for the triadic Cantor set. The two curves display the same behaviour. The differences between these graphs come from two facts : first, in the case of the Fibonacci potential, $C(t)$ was calculated numerically whereas the expression is exact for the triadic Cantor set ; second, the spectrum associated to the Fibonacci potential is in fact a superposition of Cantor sets, not a pure triadic Cantor set.

Root mean displacement (RMS)

A numerical study of the Fibonacci potential also indicates that the RMS carries the signature of the fractal feature of the spectrum. However, the methods used for the return probability did not apply well for the study of the RMS. To understand its behaviour, it was useful to make a connection with dynamical systems and iterated function systems (IFS) ([2]). We use the tight-binding formalism : take a 1D lattice of N sites, and consider the following discrete Schroedinger

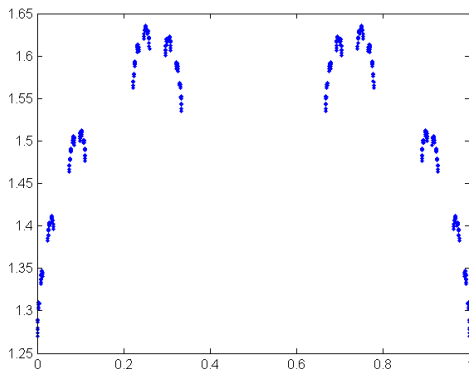


FIGURE 3 – Thouless coefficient for a triadic Cantor spectrum.

equation :

$$-\phi_n(k+1) - \phi_n(k-1) + 2\phi_n(k) + V(k)\phi_n(k) = E_n\phi_n(k)$$

with $\phi_n = (\phi_n(0), \phi_n(1), \dots, \phi_n(N))$ an eigenfunction associated to the eigenvalue E_n . One can relate $\phi_n(k)$ to a set of orthogonal polynomials with respect to the measure μ :

$$\int_K p_m(x)p_n(x)d\mu(x) = \delta_{n,m}$$

The study of the polynomials led to the conclusion that the RMS too has a power law behaviour modulated by log-periodic oscillations. A numerical study is now in process.

Thouless coefficient

The Thouless coefficient (see [3],[4]) associated to the system is defined by :

$$\gamma(E) = \int_{\text{spectrum}} \ln |y - E| d\mu(y)$$

for $E \in \text{spectrum}$ In the case of a Cantor spectrum, we find that $\gamma(E)$ displays a self-similarity, which translates the presence of a self-similar fractal spectrum (see fig. 3), and verifies the following functional equation :

$$\gamma(x) = -\ln 3 + \frac{1}{2}\gamma(3x) + \frac{1}{2}\gamma(3x+2)$$

We studied more deeply the connection between $\gamma(E)$ and the dynamics of the system in the case of the tight-binding formalism.

Following [5], we established the following formula :

$$\frac{1}{N} \ln |\phi_n(1)\phi_n(N)| = -\frac{1}{N} \sum_{k \neq n} \ln |E_k - E_n| \approx \gamma(E_n)$$

We now wish to obtain a generalization of the Thouless formula and use it to obtain further information about some dynamical quantities such as the RMS.

Spontaneous emission and superradiance

We consider here a system of one or two two-level atoms coupled to a fractal spectrum, and are interested in two phenomena : spontaneous emission (1 atom) and superradiance (two atoms). Previous work led by my group [6] showed that, in the case of a fractal spectrum, the decay probability does not follow a Wigner-Weisskopf exponential decrease but has a power law behaviour, modulated by log-periodic oscillations. We therefore suspect to observe some anomalous behaviour for the case of two atoms as well.

Perspectives

Regarding the quantum dynamics for a Cantor spectrum, the connection with dynamical systems and IFS formalism opened new perspectives and should give further information on the RMS and the participation ratio (I did not give further details on the later since we only have numerical results so far). Furthermore, we will explore more deeply how the Thouless formula could be connected to those relevant dynamical quantities. The results which this approach will lead to should be added to an ongoing paper on quantum dynamics for a Cantor spectrum, so far focusing on the return probability.

We also wish to apply the methods developed to the study of superradiance. In fact, the techniques used to study the dynamics of a wave packet for a fractal spectrum rely essentially on the scaling symmetry of the spectrum, and should easily be translated to study analytically the resolvent of any Hamiltonian, which, in the case of spontaneous emission and superradiance, gives precious informations on the evolution of the system.

References

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