

# Quantum dynamics for a triadic Cantor set spectrum

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We study the dynamics of a wave packet for a triadic Cantor spectrum  $K$ . The scaling symmetry of the spectrum translates into the characteristic log-periodic feature of relevant physical quantities. Using Mellin transform, we derive the analytical expression of the time-averaged return probability, which is universal for any system with a Cantor spectrum. This result is supported by numerical results obtained recently for polaritons in a Fibonacci potential. We give a more thorough analysis in the case of the tight-binding model, in which case the dynamics of the system depend essentially on the Fourier transform on a Cantor set of monomes  $x^k$ , namely :  $h_k(t) = \int_K e^{-ixt} x^k d\mu$ ,  $k \in \mathbb{Z}$ , which carry the fingerprint of the spectral fractal feature.

## Introduction

During the past decades, there has been a growing interest for Schroedinger operators  $H = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})$  with a Cantor set like spectrum, e.g. a perfect, nowhere dense set (for a review, see [1], and more recently [2]). In the context of quantum dynamics for systems verifying such a Hamiltonian, a typical behaviour for some physical quantities, such as the time-averaged return probability (auto-correlation function), has been identified. More precisely, it has been shown ([3],[4]) that, in the case of a Cantor set spectrum, the time-averaged return probability  $C(t)$ , defined as :

$$C(t) = \langle |\langle \psi(0) | \psi(t) \rangle|^2 \rangle = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$$

decreases as :  $t^{-D_2^\mu}$ , where  $D_2^\mu$  is a correlation dimension with respect to the measure  $\mu$  of the spectrum, although a full analytical expression had not been derived so far. The spatial correlation function  $C_s(l, \mu) = \int_{K \times K} \theta(l - |x - y|) d\mu(x) d\mu(y)$ , with  $\theta$  the Heaviside function, has been derived more precisely for a Cantor set spectrum ([5]), showing that this quantity is a power law  $t^{\ln 2 / \ln 3}$  ( $\frac{\ln 2}{\ln 3}$  being the fractal dimension of the Cantor set) modulated by log-periodic oscillations.

In this paper, we study the dynamics of a quantum system whose energy spectrum is a triadic Cantor set  $K$ . In section I, we describe the scaling symmetry of the spectrum and analyze how it translates in terms of measurable physical quantities ; using Mellin transform, we derive the exact analytical expression of the time-averaged return probability, and we obtain the following asymptotic expression for  $t \rightarrow \infty$  :

$$C(t) \sim_{t \rightarrow \infty} t^{-d_K} G_{per}(\ln t)$$

where  $d_K = \frac{\ln 2}{\ln 3}$  is the fractal dimension [7] of the triadic Cantor set and  $G_{per}$  a periodic function. In this paper, when not specified,  $G_{per}$  will denote a periodic function of the logarithm of the argument. This means that  $C(t)$  is asymptotically a scaling function, carrying the fingerprint of the self-similar fractal spectrum.

We also present numerical results obtained with polaritons in a Fibonacci cavity [6], which support the theory. We then discuss how the scaling symmetry of the spectrum affects other dynamical quantities. The typical log-periodic feature should naturally appear when taking the time-average of dynamical quantities. However, the fingerprint of the fractal is also visible on some instantaneous quantities, such as the root-mean displacement (RMS). To show this, we study in Section II the tight-binding model ; we show that in this case, the eigenfunctions can be seen as polynomials of the energy, and that the dynamics depend essentially on the Fourier transform (on a Cantor set) of monomials :  $h_k(t) = \int_K e^{-ixt} x^k d\mu$ ,  $k \in \mathbb{Z}$ . In fact, the wave-function is a linear combination of the  $h_k(t)$ . Using Mellin transform, we show that :  $\langle h_k(t)^* h_l(t) \rangle \sim_{t \rightarrow \infty} t^{-d_K} G_{k,l}(\ln t)$  with  $G_{k,l}(\ln t)$  a periodic function. This implies that time-averaged physical dynamical quantities, such as the time-averaged root mean displacement  $\langle \Delta x(t) \rangle = \frac{1}{t} \int_0^t \sqrt{\sum_k k^2 |\psi(k, t')|^2} dt'$  or the participation ratio,  $\langle P_r(t) \rangle = \frac{1}{t} \int_0^t \frac{dt'}{\sum_k |\psi(k, t')|^4}$ , display the characteristic behaviour discussed earlier and carry the fingerprint of the scaling symmetry of the spectrum.

## I

The triadic Cantor set, denoted  $K$  in this paper, is constructed as follows : start with the segment  $[0,1]$ , divide it in three equal parts, and remove the central one, e.g. the segment  $[1/3, 2/3]$ , then repeat the process with the two remaining segments, and so on. After an infinite number of iterations, we are left with a set of separated points, of zero Lebesgue measure.

We now consider a one dimensional quantum system described by the state vector  $|\psi(t)\rangle$  and evolving according to the Hamiltonian  $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$ . We assume that the energy spectrum of  $H$  is a triadic Cantor set.

The system is initially in a state described by  $|\psi(0)\rangle = |\psi_0\rangle$ . After a time  $t$ , the state vector  $|\psi(t)\rangle$  is given by :

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi_0\rangle \quad (1)$$

We shall also use the spectral decomposition of  $|\psi(t)\rangle$  over an orthonormal basis of eigenfunctions  $\{|\phi(\epsilon)\rangle\}$  of  $H$ .  $|\phi(\epsilon)\rangle$  is an eigenfunction associated with the energy  $\epsilon$  :

$$|\psi(t)\rangle = \int_K g(\epsilon) e^{-i\epsilon t/\hbar} |\phi(\epsilon)\rangle d\tilde{\mu}(\epsilon)$$

with  $g(\epsilon) = \langle \psi_0 | \phi(\epsilon) \rangle$ , and  $d\tilde{\mu}$  a measure defined on the spectrum. To alleviate the notations, we shall set  $\hbar = 1$ .

To go further, we need to define an appropriate measure on the spectrum. The triadic Cantor set  $K$  has a zero Lebesgue measure. However, in our case, the integrated density of states, e.g. the integral of the density of states  $\rho(\epsilon)$  over the spectrum  $K$ , is non zero. Thus,  $\rho(\epsilon)$  must be infinite in every point of  $K$ .

We define a measure  $d\mu$  on  $K$  in the following way : we suppose that the density of states is uniform, and that the integrated density of states,  $\mathcal{N}(\epsilon) = \int_K \theta(\epsilon - \epsilon') d\mu(\epsilon')$  with  $\theta$  the Heaviside function, is bounded. Without loss of generality, we impose :  $\int_K d\mu(\epsilon) = 1$ . We then proceed iteratively : starting from a segment  $[0, 1]$ , we set  $\rho(\epsilon) = 1$  everywhere ; the integral thus equals 1. Now, divide the segment in three, remove the central part and define  $\rho(\epsilon)$  to be equal to  $\frac{3}{2}$  on  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$ , and null on  $[\frac{1}{3}, \frac{2}{3}]$ . We still have :  $\int_0^1 \rho(x) dx = 1$ . We repeat this process infinitely, and thus define  $d\mu$  as a limit : for any function  $f$  defined on  $[0, 1]$ , the integral of  $f$  over  $K$  is :

$$\int_K f(x) d\mu(x) = \lim_{n \rightarrow \infty} \left(\frac{3}{2}\right)^n \sum_{a_{j_n} \in P_n} \int_{a_{j_n}}^{a_{j_n} + 3^{-n}} f(x) dx$$

where  $\{P_n\}$  is the set of the left edges of the remaining segments after  $n$  iterations in the construction of the Cantor set.

One can show (see appendix) that  $d\mu$  verifies the following important property :

$$\int_K f(x) d\mu(x) = \frac{1}{2} \int_K f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_K f\left(\frac{x+2}{3}\right) d\mu(x) \quad (2)$$

This property still holds for multiple integrals ; it becomes, for a Cantor product set (Cantor dust<sup>1</sup>, see fig. (2)) :

$$\begin{aligned} & \int_{K \times K} f(x, y) d\mu(x) d\mu(y) \\ &= \frac{1}{4} \int_{K \times K} f\left(\frac{x}{3}, \frac{y}{3}\right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4} \int_{K \times K} f\left(\frac{x+2}{3}, \frac{y}{3}\right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4} \int_{K \times K} f\left(\frac{x}{3}, \frac{y+2}{3}\right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4} \int_{K \times K} f\left(\frac{x+2}{3}, \frac{y+2}{3}\right) d\mu(x) d\mu(y) \end{aligned}$$

(2bis)

Note that the measure  $\mu$  is a special case of a self-similar measures  $m$ , e.g. such that there exists a set of contractive similarities  $\{\phi_j\}$  and probability weights  $\{\pi_j\}$  such that for any continuous function  $f$ , one has :

$$\int f(x) dm(x) = \sum_j \pi_j \int f(\phi_j(x)) dm(x)$$

Cantor sets correspond to the case of two linear similarities, with  $\pi_1 = \pi_2 = \frac{1}{2}$  :

$$\int f(x) dm(x) = \frac{1}{2} \int f(a_1 x) dm(x) + \frac{1}{2} \int f(a_2 x + b) dm(x)$$

The methods developed hereafter can easily be translated for any self-similar measure.

Let us now study the dynamics of the system described by :

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle$$

There are several physical quantities which are useful to characterize the evolution of the system, such as the RMS displacement, the participation ratio or the return probability.

We will focus first on the time-averaged return probability.

The time averaged return probability, or auto-correlation function,  $C(t)$ , is by definition the time average of the probability  $p(t) = |\langle \psi(0) | \psi(t) \rangle|^2$  to find the system in its initial state after a time  $t$  :

$$C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$$

We could directly try to calculate  $p(t)$  to study the evolution in time of the system ; however this quantity often exhibits fast local oscillations, which could hide the effects of the fractal feature. Its time average, on the other hand, yields interesting results.

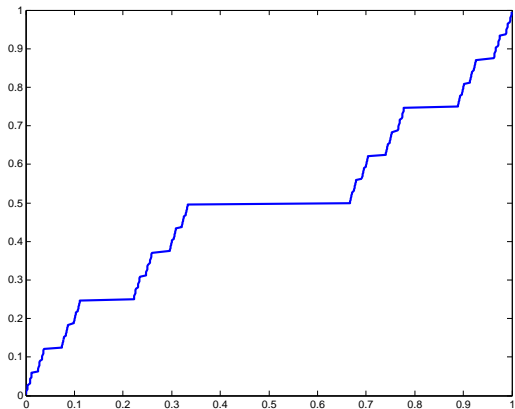


FIG. 1: Integrated density of states (IDOS) in the case of a triadic Cantor set energy spectrum. The energies are expressed in arbitrary unit and the IDOS has been normalised.

We start with the spectral decomposition of  $|\psi(t)\rangle$  :

$$|\psi(t)\rangle = \int_K g(\epsilon) e^{-i\epsilon t} |\phi(\epsilon)\rangle d\tilde{\mu}(\epsilon)$$

We impose that  $|\psi(t)\rangle$  is normalized to unity :

$$\langle \psi(t) | \psi(t) \rangle = \int_K |g(\epsilon)|^2 d\tilde{\mu}(\epsilon) = 1$$

We shall assume that  $|g(\epsilon)|$  is non-zero and constant on a bounded subset of the spectrum, and that this subset is the triadic Cantor set. We therefore consider that  $|g(\epsilon)|^2 d\tilde{\mu}(\epsilon)$  (the spectral measure of the initial state) is the measure  $d\mu$  introduced in the previous paragraph.

Thus :

$$\begin{aligned} C(t) &= \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt' \\ &= \frac{1}{t} \int_0^t \left| \int_K e^{-i\epsilon t'} d\mu(\epsilon) \right|^2 dt' \end{aligned}$$

A standard computation then leads to :

$$C(t) = \iint_{K \times K} \text{sinc}((\epsilon' - \epsilon)t) d\mu(\epsilon) d\mu(\epsilon') \quad (3)$$

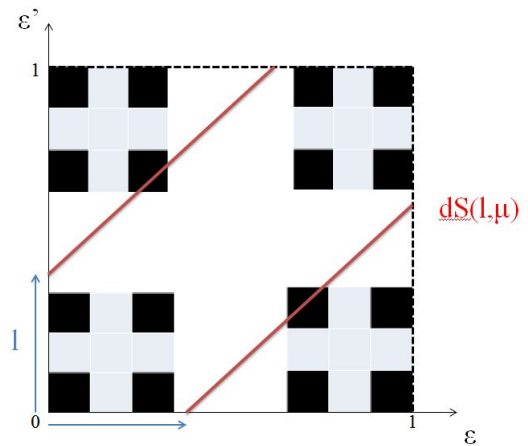


FIG. 2: 2 dimensional Cantor dust (represented here at the second step of construction) ; the infinitesimal surface  $dS(l, \mu)$  corresponds to the surface of the Cantor dust, in the sense of  $d\mu$ , contained in the two red strips.

The idea is now to transform this double integral into a one variable integral, and then to use Mellin transform and the scaling property (2) of  $d\mu$ .

Let us do the change of variables :  $l = |\epsilon' - \epsilon|$ ,  $0 \leq l \leq 1$ , and let  $dS(l, \mu)$  be the area (in the sense of  $d\mu$ ) of the 2-dimensional Cantor dust contained in the two strips of infinitesimal width located at  $\epsilon = \epsilon' + l$  and  $\epsilon = \epsilon' - l$  (fig. 2).

Since  $\text{sinc}((\epsilon' - \epsilon)t)$  can be considered constant on these strips, we obtain :

$$C(t) = \int_0^1 \text{sinc}(lt) dS(l, \mu) \quad (4)$$

We now have to determine  $dS(l, \mu)$ . For this we shall use Mellin transforms.

Let :  $M_{dS}(s) = \int_0^1 l^{s-1} dS(l, \mu) = \int_0^1 l^{s-1} \frac{dS(l, \mu)}{dl} dl$  be the Mellin transform of  $dS(l, \mu)$ , which is sometimes referred to in the literature as the energy integral ([5], [8]) (note that we wrote  $\int_0^1$  and not  $\int_0^\infty$  :  $dS(l, \mu)$  is defined on  $[0, 1]$ , but can be extended on  $[0, \infty[$  if we define it as equal to 0 outside  $[0, 1]$ ).

Now, using the definition of  $dS(l, \mu)$  and the property

(2) :

$$\begin{aligned}
M_{dS}(s) &= \int_0^1 l^{s-1} dS(l, \mu) \\
&= \int_{K \times K} |x-y|^{s-1} d\mu(x) d\mu(y) \\
&= \frac{1}{4} \int_{K \times K} \left( \left| \frac{x}{3} - \frac{y}{3} \right|^{s-1} + \left| \frac{x+2}{3} - \frac{y+2}{3} \right|^{s-1} \right) d\mu(x) d\mu(y) \\
&+ \frac{1}{4} \int_{K \times K} \left( \left| \frac{x}{3} - \frac{y+2}{3} \right|^{s-1} + \left| \frac{x+2}{3} - \frac{y}{3} \right|^{s-1} \right) d\mu(x) d\mu(y) \\
&= \frac{3^{1-s}}{2} M_{dS}(s) + \gamma(s)
\end{aligned}$$

with

$$\gamma(s) = \frac{3^{1-s}}{2} \int_{K \times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)$$

Thus :

$$M_{dS}(s) = \frac{\gamma(s)}{1 - 3^{1-s}/2}$$

The inverse Mellin transform then gives  $\frac{dS(l, \mu)}{dl}$  :

$$\begin{aligned}
\frac{dS(l, \mu)}{dl} &= \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} M_{dS}(s) l^{-s} ds \\
&= \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\gamma(s)}{1 - \frac{3^{1-s}}{2}} l^{-s} ds
\end{aligned}$$

It is important to note that  $\gamma(s)$  has no poles. This is clear from its definition : since for all  $x, y \in [0, 1]$ , the inequality  $1 \leq 2 - x + y \leq 3$  holds,  $|x - y + 2|^{s-1}$  is well defined and bounded on  $K \times K$  for any  $s \in \mathbb{C}$ , and thus  $\gamma(s)$  is well defined and has no poles. Hence, the only poles of  $\frac{\gamma(s)}{1 - \frac{3^{1-s}}{2}} l^{-s}$  are the  $\{s_n\}_{n \in \mathbb{Z}}$  such that :  $s_n = 1 - d_K - \frac{2i\pi n}{\ln 3}$ . To alleviate the notations, we write  $d_{K,n} = d_K + \frac{2i\pi n}{\ln 3}$ . We now apply the residue theorem

$$\frac{dS(l, \mu)}{dl} = \sum_n \left[ \frac{s - s_n}{1 - \frac{3^{1-s}}{2}} l^{-s} \gamma(s) \right]_{s=s_n} \quad (5)$$

$$= \frac{l^{d_K-1}}{\ln 3} \sum_n l^{\frac{2i\pi n}{\ln 3}} \underbrace{\gamma(1 - d_{K,n})}_{\gamma_n} \quad (6)$$

where we used :

$$\lim_{s \rightarrow s_n} \frac{s - s_n}{1 - \frac{3^{1-s}}{2}} = \frac{1}{\ln 3}$$

and where

$$\gamma_n = \int_{K \times K} |x - y + 2|^{-d_{K,n}} d\mu(x) d\mu(y) \quad (7)$$

Inserting (5) in (4), and with the change of variable  $v = tl$ , we get :

$$\begin{aligned}
C(t) &= \frac{1}{\ln 3} \sum_{n \in \mathbb{Z}} \gamma_n \int_0^1 l^{d_K-1} l^{2i\pi n / \ln 3} \text{sinc}(lt) dl \\
&= \frac{t^{-d_K}}{\ln 3} \sum_{n \in \mathbb{Z}} \gamma_n t^{-2i\pi n / \ln 3} \int_0^t v^{d_{K,n}-1} \text{sinc}(v) dv
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^t v^{d_{K,n}-1} \text{sinc}(v) dv \\
&= \int_0^\infty v^{d_{K,n}-1} \text{sinc}(v) dv + \mathcal{O}(t^{d_K-1})
\end{aligned}$$

and since :

$$\begin{aligned}
&\int_0^\infty v^{d_{K,n} + \frac{2i\pi n}{\ln 3} - 1} \text{sinc}(v) dv \\
&= \sin\left(\frac{\pi}{2}(d_{K,n} - 1)\right) \Gamma(d_{K,n} - 1)
\end{aligned}$$

with  $\Gamma$  the Euler gamma function, we obtain finally :

$$C(t) = t^{-d_K} \sum_{n \in \mathbb{Z}} t^{-\frac{2i\pi n}{\ln 3}} a_n [1 + \mathcal{O}(t^{-1})]$$

with  $a_n = \frac{1}{\ln 3} \gamma_n \sin\left(\frac{\pi}{2}(d_{K,n} - 1)\right) \Gamma(d_{K,n} - 1)$ , and asymptotically (see fig. (3), (4)):

$$C(t) \sim_{t \rightarrow \infty} t^{-d_K} \sum_{n \in \mathbb{Z}} a_n t^{-\frac{2i\pi n}{\ln 3}}$$

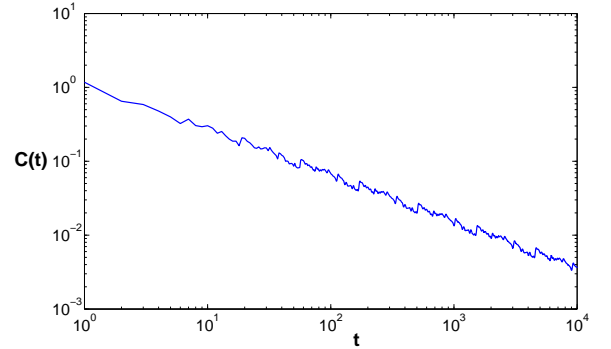


FIG. 3: Auto-correlation function in the case of a triadic Cantor set spectrum, in log-scale (log base 10).

We can prove by standard computation that :

$$a_n =_{|n| \rightarrow \infty} \mathcal{O}\left(\frac{1}{|n|^{3/2-d_K}}\right)$$

Let us now discuss the expression of  $C(t)$ . We find that, asymptotically,  $C(t)$  is of the form :

$$C(t) = t^{-d_K} G_C(\ln t)$$

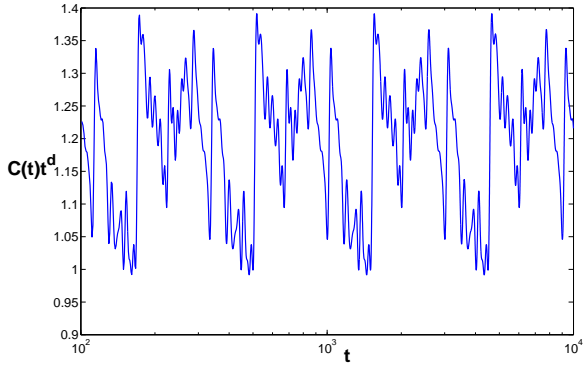


FIG. 4: Auto-correlation function multiplied by  $u^d$  in the case of a triadic Cantor set spectrum, in log-scale (log base 10). This function is log-periodic in  $u$ , of period  $\ln 3$ .

with

$$G_C(\ln t) = \sum_{n \in \mathbb{Z}} a_n e^{-\frac{2i\pi n}{\ln 3} \ln t}$$

a periodic function of  $\ln t$ . The presence of the fractal dimension  $d_K$  in the exponent is consistent with previous work on the asymptotic behaviour of the time-averaged return probability for a Cantor spectrum :  $C(t) \sim_{t \rightarrow \infty} t^{-d_K}$  ([3], [4]).

The asymptotic behaviour of  $C(t)$  could have been intuited by the following observation : using the scaling property of the measure, we find that :

$$C(t) = \frac{1}{2}C\left(\frac{t}{3}\right) + \frac{1}{2} \iint_{K \times K} \text{sinc}\left(\frac{x-x'+2}{3}t\right) d\mu(x)d\mu(x') \quad (8)$$

When  $t \rightarrow \infty$ , the second term of the sum on the right tends to zero, since for all  $x, x'$  in  $K$ ,  $1 < x - x' + 2 < 3$ . Thus, asymptotically :

$$C(t) \sim_{t \rightarrow \infty} \frac{1}{2}C\left(\frac{t}{3}\right)$$

which leads naturally to having  $C(t)$  of the form :

$$C(t) \sim_{t \rightarrow \infty} t^{-d_K} G_{per}(\ln t)$$

This means that  $C(t)$  is asymptotically a scaling function, with the scaling numbers of the Cantor set.

Qualitatively speaking,

$$C(t) = \int_{K \times K} \text{sinc}((\epsilon' - \epsilon)t) d\mu(\epsilon) d\mu(\epsilon')$$

is the wavelet transform of the spectrum, acting like a magnifying glass ; as  $t$  goes to infinity, the surface of integration is narrowed due to the sinc function, and decreases like  $t^{-d_K} G_{per}(\ln t)$ , as we saw by calculating the area variation  $dS(l, \mu)$ . Thus, we expect to observe this typical behaviour in other dynamical quantities if they

can be expressed as integrals over the spectrum ; in particular, this should be the case if we look at their time-average.

Before discussing other quantities, let us highlight the similarity between the theoretical graph of  $C(t)$  for a triadic Cantor set spectrum (fig. (3), (4)) and the numerical results obtained with the Fibonacci cavity (more details are available in [6]) fig. (5). In this experiment, cavity polaritons are confined in wire cavities, consisting of  $\lambda/2$  layers, using Bragg mirrors for the confinement in the vertical direction. The lateral dimension of the  $200\mu\text{m}$  long wires are modulated quasi-periodically : the modulation consists in two wire sections (letters)  $A$  and  $B$ , of equal length but different width ; these letters are arranged in a finite sequence,  $S_j$ , obtained recursively using the following Fibonacci like algorithm :  $S_{j>2} = [S_{j-1}S_{j-2}]$  and  $S_1 = B$ ,  $S_2 = A$  where  $[S_{j-1}S_{j-2}]$  is the concatenation of the sequences  $S_{j-1}$  and  $S_{j-2}$ . The sequence  $S_\infty$  becomes rigorously quasi-periodic as  $j$  tends to infinity. We used a finite sized cavity, however, we observed the features of the fractal spectrum predicted by the theory : namely, gaps densely distributed and an integrated density of states  $\mathcal{N}(\epsilon)$  well described by a scaling form of the type  $\mathcal{N}(\epsilon) = \epsilon^{\ln a / \ln b} \mathcal{F}\left(\frac{\ln \epsilon}{\ln b}\right)$  and which is given by the gap labelling theorem [1]. The photon modes are described by a 2D scalar wave equation with vanishing boundary conditions on the boundary of the wire. This 2D problem is then reduced to a 1D Schrodinger equation with an effective quasi-periodic potential, translating the geometry of the cavity. The equation was then solved numerically (using the transfer matrix formalism) and some useful quantities were plotted : the IDOS, the participation ration, the RMS displacement and the time-averaged return probability. The latter is given fig. (5).

We also found that the participation ratio and the return probability evidence the characteristic log-periodicity.

Let us now discuss the method used in a more general way, to identify the fingerprint of a scaling symmetry in certain physical quantities. We see from the calculation that the scaling property of the spectrum leads naturally to a specific behaviour of some physical quantities defined as integrals over the spectrum - power law modulated by log-periodic oscillations.

In the case of a fractal  $F$  described by a measure  $\mu$  defined by the following relation (the balance equation [22]):

$$\int_F f(x) d\mu(x) = \sum_{n=1}^N \int_F \left( \frac{1}{b_n} f(a_n x) + \frac{1}{c_n} f(d_n x + e_n) \right) d\mu(x)$$

for some  $a_n, d_n, e_n \in \mathbb{R}$ ,  $b_n, c_n \in \mathbb{R}^*$ , any function  $\Psi(t)$  defined by an integral :  $\Psi(t) = \int_F h(x, t) d\mu(x)$  will satisfy :

$$\Psi(t) = \sum_{n=1}^N \int_F \left( \frac{1}{b_n} h(a_n x, t) + \frac{1}{c_n} h(d_n x + e_n, t) \right) d\mu(x)$$

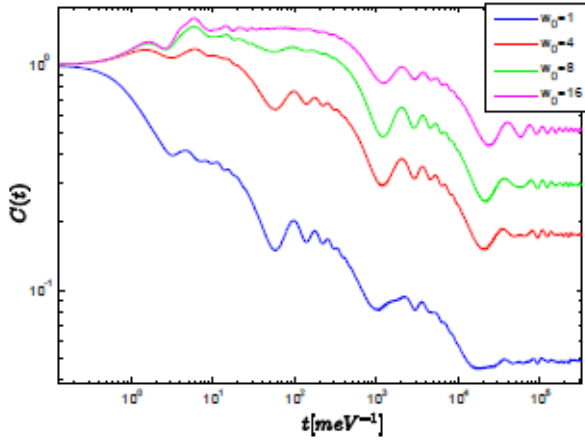


FIG. 5: Auto-correlation function of the wave packet in the Fibonacci cavity ; numerical results for an initial gaussian wave packet :  $\psi(x, t = 0) \sim e^{-2(x-x_0)^2/w_0^2}$ . The graphs correspond to different values of  $w_0$ , with identical  $x_0$ .

If furthermore  $h(x, t)$  can be expressed as a function of  $x^\alpha t^\beta$ ,  $\alpha, \beta \in \mathbb{C}$ ,  $\beta \neq 0$  :  $h(x, t) = \psi(x^\alpha t^\beta)$ , then :

$$\begin{aligned} \Psi(t) &= \sum_{n=1}^N \int_F \frac{1}{b_n} \psi(a_n^\alpha x^\alpha t^\beta) \\ &+ \underbrace{\sum_{n=1}^N \int_F \frac{1}{c_n} \psi((d_n x + e_n)^\alpha t^\beta) d\mu(x)}_{\Phi(t)} \\ &= \sum_{n=1}^N \frac{1}{b_n} \int_F h(x, a_n^{\alpha/\beta} t) d\mu(x) + \Phi(t) \\ &= \sum_{n=1}^N \frac{1}{b_n} \Psi(a_n^{\alpha/\beta} t) + \Phi(t) \end{aligned}$$

Now, assuming that the Mellin transforms of  $\Phi(t)$  and  $\Psi(t)$  (respectively  $m_\Phi(s)$  and  $m_\Psi(s)$ ) exist and have overlapping domains of definition, we can take the Mellin transform of the above equation :

$$m_\Psi(s) = m_\Psi(s) \sum_{n=1}^N \frac{a_n^{-s\alpha/\beta}}{b_n} + m_\Phi(s)$$

thus :

$$m_\Psi(s) = \frac{m_\Phi(s)}{1 - \sum_{n=1}^N \frac{a_n^{-s\alpha/\beta}}{b_n}}$$

Taking the inverse Mellin transform and using the residue formula, we may deduce  $\Psi(t)$  from the poles of  $\frac{m_\Phi(s)}{1 - \sum_{n=1}^N \frac{a_n^{-s\alpha/\beta}}{b_n}}$  (provided that the inverse Mellin transform converges and that we find a proper contour). Two cases in particular are of interest :

- if  $m_\Phi(s)$  has no poles, and if the poles  $\{s_m\}_n$  of

$\frac{1}{1 - \sum_{n=1}^N \frac{a_n^{-s\alpha/\beta}}{b_n}}$  are of the form  $s_n = \gamma + in\gamma'$ , then  $\Psi(t)$  is of the form :

$$\Psi(t) = t^{-\gamma} G_{per}(\ln t)$$

- if  $m_\Phi(s)$  has poles, if the poles  $\{s_m\}_n$  of  $\frac{1}{1 - \sum_{n=1}^N \frac{a_n^{-s\alpha/\beta}}{b_n}}$  are of the form  $s_n = \gamma + in\gamma'$  and if  $\Phi(t)$  goes to zero faster than  $t^{-\gamma}$  when  $t \rightarrow \infty$ , then  $\Psi(t)$  is asymptotically of the form :

$$\Psi(t) \sim_{t \rightarrow \infty} t^{-\gamma} G_{per}(\ln t)$$

More generally, this also occurs whenever we can rearrange the integral in order to have :  $\int_F h(\alpha x, t) d\mu(x) \propto \int_F h(x, \beta t) d\mu(x)$ .

Thus, we see in the case of quantum dynamics that, since the energy  $\epsilon$  and the time  $t$  are conjugate variables, it is likely that a time dependent quantity will be of the form  $t^\alpha G_{per}(\ln t)$ . This is well verified for  $C(t)$  : in this case,  $\Psi(t) = C(t)$ ,  $\Phi(t) = \frac{1}{2} \iint_{K \times K} \text{sinc}\left(\frac{x-x'+2t}{3}\right) d\mu(x) d\mu(x')$ , and we are in the case where  $\Phi(t)$  decreases to zero faster than  $t^{-d\kappa}$ . We also pointed out that time-averaging is equivalent to taking the wavelet transform of the spectrum. For instance, the time-average of the amplitude of the wave function at some point  $x$  in space,  $\langle |\psi(x, t)|^2 \rangle = \frac{1}{t} \int_0^t |\psi(x, t')|^2 dt'$ , is given by :

$$\langle |\psi(x, t)|^2 \rangle = \iint_K g(\epsilon')^* g(\epsilon) \phi_{\epsilon'}(x)^* \phi_\epsilon(x) \text{sinc}((\epsilon - \epsilon')t) d\mu d\mu$$

(this is easily seen by using the following expression for the wave function :

$$|\psi(x, t)| = \left| \int_{\text{space spectrum}} \int e^{-i\epsilon t} \phi(x, \epsilon)^* \phi(x', \epsilon) \psi(x', 0) dx' d\mu(\epsilon) \right|$$

derived in [10]). Since the surface of integration decreases like  $t^{-d\kappa} G_{per}(\ln t)$  as  $t \rightarrow \infty$  (due to the *sinc*), we expect to have a similar time-dependence for the global quantity. Considering that any dynamical quantity can be expressed using the wave function, this implies that this typical behaviour should be common to many quantities. However, more information about the eigenfunctions is usually required to study rigorously the dynamics of the system. We shall illustrate this by discussing now two other quantities : the root-mean displacement (RMS) and the participation ratio.

The time-averaged participation ratio is usually studied to get information on the dynamics of a quantum system. It was studied numerically for the Fibonacci cavity and the results support the idea of a log-periodic feature. By definition, the participation ratio is :

$$P_\psi(t) = \frac{1}{\int_{\text{space}} |\langle x | \psi(t) \rangle|^4 dx}$$

It reflects the localization in space of the wave function  $\psi(x, t)$  : if the state is localized, then  $P_\psi \simeq 0$  ; if it is extended, then  $P_\psi \simeq 1$  [9]. Taking its time-average, to have a time-dependent quantity, we define :

$$\langle P_\psi(t) \rangle = \frac{1}{t} \int_0^t \frac{dt'}{\int_{space} |\langle x | \psi(t') \rangle|^4 dx}$$

The RMS displacement is commonly studied both in quantum dynamics - spreading of the wave packet - and in diffusive processes.

$$\Delta x = \sqrt{\langle (x(t) - x_0)^2 \rangle} = \sqrt{\int_{space} (x(t) - x_0)^2 |\psi(t, x)|^2 dx}$$

Neither of these quantities are pure spectral quantities, as opposed to  $C(t)$  which we managed to express only in terms of the eigenvalues of the system. Here, more information about the eigenfunctions  $\phi_\epsilon$  is required to go further. To do so, we focused on the tight-binding model.

## II

We consider an infinite 1D lattice, with sites  $k$  labelled from 0 to infinity. We define a potential  $V$  on this system such that the discrete Schroedinger operator :

$$H = -\Delta + V$$

which acts on  $l_2(\mathbb{Z})$ , has a triadic Cantor set spectrum. The eigenfunctions of  $H$  verify :

$$H\phi_\epsilon(k) = \epsilon\phi_\epsilon(k)$$

which becomes, in the tight-binding model :

$$-\phi_\epsilon(k+1) - \phi_\epsilon(k-1) + 2\phi_\epsilon(k) + V(k)\phi_\epsilon(k) = \epsilon\phi_\epsilon(k)$$

or, using transfer matrices :

$$\begin{pmatrix} \phi_\epsilon(k+1) \\ \phi_\epsilon(k) \end{pmatrix} = \begin{pmatrix} 2 + V(k) - \epsilon & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_\epsilon(k) \\ \phi_\epsilon(k-1) \end{pmatrix}$$

Imposing  $\phi_\epsilon(0) = 0$ , we find that the eigenfunctions  $\{\phi_\epsilon\}_{\epsilon \in K}$ , evaluated on the sites  $k$ , can be seen as polynomials in  $\epsilon$  :

$$\phi_\epsilon(k) = \phi_\epsilon(1)p_k(\epsilon) \quad (9)$$

where the polynomials  $\{p_k(\epsilon)\}$  are obtained by multiplying the transfer matrices :

$$\begin{pmatrix} p_k(\epsilon) & * \\ p_{k-1}(\epsilon) & * \end{pmatrix} = \prod_{j=1}^k \begin{pmatrix} 2 + V(k-j) - \epsilon & -1 \\ 1 & 0 \end{pmatrix}$$

The  $\{p_k(\epsilon)\}$  verify the recurrence relation :

$$p_{k+1}(\epsilon) = (2 + V(k) - \epsilon)p_k(\epsilon) - p_{k-1}(\epsilon)$$

with initial conditions :  $\begin{cases} p_1(\epsilon) = 1 \\ p_2(\epsilon) = 2 + V(1) - \epsilon \end{cases}$  We are interested in the evolution of a wave packet, described by the normalized wave function  $\psi(t)$ , evolving according to :

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle$$

The wave function is initially localized at the site 1 :  $\langle k | \psi(0) \rangle = \delta(k-1)$ . At time  $t$ , the system is in the state :  $|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$ , and the value of the wave function at site  $k$  is given by :

$$|\psi(k, t)| = \int_K \phi_\epsilon(1)^* \phi_\epsilon(k) e^{-i\epsilon t} d\mu$$

Using (8), we obtain :

$$\begin{aligned} \psi(k, t) &= \int_K \phi_\epsilon(1)^* \phi_\epsilon(k) e^{-i\epsilon t} d\tilde{\mu}(\epsilon) \\ &= \int_K |\phi_\epsilon(1)|^2 p_k(\epsilon) e^{-i\epsilon t} d\tilde{\mu}(\epsilon) \\ &= \sum_{j \leq k} c_{j,k} \int_K \epsilon^j e^{-i\epsilon t} d\mu(\epsilon) \end{aligned}$$

with  $p_k(x) = \sum_{0 \leq j \leq k} c_{j,k} x^j$  and where we defined, as in the previous section, the local measure associated to the state  $|\psi\rangle$  :  $d\mu(\epsilon) = |\phi_\epsilon(1)|^2 d\tilde{\mu}(\epsilon)$ . Thus, the dynamical behaviour of the system depends essentially on the functions :

$$h_k(t) = \int_K e^{-i\epsilon t} \epsilon^k d\mu(\epsilon)$$

aka the Fourier transforms on  $K$  of the moments  $x^k$ . The wave function is therefore given by :

$$\psi(k, t) = \sum_{j \leq k} c_{j,k} h_j(t) \quad (10)$$

Expanding the exponential in  $h_k(t)$ , we obtain :

$$h_k(t) = \sum_{n \geq 0} \frac{(-it)^n}{n!} \mu_{n+k} \quad (11)$$

with  $\mu_n = \int_K x^n d\mu$ .

The fingerprint of the scaling symmetry of the spectrum is carried by the  $\mu_n = \int_K x^n d\mu$ . It can be shown that, for  $n \rightarrow \infty$  (see [20])

$$\mu_n = n^{-\ln 2 / \ln 3} \mathcal{F} \left( \frac{\ln n}{\ln 3} \right) \left( 1 + \mathcal{O} \left( \frac{1}{n} \right) \right)$$

where  $\mathcal{F}$  is a log-periodic function of period 1.

The  $h_k(t)$  do not verify a scaling property (of the form  $f(x) = bf(ax)$ ) ; however, there is a structure and there

are some scaling invariance.

First, the  $h_k(t)$  verify the recurrence relation :

$$\begin{aligned} h_k(t) &= \int_K e^{-i\epsilon t} \epsilon^k d\mu \\ &= \frac{3^{-k}}{2} \int_K e^{-i\epsilon t/3} \epsilon^k d\mu + \frac{3^{-k} e^{-i2t/3}}{2} \int_K e^{-i\epsilon t/3} (\epsilon + 2)^k d\mu \\ &= \frac{3^{-k}}{2} h_k(t/3) + \frac{3^{-k}}{2} e^{-i2t/3} \sum_{j=0}^k \binom{k}{j} h_j(t/3) 2^{k-j} \end{aligned}$$

From this relation, we can show by recurrence that the zeroes of  $h_k(t)$  is the set :

$$Z_k = \{t_{m,k} = \frac{\pi}{2}(2m+1)3^{k+1}, m \in \mathbb{Z}\}$$

Thus, the zeroes of  $h_k(t)$  are those of  $h_{k-1}(t/3)$ , as illustrated in fig. (6).

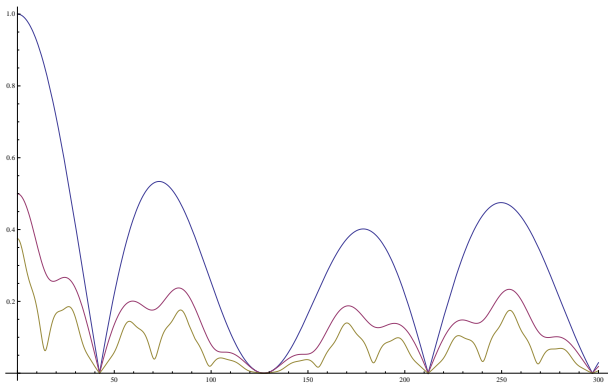


FIG. 6: Numerical curves of  $|h_0(t/9)|$  (blue),  $|h_1(t/3)|$  (red) and  $|h_2(t)|$  (yellow).

Moreover, the  $h_k(t)$  all have local maxima at points of the form  $m_{n,l} = n\pi 3^l$ . It was not proven rigorously for  $k > 0$  (the case  $k = 0$  is straightforward, the detail is given in the appendix), but the numerics support this assumption and one can intuitively understand why it is so from the definition of  $h_k(t)$  :

$$h_k(t) = \int_K x^k e^{-ixt} d\mu = \lim_{n \rightarrow \infty} \frac{3^n}{2^n} \sum_{a_{j_n} \in K_n} \int_{a_{j_n}}^{a_{j_n} + 3^{-n}} x^k e^{-ixt} dx$$

$|h_k(t)|$  is maximal at points  $t$  such that the phase  $e^{-ixt}$  varies as little as possible as  $x$  goes through  $K$ .  $K$  is composed of real numbers which have only 0 and 2 in their decomposition in base 3, e.g.  $x = \sum_{k \geq 1} \frac{\beta_k}{3^k}$  with  $\beta_k \in \{0, 2\}$ . Now, for  $t = n\pi 3^l$ , we see that for any  $x \in K$  whose decomposition in base 3 has only 0 after the  $l$  first terms, we have  $e^{-ixt} = 1$  : the phase does not change.

Using (11), we can show that, at these local maxima, the amplitude of the functions  $h_k(t)$  decreases with  $k$ . We wish now to examine how this structure translates in terms of physical dynamical quantities.

Although the  $h_k(t)$  are not scaling functions, the relations between the zeroes of  $h_k$  and the behaviour at the local maxima justify the numerical observation that, approximately,  $h_k(t) \propto h_{k-1}(t/3)$ .

This almost scaling relation translates to the RMS. In fact, since  $\psi(k, t) = \sum_{j \leq k} c_{j,k} h_j(t)$ , we can deduce that :  $\psi(k, 3t) \approx \sum_{j \leq k} c_{j,k} h_{j-1}(t) \approx \psi(k-1, t)$ . Thus :

$$\begin{aligned} \Delta x(3t)^2 &= \sum_{k \leq 1} k^2 |\psi(k, 3t)|^2 \\ &\approx \sum_{k \geq 1} k^2 |\psi(k-1, t)|^2 \\ &\approx \Delta x(t)^2 \end{aligned}$$

This implies that  $\Delta x(t)^2$  should display a log-periodic structure. Numerically,  $\Delta x(t)$  is a power-law modulated by oscillations, which have a log-periodic envelope of period  $\ln 3$  ; inside the envelope, the period of the oscillations increases with time, which is consistent with the behaviour of  $h_k(t)$  under the transformation  $t \rightarrow \frac{t}{3}$ . The numerics show that :  $\Delta x(t) \approx \sqrt{2} \Delta x(t/3)$  (see fig. (7)).

However, it is difficult to derive exactly the expression of  $\Delta x(t)^2$ , since it requires more information on the wave function  $\psi(k, t)$  (more precisely, on the coefficients  $c_k$ ). More can be said about the dynamics if we take the average in time, as we shall now see.

We study first  $\frac{1}{t} \int_0^t |\psi(k, t')|^2 dt'$ , which narrows down to the study of the quantities :

$$\frac{1}{t} \int_0^t h_k^*(t') h_l(t') dt'$$

Since  $|\psi(k, t)|^2$  is real, we need only to consider the real part of  $\frac{1}{t} \int_0^t h_k^*(t') h_l(t') dt'$ , which we note  $f_{k,l}(t)$ . A standard calculation leads to :

$$\begin{aligned} f_{k,l}(t) &= \Re \left( \frac{1}{t} \int_0^t h_k^*(t') h_l(t') dt' \right) \\ &= \iint_{K \times K} x^k x^l \text{sinc}((x' - x)t) d\mu(x) d\mu(x') \end{aligned}$$

Note that  $f_{0,0}(t)$  is the time-averaged return probability  $C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$ , which we derived earlier.

For the calculation of  $C(t)$ , we used the change of variable  $l = |x - x'|$  and reduced the problem to calculating the area  $dS(l, \mu)$ . The log-periodic behaviour comes from the expression of  $dS(l, \mu)$ .

If we make the approximation that :  $\text{sinc}(xt) = 1$  for  $|xt| < \pi$  and  $\text{sinc}(xt) = 0$  elsewhere, we get the following approximate expression for  $C(t)$  :

$$C(t) = \int_{l=0}^{\pi/t} dS(l, \mu) = t^{-\ln 2 / \ln 3} \sum_n \alpha_n t^{-2i\pi n / \ln 3}$$



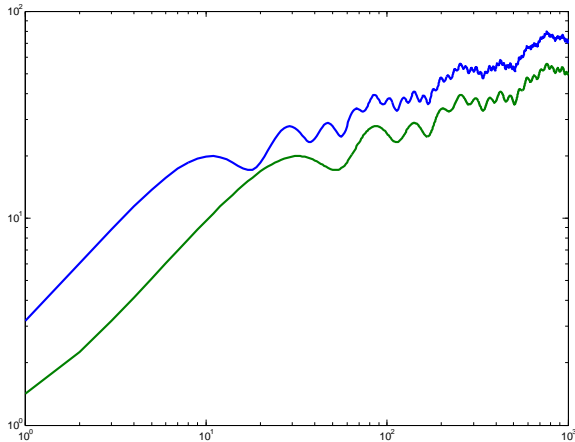


FIG. 7: RMS for a triadic Cantor spectrum, in loglog scale ; blue :  $\Delta x(t)$ , green :  $\Delta x(t/3)$ .

which accounts for the typical asymptotic behaviour, although the coefficients  $\alpha_n$  are a little bit different from the coefficients in the exact expression.

If we apply the same reasoning for

$$f_{k,l}(t) = \iint_{K \times K} x^k x'^l \text{sinc}((x' - x)t) d\mu(x) d\mu(x')$$

we find :

$$f_{k,l}(t) \approx \iint_{|x-x'| < \frac{\pi}{t}} x^k x'^l d\mu(x) d\mu(x')$$

Applying now a similar reasoning as for  $C(t)$  (see appendix), we find that :

$$\iint_{|x-x'| < \frac{\pi}{t}} x^k x'^l d\mu(x) d\mu(x') = t^{-d_K} \left[ G_{k,l}(\ln t) + \mathcal{O}\left(\frac{1}{t}\right) \right] \quad (12)$$

with  $G_{k,l}(\ln t) = \sum_{n \in \mathbb{Z}} \frac{t^{2i\pi n / \ln 3}}{\alpha - 2i\pi n / \ln 3} c_{0,n,k,l}$  a log-periodic function of period  $\ln 3$ . Thus,  $f_{k,l}(t)$  satisfies, asymptotically, the scaling relation (see fig. (10)) :

$$f_{k,l}(t) \sim_{t \rightarrow \infty} \frac{1}{2} f_{k,l}(t/3)$$

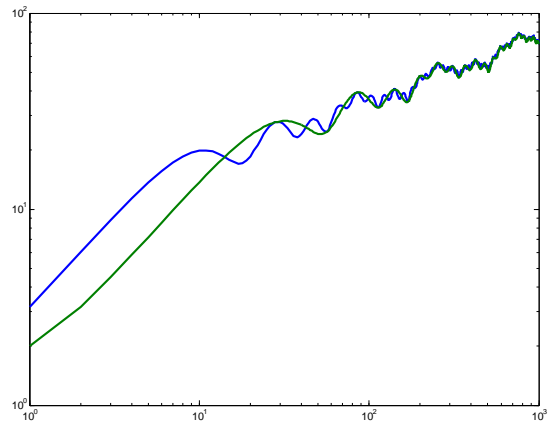


FIG. 8: RMS for a triadic Cantor spectrum, in loglog scale ; blue :  $\Delta x(t)$ , green :  $\sqrt{2}\Delta x(t/3)$ .

Since the  $G_{k,l}(\ln t)$  are all log-periodic with the same period, we conclude that the wave function has the following asymptotic behaviour :

$$\frac{1}{t} \int_0^t |\psi(k, t')|^2 dt' \sim t^{-d_K} G_{per}(\ln t)$$

with  $G_{per}(\ln t)$  of period  $\ln 3$  (see fig. (11)).

Let us now study the time-averaged RMS :

$$\langle \Delta x^2(t) \rangle = \frac{1}{t} \int_0^t \Delta x^2(t') dt' = \sum_{k \geq 0} k^2 \langle |\psi(k, t)|^2 \rangle$$

Since  $\langle |\psi(k, t)|^2 \rangle$  has, asymptotically, a log-periodic component, so does  $\langle \Delta x^2(t) \rangle$ . A scaling analysis adapted from [12] shows furthermore that :  $\langle \Delta x^2(t) \rangle \sim t^{d_K}$ , which is consistent with the numerics (fig. (12)). Thus, we find that the time-averaged RMS is a scaling function, and has the asymptotic form :

$$\langle \Delta x^2(t) \rangle \sim_{t \rightarrow \infty} t^{d_K} G_{per}\left(\frac{\ln t}{\ln 3}\right)$$

To finish, we discuss briefly the participation ratio. The numerics suggest that the instantaneous participation ratio does not seem to have a log-periodic structure, but its time-average displays a log-periodicity (see fig. (9)).

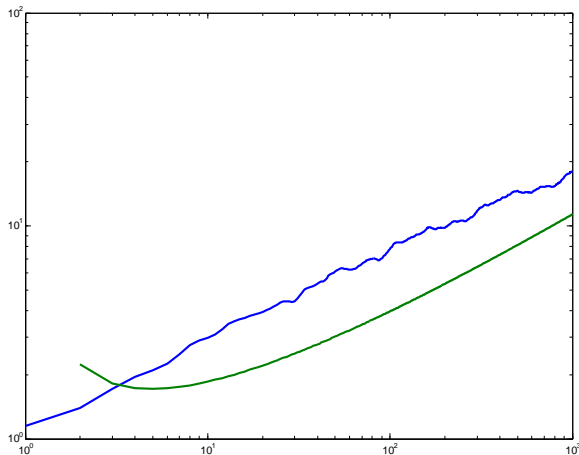


FIG. 9: Time-averaged participation ratio for a triadic Cantor spectrum (blue) ; lower-bound  $t^{-\ln 2/\ln 3}/\ln t$  (green) ; loglog scale.

### Conclusion

We identified the fingerprint of a self-similar spectrum in the dynamics of a quantum system. Due to the scaling symmetry of the spectrum, some physical dynamical quantities are scaling functions, meaning they are of the form :  $f(t) = t^\alpha G_{per}(\ln t)$  with  $\alpha \in \mathbb{R}$  and  $G_{per}$  periodic. Thus, the scaling symmetry of the spectrum is visible in dynamical quantities through the presence of a power-law modulated by log-periodic periodic oscillations. Using the Mellin transform, we derived the analytical expression of the time-averaged return probability  $C(t)$ , which is universal for any system with a Cantor spectrum. We found that  $C(t)$  is asymptotically a scaling function, of the form :

$$C(t) =_{t \rightarrow \infty} t^{-d_K} G_C(\ln t)$$

where  $d_K = \frac{\ln 2}{\ln 3}$  is the fractal dimension of the triadic Cantor set and  $G_C$  a periodic function which we calculate exactly. This result is supported by numerical results obtained recently for polaritons in a Fibonacci potential.

We gave a more thorough analysis in the case of the tight-binding model, in which case the dynamics of the system depend essentially on the Fourier transform

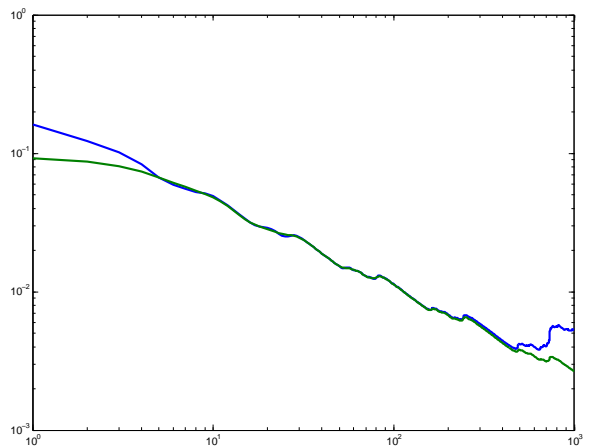


FIG. 10: Loglog scale graphs of  $f_{1,2}(t)$  (green) and  $\frac{1}{2}f_{1,2}(t/3)$  (blue)

on a Cantor set of monomials  $x^k$ , namely :  $h_k(t) = \int_K e^{-ixt} x^k d\mu$ ,  $k \in \mathbb{Z}$ , which carry the fingerprint of the spectral fractal feature. We showed that, in this model, the time-average of the wave function has, asymptotically, the expression of a scaling function, which accounts for the log-periodic behaviour of the time-averaged RMS and participation ratio.

### Appendix

#### Maxima of $h_0(t)$ for the triadic Cantor set

Using the scaling symmetry of the spectrum :

$$\int_k f(x) d\mu(x) = \frac{1}{2} \int_K f\left(\frac{x}{3}\right) d\mu(x) + \frac{1}{2} \int_K f\left(\frac{x+2}{3}\right) d\mu(x) \quad (13)$$

we find that :

$$\begin{aligned} h_0(t) &= \int_K e^{-iet} d\mu \\ &= \frac{1}{2} \int_K e^{-iet/3} d\mu + \frac{e^{-i2t/3}}{2} \int_K e^{-iet/3} d\mu \\ &= \frac{1 + e^{-i2t/3}}{2} h_0(t/3) \end{aligned}$$

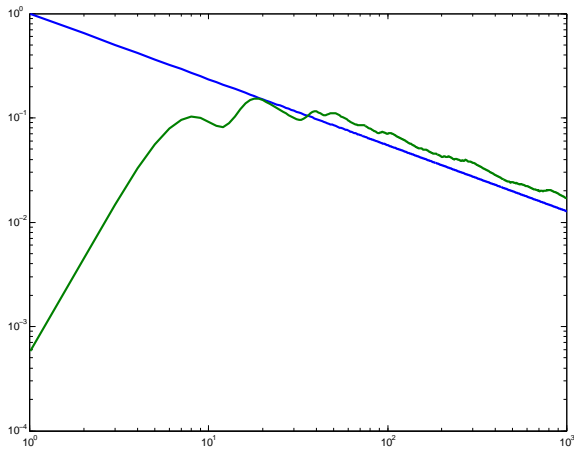


FIG. 11: Loglog scale graphs of  $|\psi(3,t)(t)|^2$  (green) and  $t^{-\ln 2/\ln 3}$  (blue)

By repeating this process, and using  $h_0(0) = \int_K 1d\mu = 1$ , we find the exact expression of  $h_0(t)$  :

$$h_0(t) = \prod_{k \geq 1} \frac{1 + e^{-i2t/3^k}}{2} \quad (14)$$

thus :

$$|h_0(t)| = \prod_{k \geq 1} \cos t/3^k \quad (15)$$

The derivative :

$$|h_0(t)|' = - \sum_{k \geq 1} 3^{-k} \sin t/3^k \prod_{j \neq k} \cos t/3^j \quad (16)$$

is null for  $t$  of the form :  $t = m_{n,l} = \pi n 3^l$ , with  $n \in \mathbb{Z}$  and  $l \in \mathbb{N}^*$ .

**Derivation of**  $\iint_{|x-x'| < \frac{\pi}{t}} x^k x'^l d\mu(x) d\mu(x')$

Let us note :  $r_{k,l}(t) = \iint_{|x-x'| < \frac{\pi}{t}} x^k x'^l d\mu(x) d\mu(x')$ . We give a full analytical expression for  $r_{k,l}(t)$  and show

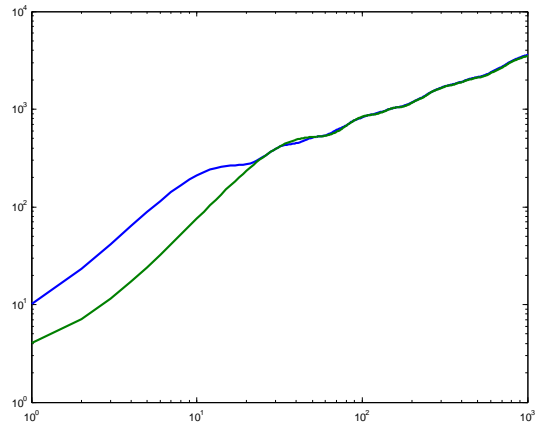


FIG. 12: Loglog scale graphs of  $\langle \Delta x^2(t) \rangle$  (blue) and  $2\langle \Delta x^2(t/3) \rangle$  (green)

that :

$$r_{k,l}(t) = t^{-\ln 2/\ln 3} \left[ G(\ln t) + \mathcal{O}\left(\frac{1}{t}\right) \right]$$

We follow a similar method as for  $C(t)$ .

First, we calculate the infinitesimal area  $dS_{k,l}(u, \mu)$ , which is the integral of  $x^k y^l$ , for the measure  $d\mu$ , on the two strips  $y = x + u$  and  $y = x - u$  (red strips on fig. 1) :

$$dS_{k,l}(u, \mu) = \int_K (x^k (x+u)^l \theta(1-u-x) + x^k (x-u)^l \theta(x-u)) d\mu(x)$$

To derive  $dS_{k,l}(u, \mu)$ , we study the poles of its Mellin transform and apply the residue theorem. The Mellin transform is :

$$m_{k,l}(s) = \int_0^1 u^{s-1} dS_{k,l}(u, \mu) = \iint_{K \times K} x^k y^l |x-y|^{s-1} d\mu d\mu$$

We now show by recurrence that the poles of  $m_{k,l}(s)$  is the set :

$$S_{k,l} = \left\{ 1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3}; j \in [0, k+l], n \in \mathbb{Z} \right\}$$

Initialization :

$$m_{0,0}(s) = \iint_{K \times K} |x-y|^{s-1} has been calculated for C(t)$$

and has poles at

$$S_{0,0} = \left\{ 1 - \frac{\ln 2}{\ln 3} + \frac{2i\pi n}{\ln 3}; n \in \mathbb{Z} \right\}$$

Inductive step :  
using the symmetry of the measure, we find :

$$\begin{aligned} m_{k,l}(s) &= \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu \\ &= \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu \\ &+ \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} (x+2)^k (y+2)^l |x - y|^{s-1} d\mu d\mu + R_{k,l}(s) \end{aligned}$$

with

$$\begin{aligned} R_{k,l}(s) &= \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} (x+2)^k y^l |x - y + 2|^{s-1} d\mu d\mu \\ &+ \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} x^k (y+2)^l |x - y - 2|^{s-1} d\mu d\mu \end{aligned}$$

which has no poles. Expanding the second term :

$$\begin{aligned} m_{k,l}(s) &= \frac{3^{1-s-(k+l)}}{4} \iint_{K \times K} x^k y^l |x - y|^{s-1} d\mu d\mu \\ &+ \frac{3^{-(k+l)}}{3^{s-1} 4} \sum_{i=0}^k \sum_{j=0}^l \binom{k}{i} \binom{l}{j} 2^{k+l-i-j} \iint x^i y^j |x - y|^{s-1} d\mu d\mu \\ &+ R_{k,l}(s) \\ &= \frac{3^{1-s-(k+l)}}{2} m_{k,l}(s) \\ &+ \frac{3^{1-s-(k+l)}}{4} \sum_{i=1}^k \sum_{j=1}^l \binom{k}{i} \binom{l}{j} 2^{k+l-i-j} m_{i,j}(s) + R_{k,l}(s) \end{aligned}$$

Thus, the poles of  $m_{k,l}(s)$  is the set of the poles of all the  $m_{i,j}(s)$  with  $i \leq k$  and  $j \leq l$ , plus the set of complex points  $\left\{ 1 - \frac{\ln 2}{\ln 3} - (k+l) + \frac{2i\pi n}{\ln 3}; n \in \mathbb{Z} \right\}$ . Using the induction hypothesis, we conclude that the poles of  $m_{k,l}(s)$  is the set :

$$S_{k,l} = \left\{ 1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3}; j \in [0, k+l], n \in \mathbb{Z} \right\}$$

which completes the proof.

Now, we apply the inverse Mellin transform and use the residue formula to find  $dS_{k,l}(u, \mu)$  :

$$\begin{aligned} dS_{k,l}(u, \mu) &= \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} u^{-s} m_{k,l}(s) ds \\ &= \sum_{j=0}^{k+l} \sum_{n \in \mathbb{Z}} c_{j,n} u^{j+\alpha-1-2i\pi n/\ln 3} \end{aligned}$$

with  $\alpha = \frac{\ln 2}{\ln 3}$  and

$$\begin{aligned} c_{j,n,k,l} &= R_{k,l} \left( 1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3} \right) \\ &+ \sum_{i=1}^k \sum_{i'=1}^l \binom{k}{i} \binom{l}{i'} 2^{k+l-i-j} m_{i,j} \left( 1 - \frac{\ln 2}{\ln 3} - j + \frac{2i\pi n}{\ln 3} \right) \end{aligned}$$

By integration, we obtain  $r_{k,l}(t)$  :

$$\begin{aligned} r_{k,l}(t) &= \int_0^{1/t} dS(u, \mu) \\ &= \sum_{j=0}^{k+l} \sum_{n \in \mathbb{Z}} \int_0^{1/t} c_{j,n,k,l} u^{j+\alpha-1-2i\pi n/\ln 3} \\ &= t^{-\alpha} \sum_{j=0}^{k+l} \sum_{n \in \mathbb{Z}} c_{j,n,k,l} \frac{t^{-j+2i\pi n/\ln 3}}{\alpha + j - 2i\pi n/\ln 3} \end{aligned}$$

Thus :

$$r_{k,l}(t) = t^{-\alpha} \left[ G(\ln t) + \mathcal{O}\left(\frac{1}{t}\right) \right]$$

with  $G(\ln t) = \sum_{n \in \mathbb{Z}} \frac{t^{2i\pi n/\ln 3}}{\alpha - 2i\pi n/\ln 3} c_{0,n,k,l}$ .

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<sup>1</sup> obtained by iteration of the following process : start with a full square, divide it in 9 equal squares and keep only the

four at the corners of the initial square