

Derivation of the auto-correlation function for a triadic Cantor set spectrum

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We derive the analytical expression of the time-averaged correlation function of a quantum system in the case of a triadic Cantor set energy spectrum. We generalize the method used and discuss the connection between the presence of scaling symmetries in physical systems and the typical log-periodic behaviour of some quantities. As examples, we discuss the case of the participation ratio and of the RMS displacement for a triadic Cantor set spectrum, as well as the spontaneous emission for a two level atom coupled to a self-similar fractal spectrum. We also present some numerical results obtained recently for polaritons in a Fibonacci potential with an associated Cantor like spectrum, which support the theoretical results.

I. INTRODUCTION

During the past decades, there has been a growing interest for Schroedinger operators $H = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})$ with a Cantor set like spectrum, e.g. a perfect, nowhere dense set (for a review, see [1], and more recently [2]). In the context of quantum dynamics for systems verifying such a Hamiltonian, a typical behaviour for some physical quantities, such as the auto-correlation function (time-averaged return probability), has been identified. More precisely, it has been shown ([3],[4]) that, in the case of a Cantor set spectrum, the auto-correlation function $C(t)$ decreases as : $t^{-D_2^\mu}$, where D_2^μ is a correlation dimension with respect to the measure μ of the spectrum, although a full analytical expression had not been derived so far. A slightly different quantity, the spatial correlation function $C_s(l, \mu) = \int_{K \times K} \theta(l - |x - y|) d\mu(x) d\mu(y)$, with θ the

Heaviside function, K the spectrum and $d\mu$ the measure associated to the K , has been derived more precisely for a Cantor set spectrum ([5]), showing that this quantity is a power law t^{d_K} (d_K being the fractal dimension of the Cantor set) modulated by log-periodic oscillations.

We recently studied numerically the auto-correlation function for a gas of cavity polaritons placed in a Fibonacci potential and with a Cantor set like energy spectrum [6]; we found in this case that $C(t)$ has a log-periodic behaviour.

In this paper, we study the dynamics of a quantum system whose energy spectrum is described by a self-similar measure μ , e.g. such that there exists a set of contractive similarities $\{\phi_j\}$ and probability weights $\{\pi_j\}$ such that for any continuous function f , one has :

$$\int f(x) d\mu(x) = \sum_j \pi_j \int f(\phi_j(x)) d\mu(x)$$

We shall focus on the case of a set of two linear similarities, with $\pi_1 = \pi_2 = \frac{1}{2}$:

$$\int f(x) d\mu(x) = \frac{1}{2} \int f(a_1 x) + \frac{1}{2} \int f(a_2 x + b)$$

Cantor sets are examples of such spectra, and occur in many cases. We will therefore analyse in detail the case

of a triadic Cantor set ; we shall derive the exact analytical expression of the auto-correlation function $C(t)$, and show that it is of the form :

$$C(t) = \left(\frac{t}{\tau}\right)^{-d_K} g(\ln(t/\tau))$$

where $d_K = \frac{\ln(2)}{\ln(3)}$ is the fractal dimension [7] of the triadic Cantor set, τ a time scale and g a periodic function.

We first give a detailed calculation of $C(t)$, then discuss its behaviour and the similarity with numerical results obtained with the Fibonacci cavity. Furthermore, we will generalize the method, which is in fact appropriate for the study of systems described by a self-similar measure (fractals), and gives an intuition of how the fractal properties translate through these quantities or functions. We will finally discuss a few examples in quantum dynamics.

II.

A triadic Cantor set is constructed as follows : start with a segment $[0, \epsilon_{max}]$, divide it in three equal parts, and remove the central one, e.g. the segment $[\epsilon_{max}/3, 2\epsilon_{max}/3]$, then repeat the process with the two remaining segments, and so on. After an infinite number of iterations, we are left with a set of separated points, of zero Lebesgue measure.

In this paper, K will denote a triadic Cantor set constructed from $[0, 1]$.

We now consider a one dimensional quantum system described by the state vector $|\psi(t)\rangle$ and evolving with the Hamiltonian $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$. We assume that the energy spectrum of H is a triadic Cantor set. Note that, since all Cantor sets are homeomorphic to the triadic Cantor set ([17]), the results and methods can be adapted to any Cantor set spectrum.

The system is initially in a state described by $|\psi(t=0)\rangle = |\psi_0\rangle$. After a time t , the state vector $|\psi(t)\rangle$ is given by :

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi_0\rangle$$

We shall also use the spectral decomposition of $|\psi(t)\rangle$ over an orthonormal basis of eigenfunctions $\{|\phi(\epsilon)\rangle\}$ of H , $|\phi(\epsilon)\rangle$ being the eigenfunction associated with the energy ϵ :

$$|\psi(t)\rangle = \int_{\text{spectrum}} g(\epsilon) e^{-i\epsilon t/\hbar} |\phi(\epsilon)\rangle d\tilde{\mu}(\epsilon)$$

where $g(\epsilon)$ is the projection of $|\psi_0\rangle$ on $|\phi(\epsilon)\rangle$, and $d\tilde{\mu}$ a measure defined on the spectrum.

To go further, we need to define an appropriate measure on the spectrum. The triadic Cantor set K has a zero Lebesgue measure. However, in our case, the integrated density of states, e.g. the integral of $\rho(\epsilon)$ over the spectrum K (we shall generalize to the case of an infinite Cantor set later), is non zero. Thus, $\rho(\epsilon)$ must be infinite in every point of K .

We define a measure $d\mu$ on K , such that the integrated density of states $\mathcal{N}(\epsilon)$ is given by :

$$\mathcal{N}(\epsilon) = \int_K \theta(\epsilon - \epsilon') d\mu(\epsilon')$$

with θ the Heaviside function. We furthermore suppose that the density of states is uniform.

Without the loss of generality, we suppose that the integral of $d\mu$ over K equals 1 : $\int_K d\mu = 1$. We then proceed iteratively : starting from a segment $[0, 1]$, we set $\rho(\epsilon) = 1$ everywhere ; the integral thus equals 1. Now, divide the segment in three, remove the central part and define $\rho(\epsilon)$ to be equal to $\frac{3}{2}$ on $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$, and null on $[\frac{1}{3}, \frac{2}{3}]$. We still have : $\int_0^1 \rho(x) dx = 1$. We repeat this process infinitely, and thus define $d\mu$ as a limit : for any function f defined on $[0, 1]$, the integral of f over K is :

$$\int_K f(x) d\mu(x) = \lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n \sum_{a_{j_n} \in P_n} \int_{a_{j_n}}^{a_{j_n} + 3^{-n}} f(x) dx$$

where $\{P_n\}$ is the set of the left edges of the remaining segments after n iterations in the construction of the Cantor set.

One can show (see annexe) that $d\mu$ verifies the following important property :

$$\int_K f(x) d\mu(x) = \frac{1}{2} \int_K f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_K f\left(\frac{x+2}{3}\right) d\mu(x) \quad (1)$$

This property still holds for multiple integrals ; it becomes, for a double integral :

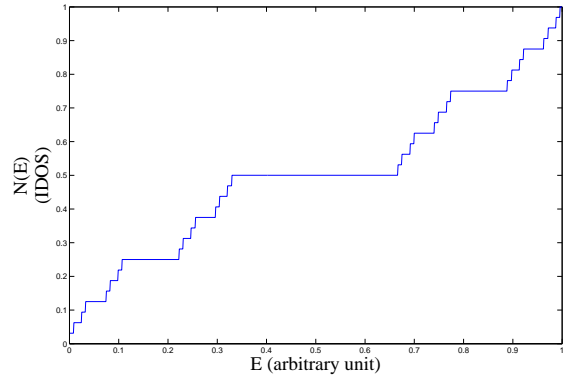


FIG. 1: Integrated density of states (IDOS) in the case of a triadic Cantor set energy spectrum. The energies are expressed in arbitrary unit and the IDOS has been normalised.

$$\begin{aligned} & \int_{K \times K} f(x, y) d\mu(x) d\mu(y) \\ &= \frac{1}{4} \int_{K \times K} f\left(\frac{x}{3}, \frac{y}{3}\right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4} \int_{K \times K} f\left(\frac{x+2}{3}, \frac{y}{3}\right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4} \int_{K \times K} f\left(\frac{x}{3}, \frac{y+2}{3}\right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4} \int_{K \times K} f\left(\frac{x+2}{3}, \frac{y+2}{3}\right) d\mu(x) d\mu(y) \end{aligned}$$

III.

Let us now study the dynamics of the system described in the previous paragraph. There are several physical quantities which are useful to characterize the evolution of the system, such as the RMS displacement, the participation ratio or the auto-correlation function.

We will focus here on the auto-correlation function, then discuss briefly the participation ratio and the RMS displacement.

The auto-correlation function, or time averaged return probability, $C(t)$, is by definition the time average of the probability $p(t) = |\langle \psi(0) | \psi(t) \rangle|^2$ to find the system in its initial state after a time t :

$$C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$$

We could directly try to calculate $p(t)$ to study the evolution in time of the system ; however this quantity of-

ten exhibits fast local oscillations, which could hide the effects of the fractal feature of the spectrum. Its time average, on the other hand, yields interesting results, as we shall see.

We start with the spectral decomposition of $|\psi(t)\rangle$:

$$|\psi(t)\rangle = \int_{\text{spectrum}} g(\epsilon) e^{-i\epsilon t/\hbar} |\phi(\epsilon)\rangle d\tilde{\mu}(\epsilon)$$

The first step is to obtain an integral over K , with a dimensionless variable, in order to use the measure introduced earlier.

We impose that $|\psi(t)\rangle$ is normalized to unity :

$$\langle \psi(t) | \psi(t) \rangle = \int_{\text{spectrum}} |g(\epsilon)|^2 d\tilde{\mu}(\epsilon) = 1$$

This implies that $|g(\epsilon)|^2 \rightarrow_{\epsilon \rightarrow \infty} 0$. We will therefore assume that $|g(\epsilon)|^2$ is non-zero only for a finite number of triadic Cantor sets, and that it is constant on these sets. By construction, a finite union of triadic Cantor sets is also a triadic Cantor set embedded in a larger segment, say of - Lebesgue - measure ϵ_{max} , which without loss of generality we will assume to originate at zero. Let us note $K_{\epsilon_{max}}$ this new set, obtained from K by applying a dilatation of factor ϵ_{max} . We can make the change of variable : $\lambda = \epsilon/\epsilon_{max}$ with $\lambda \in K$ and define a new measure :

$$d\tilde{\mu}_{\epsilon_{max}}(\lambda) = |g(\epsilon)|^2 d\tilde{\mu}(\lambda \epsilon_{max})$$

Assuming that $|g(\epsilon)|$ is uniform, we can consider that $d\tilde{\mu}_{\epsilon_{max}}(\lambda)$ is the measure $d\mu$ introduced in the previous paragraph. To alleviate the notations, we shall now write : $d\tilde{\mu}_{\epsilon_{max}} = d\mu$.

Thus :

$$\begin{aligned} C(t) &= \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt' \\ &= \frac{1}{t} \int_0^t \left| \int_{K_{\epsilon_{max}}} |g(\epsilon)|^2 e^{-i\epsilon t'/\hbar} d\mu_{\epsilon_{max}}(\epsilon) \right|^2 dt' \end{aligned}$$

A standard calculation then leads to :

$$C(t) = \int_{K \times K} \text{sinc}((\lambda' - \lambda)\epsilon_{max}t/\hbar) d\mu(\lambda) d\mu(\lambda')$$

We now make the change of variables : $u = \frac{\epsilon_{max}t}{\hbar}$:

$$C(t) = C(u) = \int_{K \times K} \text{sinc}((\lambda' - \lambda)u) d\mu(\lambda) d\mu(\lambda')$$

The idea is now to transform this double integral into a one variable integral, and then to use Mellin transform and the scaling property (1) of $d\mu$.

Let us do the change of variables : $l = |\lambda' - \lambda|$, $0 \leq l \leq 1$, and let $dS(l, \mu)$ be the surface (in the sense of $d\mu$) of the 2 dimensional Cantor dust¹ contained in the two strips of infinitesimal width located at $\lambda = \lambda' + l$ and $\lambda = \lambda' - l$ (fig. 1).

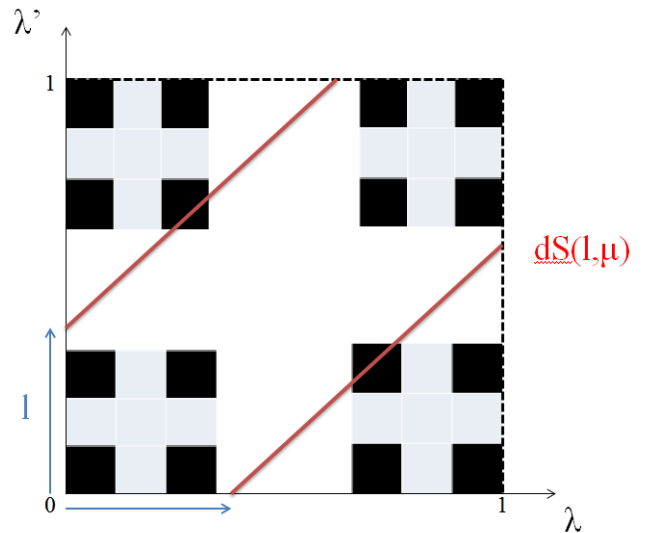


FIG. 2: 2 dimensional Cantor dust (represented here at the second step of construction) ; the infinitesimal surface $dS(l, \mu)$ corresponds to the surface of the Cantor dust, in the sense of $d\mu$, contained in the two red strips.

Since $\text{sinc}((\lambda' - \lambda)t)$ can be considered constant on these strips, we obtain :

$$C(u) = \int_0^1 \text{sinc}(lu) dS(l, \mu) \quad (2)$$

We now have to determine $dS(l, \mu)$. For this we shall use Mellin transforms.

Let : $M_{dS}(s) = \int_0^1 l^{s-1} dS(l, \mu) \equiv \int_0^1 l^{s-1} \frac{dS(l, \mu)}{dl} dl$ be the Mellin transform of $dS(l, \mu)$, which is sometimes referred to in the literature as the energy integral ([5], [8]) (note that we wrote \int_0^1 and not \int_0^∞ : $dS(l, \mu)$ is defined on $[0, 1]$, but can be extended on $[0, \infty[$ if we define it as equal to 0 outside $[0, 1]$).

Now, using the definition of $dS(l, \mu)$ and the property (1) :

$$\begin{aligned} M_{dS}(s) &= \int_0^1 l^{s-1} dS(l, \mu) \\ &= \int_{K \times K} |x - y|^{s-1} d\mu(x) d\mu(y) \\ &= \frac{1}{4} \int_{K \times K} \left(\left| \frac{x}{3} - \frac{y}{3} \right|^{s-1} + \left| \frac{x+2}{3} - \frac{y+2}{3} \right|^{s-1} \right) d\mu(x) d\mu(y) \\ &\quad + \frac{1}{4} \int_{K \times K} \left(\left| \frac{x}{3} - \frac{y+2}{3} \right|^{s-1} + \left| \frac{x+2}{3} - \frac{y}{3} \right|^{s-1} \right) d\mu(x) d\mu(y) \end{aligned}$$

$$M_{dS}(s) = \frac{3^{1-s}}{2} \int_{K \times K} |x-y|^{s-1} d\mu(x) d\mu(y) + \underbrace{\frac{3^{1-s}}{2} \int_{K \times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)}_{\gamma(s)}$$

Thus :

$$M_{dS}(s) = \frac{\gamma(s)}{1 - 3^{1-s}/2}$$

with $\gamma(s) = \frac{3^{1-s}}{2} \int_{K \times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)$.

The inverse Mellin transform then gives $\frac{dS(l, \mu)}{dl}$:

$$\begin{aligned} \frac{dS(l, \mu)}{dl} &= \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} M_{dS}(s) l^{-s} ds \\ &= \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\gamma(s)}{1 - \frac{3^{1-s}}{2}} l^{-s} ds \end{aligned}$$

It is important to note that $\gamma(s)$ has no poles. It is clear from its definition :

$$\gamma(s) = \frac{3^{1-s}}{2} \int_{K \times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)$$

Since for all $x, y \in [0, 1]$, the inequality $1 \leq 2-x+y \leq 3$ holds, $|x-y+2|^{s-1}$ is well defined and bounded on $K \times K$ for any $s \in \mathbb{C}$, and thus $\gamma(s)$ is well defined and has no poles. The only poles of $\frac{\gamma(s)}{1 - \frac{3^{1-s}}{2}} l^{-s}$ are the $\{s_n\}_{n \in \mathbb{Z}}$ such that : $1 - s_n = \frac{\ln(2)}{\ln(3)} + \frac{2i\pi n}{\ln(3)}$. We now apply the residue theorem :

$$\begin{aligned} \frac{dS(l, \mu)}{dl} &= \sum_n \left[\frac{s - s_n}{1 - \frac{3^{1-s}}{2}} l^{-s} \gamma(s) \right]_{s=s_n} \\ &= \frac{l^{d_K-1}}{\ln(3)} \sum_n l^{\frac{2i\pi n}{\ln(3)}} \underbrace{\gamma \left(1 - d_K - \frac{2i\pi n}{\ln(3)} \right)}_{\gamma_n} \end{aligned}$$

where we used :

$$\lim_{s \rightarrow s_n} \frac{s - s_n}{1 - \frac{3^{1-s}}{2}} = \frac{1}{\ln(3)}$$

and with $d_K = \frac{\ln(2)}{\ln(3)}$ and

$$\gamma_n = \int_{K \times K} |x-y+2|^{-d_K - \frac{2i\pi n}{\ln(3)}} d\mu(x) d\mu(y)$$

Inserting this expression in (2), we get :

$$\begin{aligned} C(u) &= \frac{1}{\ln(3)} \sum_{n \in \mathbb{Z}} \gamma_n \int_0^1 l^{d_K-1} l^{2i\pi n / \ln(3)} \gamma_n \text{sinc}(lu) dl \\ &= \frac{u^{-d_K}}{\ln(3)} \sum_{n \in \mathbb{Z}} \gamma_n u^{-2i\pi n / \ln(3)} \int_0^u v^{d_K-1+2i\pi n / \ln(3)} \text{sinc}(v) dv \end{aligned}$$

with the change of variable : $v = ul$. Since $\int_0^u v^{d_K + \frac{2i\pi n}{\ln(3)} - 1} \text{sinc}(v) dv$ quickly converges to $\int_0^\infty v^{d_K + \frac{2i\pi n}{\ln(3)} - 1} \text{sinc}(v) dv$, one can replace \int_0^u by \int_0^∞ .

(More precisely : $|\int_0^u v^{d_K + \frac{2i\pi n}{\ln(3)} - 1} \text{sinc}(v) dv - \int_0^\infty v^{d_K + \frac{2i\pi n}{\ln(3)} - 1} \text{sinc}(v) dv| = O(u^{d_K-1})$)

Using :

$$\begin{aligned} \int_0^\infty v^{d_{K,n} + \frac{2i\pi n}{\ln(3)} - 1} \text{sinc}(v) dv \\ = \sin \left(\frac{\pi}{2} (d_{K,n} - 1) \right) \Gamma \left(d_{K,n} + \frac{2i\pi n}{\ln(3)} - 1 \right) \end{aligned}$$

with Γ the Euler gamma function, and $d_{K,n} = d_K + \frac{2i\pi n}{\ln(3)}$ one finally finds, for $u \gg 1$ (see fig. (3)):

$$C(u) = \frac{u^{-d_K}}{\ln(3)} \sum_n \gamma_n u^{-\frac{2i\pi n}{\ln(3)}} \sin \left(\frac{\pi}{2} (d_{K,n} - 1) \right) \Gamma(d_{K,n} - 1)$$

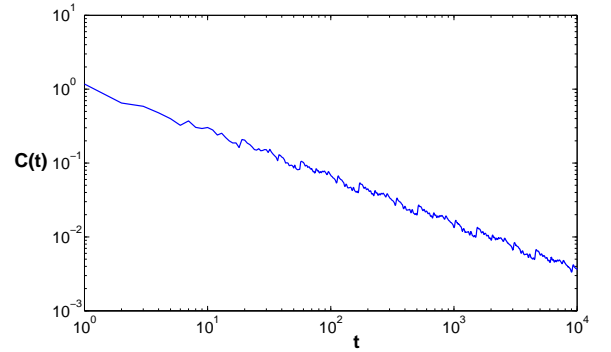


FIG. 3: Auto-correlation function in the case of a triadic Cantor set spectrum, in log-scale (log base 10).

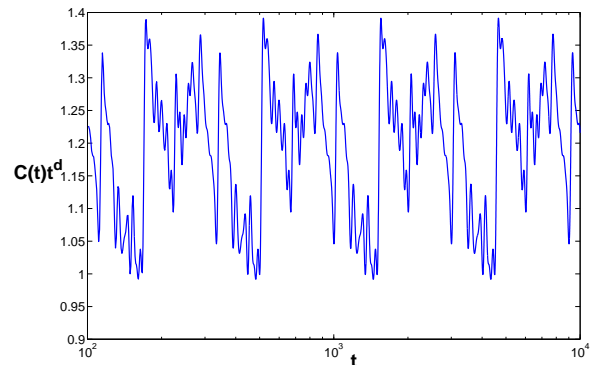


FIG. 4: Auto-correlation function multiplied by u^d in the case of a triadic Cantor set spectrum, in log-scale (log base 10). This function is log-periodic in u , of period $\ln(3)$.

We can prove (see annexe) that the coefficients in the expression (3) decay at least like $\frac{1}{n^{3/2-d}}$:

$$|\gamma_n \sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right) \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

IV.

The analytical expression (4) of the auto-correlation function is the main result of this paper which we shall now discuss.

We find that $C(u)$ is of the form : $C(u) = u^{-d_K} g(u)$, with $d_K = \frac{\ln(2)}{\ln(3)}$ the fractal dimension of the triadic Cantor set K and

$$g(u) = \sum_{n \in \mathbb{Z}} \gamma_n u^{-\frac{2i\pi n}{\ln(3)}} \sin\left(\frac{\pi}{2}(d_n - 1)\right) \Gamma(d_n - 1)$$

with $d_n = d + \frac{2i\pi n}{\ln(3)}$, a log-periodic function. The presence of the fractal dimension d_K in the exponent is consistent with previous work on the asymptotic behaviour of the auto-correlation function for a Cantor spectrum : $C(t) \approx_{t \rightarrow \infty} t^{-d_K}$ ([3], [4]).

The log-periodicity is the signature of the scaling symmetry of the spectrum. To see this, note that $C(t)$ is in fact the spectral average, for the measure $d\mu$, of the wavelet transform sinc : $C(t) = \int_{K \times K} \text{sinc}((\lambda' - \lambda)\epsilon_{max}t/\hbar) d\mu(\lambda) d\mu(\lambda')$. Since the spectrum has a scaling symmetry, one would have expected to find a log-periodic behaviour for the sinc-wavelet transform, at least locally (e.g. for the quantity : $C_{local}(t, \lambda) = \int_K \text{sinc}((\lambda' - \lambda)\epsilon_{max}t/\hbar) d\mu(\lambda)$). It was less obvious that the log-periodicity would remain after averaging over the spectrum ; this is due to the fact that we chose the density of modes and the distribution $|g(\epsilon)|^2$ to be uniform, and that the set has a unique scaling factor. In other words, in the case of a superposition of triadic Cantor sets for example, it is not certain that these oscillations would survive the averaging.

Let us now highlight the similarity between the theoretical graph of $C(t)$ for a triadic Cantor set spectrum (fig. (3), (4)) and the numerical results obtained with the Fibonacci cavity (more details are available in [6]) fig. (5). In this experiment, cavity polaritons are confined in wire cavities, consisting of $\lambda/2$ layers, using Bragg mirrors for the confinement in the vertical direction. The lateral dimension of the $200\mu\text{m}$ long wires are modulated quasi-periodically : the modulation consists in two wire sections (letters) A and B , of equal length but different width ; these letters are arranged in a finite sequence, S_j , obtained recursively using the following Fibonacci like algorithm : $S_{j>2} = [S_{j-1}S_{j-2}]$ and $S_1 = B, S_2 = A$ where $[S_{j-1}S_{j-2}]$ is the concatenation of the sequences S_{j-1} and S_{j-2} . The sequence S_∞ becomes rigorously quasi-periodic as j tends to infinity. We used a finite sized cavity, however, we observed the features of the fractal spectrum predicted by the theory : namely,

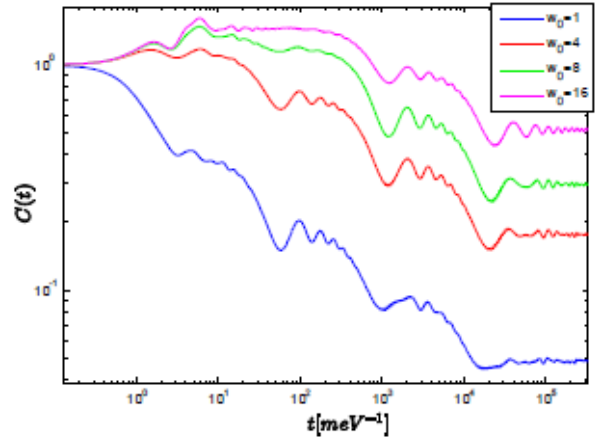


FIG. 5: Auto-correlation function of the wave packet in the Fibonacci cavity ; numerical results for an initial gaussian wave packet : $\psi(x, t = 0) \sim e^{-2(x-x_0)^2/w_0^2}$. The graphs correspond to different values of w_0 , with identical x_0 .

gaps densely distributed and an integrated density of states $N(\epsilon)$ well described by a scaling form of the type $N(\epsilon) = \epsilon^{\ln a / \ln b} F(\frac{\ln \epsilon}{\ln b})$ and which follows the gap labelling theorem. The photon modes are described by a 2D scalar wave equation with vanishing boundary conditions on the boundary of the wire. This 2D problem is then reduced to a 1D Schrodinger equation with an effective quasi-periodic potential, translating the geometry of the cavity. The equation was then solved numerically (using the transfer matrix formalism) and some useful quantities were plotted : the IDOS, the participation ratio, the RMS displacement and the time-averaged return probability. The latter is given fig. (5).

We found that the participation ratio and the return probability evidence a log-periodicity, which would be expected as a signature of the fractal spectrum.

V.

Let us now discuss the method used in a more general way, to highlight the connection between the presence of a scaling symmetry in a physical system and the typical form of certain quantities. We see from the calculation that the scaling property of the spectrum leads naturally to a specific behaviour of some physical quantities defined as integrals over the spectrum - power law modulated by log-periodic oscillations.

Consider a system having a fractal spectrum, characterized by a scaling property, which is expressed through a condition of the kind (1) verified by the measure.

One can show (see annexe) that a function f verifying a functional equation of the type :

$$f(x) = \frac{1}{b} f(ax) + g(x)$$

is (under some conditions on g) of the form :

$$f(x) = x^{\ln(b)/\ln(a)} G\left(\frac{\ln x}{\ln a}\right) \quad (3)$$

with G a 1-periodic function.

Now, in the case of a fractal F described by a measure μ such that :

$$\int_F f(x) d\mu(x) = \sum_{n \in \mathbb{N}} \frac{1}{b_n} f(a_n x) d\mu(x)$$

for some $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}^*$, then any function $\Psi(t)$ defined by an integral : $\Psi(t) = \int_F h(x, t) d\mu(x)$ will satisfy :

$$\Psi(t) = \sum_{n \in \mathbb{N}} \frac{1}{b_n} \int_F h(a_n x, t) d\mu(x)$$

If furthermore $h(x, t)$ can be expressed as a function of $x^\alpha t^\beta$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$: $h(x, t) = \psi(x^\alpha t^\beta)$, then :

$$\begin{aligned} \Psi(t) &= \sum_{n \in \mathbb{N}} \frac{1}{b_n} \int_F \psi(a_n x^\alpha t^\beta) d\mu(x) \\ &= \sum_{n \in \mathbb{N}} \frac{1}{b_n} \int_F h(x, a_n^{1/\beta} t) d\mu(x) \\ &= \sum_{n \in \mathbb{N}} \frac{1}{b_n} \Psi(a_n^{1/\beta} t) \\ &= \frac{1}{b_j} \Psi(a_j^{1/\beta} t) + \Phi(t) \end{aligned}$$

with $\Phi(t) = \sum_{n \in \mathbb{N}-j} \frac{1}{b_n} \Psi(a_n^{1/\beta} t)$. Thus, if the Mellin transforms of $\Phi(t)$ and $\Psi(t)$ exist and have overlapping domains of definition, and if the Mellin transform of $\Phi(t)$ has no poles, then $\Psi(t)$ will be of the form :

$$\Psi(t) = t^{\ln(b_j)/\ln(a_j^{1/\beta})} G\left(\frac{\ln t}{\ln a_j^{1/\beta}}\right)$$

with G 1-periodic.

More generally, this also occurs whenever we can rearrange the integral in order to have : $\int_F h(\alpha x, t) d\mu(x) \propto \int_F h(x, \beta t) d\mu(x)$.

Thus, we see in the case of quantum dynamics that, since the energy ϵ and the time t are conjugate variables, it is likely that a time dependant quantity defined as an integral over the spectrum (through the spectral decomposition for instance) of a function defined using the evolution operator $e^{iHt/\hbar}$ or of any function which couples ϵ and t , will be of the form (5). This is well verified for $C(t)$.

VI.

Let us now give an overview of the other physical quantities which do or should evidence such a behaviour ; we

also discuss the RMS displacement, commonly studied in quantum dynamics.

We discuss first the time-averaged participation ratio, usually studied to get information on the dynamics of a quantum system. Moreover, this quantity was studied numerically for the Fibonacci cavity and the results support the idea of a log-periodic feature. By definition, the participation ratio is :

$$P_\psi = \frac{1}{\int_{\text{espace}} |\langle x | \psi \rangle|^4 dx}$$

This quantity reflects the localization in space of the wave function $\psi(x, t)$: if the state is localized, then $P_\psi \simeq 0$; if it is extended, then $P_\psi \simeq 1$ [9]. Taking its time-average, to have a time-dependent quantity, we define :

$$P_\psi(t) = \frac{1}{t} \int_0^t \frac{dt'}{\int_{\text{espace}} |\langle x | \psi(t') \rangle|^4 dx}$$

Now, using Green functions, we can show [10] that :

$$|\psi(x, t)| = \left| \int_{\text{space spectrum}} \int e^{-i\epsilon t} \phi(x, \epsilon)^* \phi(x', \epsilon) \psi(x', 0) dx' d\mu(\epsilon) \right|$$

$|\phi(\epsilon)\rangle$ being the eigenfunction of the Hamiltonian associated with the energy ϵ . The calculation has not yet completely been carried through, since sufficient information on the behaviour of the space correlation term $\phi(x, \epsilon)^* \phi(x', \epsilon)$ is still missing. However, two ideas can be considered.

The first one is to obtain an approximate expression of the participation ratio by finding an upper and lower bound for $\int \phi(x, \epsilon)^* \phi(x', \epsilon) d\mu(\epsilon)$. In fact, it has been established ([11], [12]) for a lot of diffusive processes that the non-diagonal Green function $G(x, x', t)$ verifies :

$$F_t(c_1, c_2, x, x') \leq G(x, x', t) \leq F_t(c_3, c_4, x, x')$$

with

$$F_t(c_i, c_j, x, x') = \frac{c_i}{t^{d_s/2}} e^{-c_j \left(\frac{|x-x'|^{d_w}}{t} \right)^{\frac{1}{d_w-1}}}$$

where c_1, c_2, c_3, c_4 are positive constants, d_w the walk dimension which occurs in the dispersion relation $\langle x^2(t) \rangle = t^{2/d_w}$, d_s the spectral dimension such that $d_s = \frac{2d_f}{d_w}$ and d_f the Hausdorff dimension [13] of the space in which the diffusive process takes place. These bounds hold for diffusive processes ; however, it could be possible to use the same logic for the case of Schroedinger's equation, since the difference with the diffusion equation lies in the factor i in the time derivative.

The second idea is the following : it is an empirical numerical result that, in the case of a euclidean manifold, the eigenfunctions $\{\phi(\mathbf{r}, \epsilon)\}$ of the Laplacian verify (see [12] and references therein) : $\overline{\phi(\mathbf{r}, \epsilon)^* \phi(\mathbf{r}', \epsilon)} = f\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l_\epsilon}\right)$,

where $\overline{(\dots)}$ denotes the average over the points \mathbf{r}, \mathbf{r}' such as $|\mathbf{r} - \mathbf{r}'| = \text{constant}$ and f is some function. This has recently been used for a Laplacian defined on a fractal [12]. Assuming furthermore that, by analogy with the euclidean case, the relation between l_ϵ and ϵ is of the form : $l_\epsilon \propto \epsilon^\alpha$, we are able to obtain an analytical expression of the participation ratio, with the expected log-periodic feature.

A numerical simulation would therefore be interesting in order to verify this conjecture. This requires, in our case, a system such that the Laplacian has a triadic Cantor set spectrum, or any self-similar fractal spectrum.

We shall now briefly discuss the RMS displacement, another quantity commonly studied both in quantum dynamics - spreading of the wave packet - and in diffusive processes. First note that, since the above method (which led to the expression of the auto-correlation function), uses the linearity of the integral, it is obvious that it is unlikely to work for the RMS

$$\Delta x = \sqrt{\langle (x(t) - x_0)^2 \rangle} = \sqrt{\int_{space} (x(t) - x_0)^2 |\psi(t, x)|^2 dx}$$

because of the square root. However, we should obtain interesting results for the moment of order 2 :

$$(\Delta x)^2 = \langle (x(t) - x_0)^2 \rangle = \int_{space} (x(t) - x_0)^2 |\psi(t, x)|^2 dx$$

Although a numerical study, in the case of the Fibonacci cavity, did not give encouraging results, note that this quantity is very close to

$$\begin{aligned} & \int_{space} (x(t) - x_0)^2 |\psi(t, x)|^2 dx \\ &= \left| \int_{space} \int_{spectrum} e^{-i\epsilon t} \phi(x, \epsilon)^* \phi(x', \epsilon) \psi(x', 0) dx' d\mu(\epsilon) \right|^2 \end{aligned}$$

and that using the assumptions discussed for the participation ratio about the behaviour of $\phi(\mathbf{r}, \epsilon)^* \phi(\mathbf{r}', \epsilon)$ should also lead to interesting results.

Furthermore, the case of a Hamiltonian with a one-scale Cantor set spectrum has been solved numerically in the tight-binding approximation [14], and it was found that the moments $\langle (x(t) - x_0)^\alpha \rangle$ have a log-periodic behaviour.

One can also use this method to derive the decay probability of a two level atom coupled to a fractal spectrum ([15]). We show in this case that Fermi's golden rule does not hold, and that the decay probability of the atom from an energy level E_i to an energy $E_f \pm \Delta E$ is of the form (5). In this case, the log-periodic oscillations could be interpreted as a signature of the topological properties of the spectrum : a closed, nowhere dense set with no isolated point.

To finish, let us discuss briefly the question of turbulent diffusion, which was found to be related to fractals.

A powerful tool for the study of turbulent diffusion is the method of breakdown coefficients, bdc, (see [16] and references therein). If a turbulent system happens to have a physically distinguished scale factor (for instance, unstable eddies breaking up in typically always the same number of smaller eddies), then the moments of the bdc's have a log-periodic behaviour (namely, a power law modulated by log periodic oscillations) [16]. It is interesting to note that these log-periodic oscillations were derived with a very different approach - starting from a functional relation obtained in the frame of statistical physics. Thus, two very different physical problems (turbulence and quantum dynamics) led to a similar observation : there is a deep connection between the topological properties of self-similar sets and the occurrence of anomalous log-periodic behaviour of statistical and dynamical quantities.

VII. CONCLUSION

In summary, we derived the analytical expression of the auto-correlation function $C(t)$ in the case of a triadic Cantor set energy spectrum, and found that it is of the form : $C(t) = \left(\frac{t}{\tau}\right)^{-d_K} g(\ln(t/\tau))$ where $d_K = \frac{\ln(2)}{\ln(3)}$ is the fractal dimension of the triadic Cantor set, τ a time scale a g a periodic function. We generalize the method and discuss the connection between the presence of self-similar structures or features in a physical system and the typical log-periodic behaviour of certain quantities. In particular, the participation ratio and the moment of order 2 are likely to have such a behaviour for a one-scale Cantor set spectrum ; we are at the moment researching on these topics. The method developed also led to some first theoretical results for the spontaneous emission of a two level atom coupled to a fractal spectrum ; numerical and experimental work should be pursued in the near future.

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Annexe

Scaling property of the measure $d\mu$

Let f be an integrable function defined on $[0, 1]$, with real or complex values.

We derive here the following property :

$$\int_K f(x) d\mu(x) = \frac{1}{2} \int_K f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_K f\left(\frac{x+2}{3}\right) d\mu(x)$$

By definition :

$$\int_K f(x) d\mu(x) = \lim_{n \rightarrow \infty} \underbrace{\left(\frac{2}{3}\right)^n \sum_{a_{j_n} \in P_n} \int_{a_{j_n}}^{a_{j_n} + 3^{-n}} f(x) dx}_{I_n(f(x))}$$

with P_n the set of the left edges of the remaining segments at the n^{th} step of the construction of the triadic Cantor set. One can directly check that the sequence $(P_n)_n$ follows the recurrence property :

$$P_n = \frac{1}{3}P_{n-1} \cup \left\{ \frac{2}{3} + \frac{1}{3}P_{n-1} \right\}$$

Therefore :

$$\begin{aligned} I_n(f(x)) &= \left(\frac{2}{3}\right)^n \sum_{a_{j_n} \in P_n} \int_{a_{j_n}}^{a_{j_n} + 3^{-n}} f(x) dx \\ &= \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}} \in P_{n-1}} \int_{a_{j_{n-1}}/3}^{a_{j_{n-1}}/3 + 3^{-n}} f(x) dx \\ &+ \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}} \in P_{n-1}} \int_{\frac{2}{3} + a_{j_{n-1}}/3}^{\frac{2}{3} + a_{j_{n-1}}/3 + 3^{-n}} f(x) dx \\ &= \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}} \in P_{n-1}} \int_{a_{j_{n-1}}}^{a_{j_{n-1}} + 3^{-n+1}} f\left(\frac{u}{3}\right) \frac{du}{3} \\ &+ \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}} \in P_{n-1}} \int_{a_{j_{n-1}}}^{a_{j_{n-1}} + 3^{-n+1}} f\left(\frac{v+2}{3}\right) \frac{dv}{3} \\ &= \frac{3}{2} \frac{1}{3} \left[I_{n-1}\left(f\left(\frac{u}{3}\right)\right) + I_{n-1}\left(f\left(\frac{2+v}{3}\right)\right) \right] \end{aligned}$$

But : $\lim_{n \rightarrow \infty} I_{n-1}\left(f\left(\frac{u}{3}\right)\right) = \int_K f\left(\frac{u}{3}\right) d\mu(u)$ and $\lim_{n \rightarrow \infty} I_{n-1}\left(f\left(\frac{2+v}{3}\right)\right) = \int_K f\left(\frac{2+v}{3}\right) d\mu(u)$, so finally :

$$\int_K f(x) d\mu(x) = \frac{1}{2} \int_K f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_K f\left(\frac{x+2}{3}\right) d\mu(x)$$

This completes the proof.

Log-periodicity of functions verifying : $f(x) = \frac{1}{b}f(ax) + g(x)$

Let $f : \mathbb{R} \rightarrow \mathbb{C}$ verifying a functional equality of the kind :

$$f(x) = \frac{1}{b}f(ax) + g(x) \quad (4)$$

Let's show that f is of the form :

$$f(x) = x^{\ln(b)/\ln(a)} G\left(\frac{\ln x}{\ln a}\right)$$

with G 1-periodic.

We begin by taking the Mellin transform of (3) (assuming that the domain of definition of the Mellin transforms of f and g have a non empty intersection) :

$$\begin{aligned} M_f &= \int_0^\infty f(x) x^{s-1} dx \\ &= \frac{1}{b} \int_0^\infty f(ax) x^{s-1} dx + \int_0^\infty g(x) x^{s-1} dx \\ &= \frac{1}{b} \int_0^\infty f(u) \left(\frac{u}{a}\right)^{s-1} \frac{du}{a} + M_g(s) \end{aligned}$$

So :

$$M_f(s) = \frac{M_g(s)}{1 - \frac{a^{-s}}{b}}$$

and by taking the inverse Mellin transform :

$$\begin{aligned} f(x) &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} M_f(s)x^{-s} ds \\ &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{M_g(s)}{1-\frac{a^{-s}}{b}} x^{-s} ds \end{aligned}$$

Assuming now that M_g has no poles, the residue formula finally yields :

$$\begin{aligned} f(x) &= \sum_{n \in \mathbb{Z}} \left[\frac{(s - s_n) M_g(s) x^{-s}}{1 - \frac{a^{-s}}{b}} \right]_{s=s_n} \\ & \quad \left(s_n = -\frac{\ln a}{\ln b} - \frac{2i\pi n}{\ln b}, \text{ poles of } \frac{M_g(s)}{1 - \frac{a^{-s}}{b}} \right) \\ &= x^{\ln(b)/\ln(a)} \underbrace{\sum_{n \in \mathbb{Z}} c_n x^{2i\pi n / \ln b}}_{G(x)} \end{aligned}$$

which is what we wanted to prove.

Asymptotic behaviour of the coefficients

We prove here that the coefficients in the expression (??) decay at least like $\frac{1}{n^{3/2-d}}$:

$$|\gamma_n \sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right) \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

To see this, let us start with $|\sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right) \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)|$; we have ([14]) :

$$\lim_{n \rightarrow \infty} \left| \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right) \right| = \sqrt{2\pi} e^{-(d-1)} e^{-\frac{2\pi^2 n}{2\ln(3)}} \left(\frac{2\pi n}{\ln(3)}\right)^{d-\frac{3}{2}}$$

Furthermore, since $|\sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right)| \sim_{n \rightarrow \infty} e^{\frac{2\pi^2 n}{2\ln(3)}} e^{1-d}/2$, one has :

$$\left| \sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right) \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right) \right| = O\left(\frac{1}{n^{3/2-d}}\right)$$

Let us now study γ_n . By definition :

$$\begin{aligned} \gamma_n &= \int_{K \times K} \frac{d\mu(x)d\mu(y)}{|2-x+y|^{d_n}} \\ &= \lim_{k \rightarrow \infty} \left(\frac{3}{2}\right)^{2k} \sum_{a_{j_k}, a_{j'_k} \in P_k} \int_{a_{j_k}}^{a_{j_k}+3^{-k}} \int_{a_{j'_k}}^{a_{j'_k}+3^{-k}} \frac{dxdy}{|2-x+y|^{d_n}} \end{aligned}$$

with $d_n = \frac{\ln 2}{\ln 3} + \frac{2i\pi n}{\ln 3}$ and $P_k = \{a_{j_k}\}$ the set of the left edges of the remaining segments after k iterations in the construction of the triadic Cantor set K . Since K is embedded in $[0, 1]$, and since for all $x, y \in [0, 1]$ the inequality $2 > x - y$ holds, one can suppress the absolute value in the integral. Now, for all $x, y \in [0, 1]$, one has : $1 \leq 2 - x + y \leq 3$, so $|\frac{1}{|2-x+y|^{d_n}}| = \frac{1}{|2-x+y|^d} < 1$.

Thus :

$$|\gamma_n| < \int_{K \times K} \frac{d\mu(x)d\mu(y)}{|(2-x+y)^{d_n}|} < \int_{K \times K} d\mu(x)d\mu(y) = 1$$

and so : $|\gamma_n| = O(1)$. Numerically, it seems that we even have : $|\gamma_n| = O\left(\frac{1}{n}\right)$, but this has not been proven rigorously.

We therefore obtain finally that, at least :

$$|\gamma_n \sin\left(\frac{\pi}{2}\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)\right) \Gamma\left(d + \frac{2i\pi n}{\ln(3)} - 1\right)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

¹ obtained by iteration of the following process : start with a full square, divide it in 9 equal squares and keep only the

four at the corners of the initial square ; see fig.(2)