Quantum dynamics for a triadic Cantor set spectrum

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(Dated: June 1, 2015)

We study the dynamics of a wave packet for a triadic Cantor spectrum K. The scaling symmetry of the spectrum translates into the characteristic log-periodic feature of relevant physical quantities. Using Mellin transform, we derive the analytical expression of the time-averaged return probability, which is universal for any system with a Cantor spectrum. This result is supported by numerical results obtained recently for polaritons in a Fibonacci potential. We give a more thorough analysis in the case of the tight-binding model, in which case the dynamics of the system depend essentially on the Fourier transform on a Cantor set of monomes x^k, namely : $h_k(t) = \int_K e^{-ixt}x^k d\mu$, $k \in \mathbb{Z}$, which carry the fingerprint of the spectral fractal feature.

I. INTRODUCTION

During the past decades, there has been a growing interest for Schroedinger operators $H = -\frac{\hbar^2}{2m}\Delta + V(\mathbf{r})$ with a Cantor set like spectrum, e.g. a perfect, nowhere dense set (for a review, see [1], and more recently [2]). In the context of quantum dynamics for systems verifying such a Hamiltonian, a typical behaviour for some physical quantities, such as the time-averaged return probability (auto-correlation function), has been identified. More precisely, it has been shown ([3],[4]) that, in the case of a Cantor set spectrum, the time-averaged return probability $C(t)$, defined as : $C(t) = \frac{1}{t}\int_0^t |\langle\psi(0)|\psi(t')\rangle|^2 dt'$ decreases as : $t^{-D_2^\mu}$, where D_2^μ is a correlation dimension with respect to the measure μ of the spectrum, although a full analytical expression had not been derived so far. The spatial correlation function $C_s(l,\mu) = \int_{K\times K} \theta(l-|x-y|)d\mu(x)d\mu(y)$, with θ the Heaviside function, has been derived more precisely for a Cantor set spectrum ([5]), showing that this quantity is a power law t^{d_K} (d_K being the fractal dimension of the Cantor set) modulated by log-periodic oscillations.

In this paper, we study the dynamics of a quantum system whose energy spectrum is a triadic Cantor set K. In section I, we describe the scaling symetry of the spectrum and analyze how it translates in terms of measurable physical quantities ; we show, using Mellin transform, that the return probability has the form :

$$C(t) = \left(\frac{t}{\tau}\right)^{-d_K} g(\ln(t/\tau))$$

where $d_K = \frac{\ln(2)}{\ln(3)}$ is the fractal dimension [7] of the triadic Cantor set, τ a time scale a g a periodic function, which is typical for a fractal spectrum, and we derive its analytical expression. We also present numerical results obtained with polaritons in a Fibonacci cavity [6], which support the theory. In section II, we narrow our study to the tight-binding model ; we show that in this case, the eigenfunctions can be seen as polynomials of the energy, and that the dynamics depend essentially on the Fourier transform (on a Cantor set) of monomes : $h_k(t) = \int_K e^{-ixt}x^k d\mu$, $k \in \mathbb{Z}$. In fact, the time dependent Green function is a linear combination of the $h_k(t)$. As examples we consider two relevant quantities to characterize the dynamics of the system, the root mean displacement (RMS), $\Delta x(t) = \sqrt{\sum_k k^2|\psi(k,t)|^2}$, and the participation ratio, $¶_r(t) = \frac{1}{\sum_k |\psi(k,t)|^4}$.

II. I

The triadic Cantor set, denoted K in this paper, is constructed as follows : start with the segment [0,1], divide it in three equal parts, and remove the central one, e.g. the segment $[1/3, 2/3]$, then repeat the process with the two remaining segments, and so on. After an infinite number of iterations, we are left with a set of separated points, of zero Lebesgue measure.

We now consider a one dimensional quantum system described by the state vector $|\psi(t)\rangle$ and evolving with the Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. We assume that the energy spectrum of H is a triadic Cantor set. Note that, since all Cantor sets are homeomorphic to the triadic Cantor set ([17]), the results and methods can be adapted to any Cantor set spectrum.

The system is initially in a state described by $|\psi(t = 0)\rangle = |\psi_0\rangle$. After a time t, the state vector $|\psi(t)\rangle$ is given by :

$$|\psi(t)\rangle = e^{-\mathrm{i}\hat{\mathrm{H}}\mathrm{t}/\hbar}|\psi_0\rangle$$

We shall also use the spectral decomposition of $|\psi(t)\rangle$ over an orthonormal basis of eigenfunctions $\{|\phi(\epsilon)\rangle\}$ of H,$|\phi(\epsilon)\rangle$ being the eigenfunction associated with the energy ϵ :

$$|\psi(t)\rangle = \int_{spectrum} g(\epsilon)e^{-i\epsilon t/\hbar}|\phi(\epsilon)\rangle d\tilde{\mu}(\epsilon)$$

where $g(\epsilon)$ is the projection of $|\psi_0\rangle$ on $|\phi(\epsilon)\rangle$, and $d\tilde{\mu}$ a measure defined on the spectrum.

To go further, we need to define an appropriate measure on the spectrum. The triadic Cantor set K has a zero Lebesgue measure. However, in our case, the integrated density of states, e.g. the integral of $\rho(\epsilon)$ over the

spectrum K (we shall generalize to the case of an infinite Cantor set later), is non zero. Thus, $\rho(\epsilon)$ must be infinite in every point of K.

We define a measure $d\mu$ on K, such that the integrated density of states $\mathcal{N}(\epsilon)$ is given by :

$$\mathcal{N}(\epsilon) = \int_K \theta(\epsilon - \epsilon')d\mu(\epsilon')$$

with θ the Heaviside function. We furthermore suppose that the density of states is uniform.

Without the loss of generality, we suppose that the integral of $d\mu$ over K equals 1 : $\int_K d\mu = 1$. We then proceed iteratively : starting from a segment $[0,1]$, we set $\rho(\epsilon) = 1$ everywhere ; the integral thus equals 1. Now, divide the segment in three, remove the central part and define $\rho(\epsilon)$ to be equal to $\frac{3}{2}$ on $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$, and null on $[\frac{1}{3}, \frac{2}{3}]$. We still have : $\int_0^1 \rho(x)dx = 1$. We repeat this process infinitely, and thus define $d\mu$ as a limit : for any function f defined on $[0,1]$, the integral of f over K is :

$$\int_K f(x)d\mu(x) = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n \sum_{a_{j_n}\in P_n} \int_{a_{j_n}}^{a_{j_n}+3^{-n}} f(x)dx$$

where $\{P_n\}$ is the set of the left edges of the remaining segments after n iterations in the construction of the Cantor set.

One can show (see annexe) that $d\mu$ verifies the following important property :

$$\int_K f(x)d\mu(x) = \frac{1}{2}\int_K f\left(\frac{x}{3}\right)d\mu(x) + \frac{1}{2}\int_K f\left(\frac{x+2}{3}\right)d\mu(x) \tag{1}$$

This property still holds for multiple integrals ; it becomes, for a double integral :

$$\begin{aligned}
&\int_{K\times K} f(x,y)d\mu(x)d\mu(y) \\
&= \frac{1}{4}\int_{K\times K} f\left(\frac{x}{3},\frac{y}{3}\right)d\mu(x)d\mu(y) \\
&+ \frac{1}{4}\int_{K\times K} f\left(\frac{x+2}{3},\frac{y}{3}\right)d\mu(x)d\mu(y) \\
&+ \frac{1}{4}\int_{K\times K} f\left(\frac{x}{3},\frac{y+2}{3}\right)d\mu(x)d\mu(y) \\
&+ \frac{1}{4}\int_{K\times K} f\left(\frac{x+2}{3},\frac{y+2}{3}\right)d\mu(x)d\mu(y)
\end{aligned}$$

Note that the measure μ is a special case of a self-similar measures m, e.g. such that there exists a set of contractive similarities $\{\phi_j\}$ and probability weights $\{\pi_j\}$ such that for any continuous function f, one has :

$$\int f(x)dm(x) = \sum_j \pi_j \int f(\phi_j(x))dm(x)$$

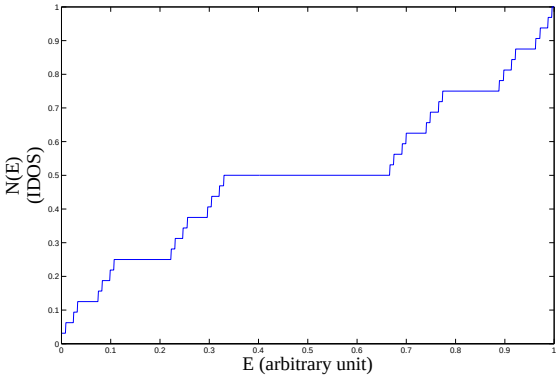

FIG. 1: Integrated density of states (IDOS) in the case of a triadic Cantor set energy spectrum. The energies are expressed in arbitrary unit and the IDOS has been normalised.

Cantor sets correspond to the case of two linear similarities, with $\pi_1 = \pi_2 = \frac{1}{2}$:

$$\int f(x)dm(x) = \frac{1}{2}\int f(a_1x)dm(x) + \frac{1}{2}\int f(a_2x+b)dm(x)$$

The methods developed hereafter can easily be translated for any self-similar measure.

III.

Let us now study the dynamics of the system described in the previous paragraph. There are several physical quantities which are useful to characterize the evolution of the system, such as the RMS displacement, the participation ratio or the auto-correlation function.

We will focus here on the auto-correlation function, as it is a very instructive example to understand how the scaling symmetry of the spectrum affects the dynamics of the system.

The auto-correlation function, or time averaged return probability, $C(t)$, is by definition the time average of the probability $p(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ to find the system in its initial state after a time t :

$$C(t) = \frac{1}{t}\int_0^t |\langle\psi(0)|\psi(t')\rangle|^2 dt'$$

We could directly try to calculate $p(t)$ to study the evolution in time of the system ; however this quantity often exhibits fast local oscillations, which could hide the effects of the fractal feature. Its time average, on the other hand, yields interesting results, as we shall see.

We start with the spectral decomposition of $|\psi(t)\rangle$:

$$|\psi(t)\rangle = \int_{spectrum} g(\epsilon)e^{-i\epsilon t/\hbar}|\phi(\epsilon)\rangle d\tilde{\mu}(\epsilon)$$

The first step is to obtain an integral over K, with a dimensionless variable, in order to use the measure introduced earlier.

We impose that $|\psi(t)\rangle$ is normalized to unity :

$$\langle\psi(t)|\psi(t)\rangle = \int_{spectrum} |g(\epsilon)|^2 d\tilde{\mu}(\epsilon) = 1$$

We shall assume that $|g(\epsilon)|$ is non-zero and uniform on a bounded subset of the spectrum, and that this subset is triadic Cantor set. We therefore consider that $|g(\epsilon)|^2 d\tilde{\mu}(\epsilon)$ (the spectral measure of the initial state) is the measure $d\mu$ introduced in the previous paragraph.

Thus :

$$\begin{aligned} C(t) &= \tfrac{1}{t}\int_0^t |\langle\psi(0)|\psi(t')\rangle|^2 dt' \\ &= \tfrac{1}{t}\int_0^t |\int_{K_{\epsilon_{max}}} |g(\epsilon)|^2 e^{-i\epsilon t'/\hbar} d\mu_{\epsilon_{max}}(\epsilon)|^2 dt' \end{aligned}$$

A standard calculation then leads to :

$$C(t) = \int_{K\times K} \mathrm{sinc}((\lambda'-\lambda)\epsilon_{max} t/\hbar) d\mu(\lambda) d\mu(\lambda')$$

We now make the change of variables : $u = \frac{\epsilon_{max} t}{\hbar}$:

$$C(t) = C(u) = \int_{K\times K} \mathrm{sinc}((\lambda'-\lambda)u) d\mu(\lambda) d\mu(\lambda')$$

The idea is now to transform this double integral into a one variable integral, and then to use Mellin transform and the scaling property (1) of $d\mu$.

Let us do the change of variables : $l = |\lambda' - \lambda|$, $0 \leq l \leq 1$, and let $dS(l,\mu)$ be the surface (in the sense of $d\mu$) of the 2 dimensional Cantor dust[1] contained in the two strips of infinitesimal width located at $\lambda = \lambda' + l$ and $\lambda = \lambda' - l$ (fig. 1).

Since $\mathrm{sinc}((\lambda'-\lambda)t)$ can be considered constant on these strips, we obtain :

$$C(u) = \int_0^1 \mathrm{sinc}(lu) dS(l,\mu) \qquad (2)$$

We now have to determine $dS(l,\mu)$. For this we shall use Mellin transforms.

Let : $M_{dS}(s) = \int_0^1 l^{s-1} dS(l,\mu) \equiv \int_0^1 l^{s-1} \frac{dS(l,\mu)}{dl} dl$ be the Mellin transform of $dS(l,\mu)$, which is sometimes referred to in the literature as the energy integral ([5], [8]) (note that we wrote $\int_0^1$ and not $\int_0^\infty$: $dS(l,\mu)$ is defined on $[0,1]$, but can be extended on $[0,\infty[$ if we define it as equal to 0 outside $[0,1]$).

Now, using the definition of $dS(l,\mu)$ and the property

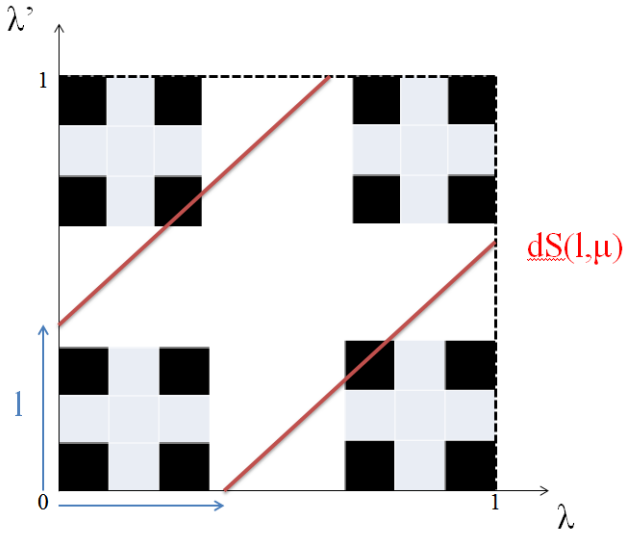

FIG. 2: 2 dimensional Cantor dust (represented here at the second step of construction) ; the infinitesimal surface $dS(l,\mu)$ corresponds to the surface of the Cantor dust, in the sense of $d\mu$, contained in the two red strips.

(1) :

$$\begin{aligned} M_{dS}(s) &= \int_0^1 l^{s-1} dS(l,\mu) \\ &= \int_{K\times K} |x-y|^{s-1} d\mu(x) d\mu(y) \\ &= \frac{1}{4}\int_{K\times K} \left(|\frac{x}{3} - \frac{y}{3}|^{s-1} + |\frac{x+2}{3} - \frac{y+2}{3}|^{s-1} \right) d\mu(x) d\mu(y) \\ &+ \frac{1}{4}\int_{K\times K} \left(|\frac{x}{3} - \frac{y+2}{3}|^{s-1} + |\frac{x+2}{3} - \frac{y}{3}|^{s-1} \right) d\mu(x) d\mu(y) \end{aligned}$$

$$\begin{aligned} M_{dS}(s) &= \frac{3^{1-s}}{2}\int_{K\times K} |x-y|^{s-1} d\mu(x) d\mu(y) \\ &+ \underbrace{\frac{3^{1-s}}{2}\int_{K\times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)}_{\gamma(s)} \end{aligned}$$

Thus :

$$M_{dS}(s) = \frac{\gamma(s)}{1 - 3^{1-s}/2}$$

with $\gamma(s) = \frac{3^{1-s}}{2}\int_{K\times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)$.

The inverse Mellin transform then gives $\frac{dS(l,\mu)}{dl}$:

$$\begin{aligned} \frac{dS(l,\mu)}{dl} &= \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} M_{dS}(s) l^{-s} ds \\ &= \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\gamma(s)}{1-\frac{3^{1-s}}{2}} l^{-s} ds \end{aligned}$$

It is important to note that $\gamma(s)$ has no poles. It is clear from its definition :

$$\gamma(s) = \frac{3^{1-s}}{2} \int_{K\times K} |x-y+2|^{s-1} d\mu(x) d\mu(y)$$

Since for all $x, y \in [0,1]$, the inequality $1 \leq 2-x+y \leq 3$ holds, $|x-y+2|^{s-1}$ is well defined and bounded on $K \times K$ for any $s \in \mathbb{C}$, and thus $\gamma(s)$ is well defined and has no poles. The only poles of $\frac{\gamma(s)}{1-\frac{3^{1-s}}{2}} l^{-s}$ are the $\{s_n\}_{n\in\mathbb{Z}}$ such that : $1 - s_n = \frac{\ln(2)}{\ln(3)} + \frac{2i\pi n}{\ln(3)}$. We now apply the residue theorem :

$$\begin{aligned} \frac{dS(l,\mu)}{dl} &= \sum_n \left[\frac{s-s_n}{1-\frac{3^{1-s}}{2}} l^{-s} \gamma(s) \right]_{s=s_n} \\ &= \frac{l^{d_K-1}}{\ln(3)} \sum_n l^{\frac{2i\pi n}{\ln(3)}} \underbrace{\gamma\left(1-d_K-\frac{2i\pi n}{\ln(3)}\right)}_{\gamma_n} \end{aligned}$$

where we used :

$$\lim_{s\to s_n} \frac{s-s_n}{1-\frac{3^{1-s}}{2}} = \frac{1}{\ln(3)}$$

and with $d_K = \frac{\ln(2)}{\ln(3)}$ and

$$\gamma_n = \int_{K\times K} |x-y+2|^{-d_K - \frac{2i\pi n}{\ln(3)}} d\mu(x) d\mu(y)$$

Inserting this expression in (2), we get :

$$\begin{aligned} C(u) &= \frac{1}{\ln(3)} \sum_{n\in\mathbb{Z}} \gamma_n \int_0^1 l^{d_K-1} l^{2i\pi n/\ln(3)} \gamma_n \mathrm{sinc}(lu) dl \\ &= \frac{u^{-d_K}}{\ln(3)} \sum_{n\in\mathbb{Z}} \gamma_n u^{-2i\pi n/\ln(3)} \int_0^u v^{d_K-1+2i\pi n/\ln(3)} \mathrm{sinc}(v) dv \end{aligned}$$

with the change of variable : $v = ul$. Since $\int_0^u v^{d+\frac{2i\pi n}{\ln(3)}-1} \mathrm{sinc}(v) dv$ quickly converges to $\int_0^\infty v^{d_K+\frac{2i\pi n}{\ln(3)}-1} \mathrm{sinc}(v) dv$, one can replace $\int_0^u$ by $\int_0^\infty$.

(More precisely : $|\int_0^u v^{d_K+\frac{2i\pi n}{\ln(3)}-1} \mathrm{sinc}(v) dv - \int_0^\infty v^{d_K+\frac{2i\pi n}{\ln(3)}-1} \mathrm{sinc}(v) dv| = O(u^{d_K-1})$)

Using :

$$\begin{aligned} &\int_0^\infty v^{d_{Kn}+\frac{2i\pi n}{\ln(3)}-1} \mathrm{sinc}(v) dv \\ &= \sin\left(\frac{\pi}{2}(d_{K,n}-1)\right) \Gamma\left(d_{K,n} + \frac{2i\pi n}{\ln(3)} - 1\right) \end{aligned}$$

with Γ the Euler gamma function, and $d_{K,n} = d_K + \frac{2i\pi n}{\ln(3)}$ one finally finds, for $u >> 1$ (see fig. (3)):

$$C(u) = \frac{u^{-d_K}}{\ln(3)} \sum_n \gamma_n u^{-\frac{2i\pi n}{\ln(3)}} \sin\left(\frac{\pi}{2}(d_{K,n}-1)\right) \Gamma(d_{K,n}-1)$$

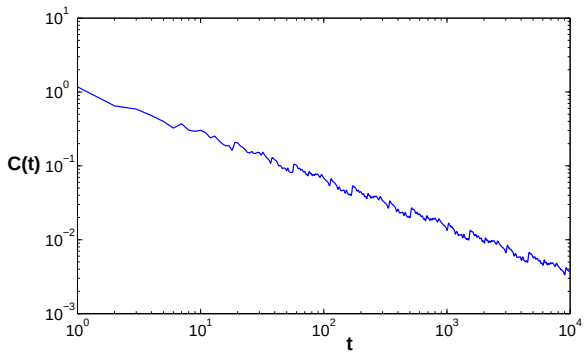

FIG. 3: Auto-correlation function in the case of a triadic Cantor set spectrum, in log-scale (log base 10).

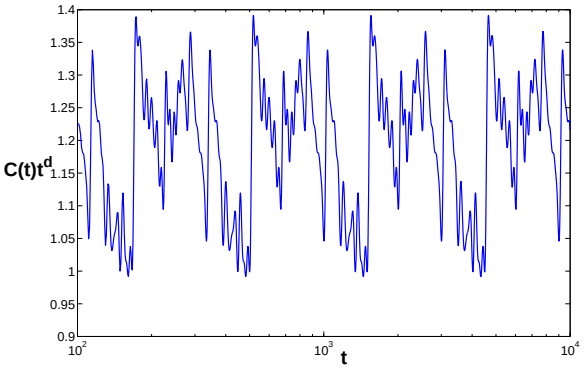

FIG. 4: Auto-correlation function multiplied by u^d in the case of a triadic Cantor set spectrum, in log-scale (log base 10). This function is log-periodic in u, of period ln(3).

We can prove (see annexe) that the coefficients in the expression (3) decay at least like $\frac{1}{n^{3/2-d}}$:

$$|\gamma_n \sin\left(\frac{\pi}{2}(d+\frac{2i\pi n}{\ln(3)}-1)\right) \Gamma(d+\frac{2i\pi n}{\ln(3)}-1)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

IV.

Let us now discuss the expression of $C(u)$.

We find that $C(u)$ is of the form : $C(u) = u^{-d_K} g(u)$, with $d_K = \frac{\ln(2)}{\ln(3)}$ the fractal dimension of the triadic Cantor set K and

$$g(u) = \sum_{n\in\mathbb{Z}} \gamma_n u^{-\frac{2i\pi n}{\ln(3)}} \sin\left(\frac{\pi}{2}(d_n-1)\right) \Gamma(d_n-1)$$

with $d_n = d + \frac{2i\pi n}{\ln(3)}$, a log-periodic function. The presence of the fractal dimension d_K in the exponent is consistent with previous work on the asymptotic behaviour of the time-averaged return probability for a Cantor spectrum : $C(t) \approx_{t\to\infty} t^{-d_k}$([3], [4]).

The log-periodicity is the signature of the scaling symmetry of the spectrum. To see this, note that $C(t)$

is in fact the spectral average, for the measure $d\mu$, of the wavelet transform sinc : $C(t) = \int_{K\times K} \mathrm{sinc}((\lambda' - \lambda)\epsilon_{max}t/\hbar)d\mu(\lambda)d\mu(\lambda')$. Since the spectrum has a scaling symmetry, one would have expected to find a log-periodic behaviour for the sinc-wavelet transform, at least locally (e.g. for the quantity : $C_{local}(t,\lambda) = \int_K \mathrm{sinc}((\lambda' - \lambda)\epsilon_{max}t/\hbar)d\mu(\lambda))$. It was less obvious that the log-periodicity would remain after averaging over the spectrum ; this is due to the fact that we chose the density of modes and the distribution $|g(\epsilon)|^2$ to be uniform, and that the set has a unique scaling factor. In other words, in the case of a superposition of triadic Cantor sets for example, it is not certain that these oscillations would survive the averaging.

Let us now highlight the similarity between the theoretical graph of $C(t)$ for a triadic Cantor set spectrum (fig. (3), (4)) and the numerical results obtained with the Fibonacci cavity (more details are available in [6]) fig. (5). In this experiment, cavity polaritons are confined in wire cavities, consisting of $\lambda/2$ layers, using Bragg mirrors for the confinement in the vertical direction. The lateral dimension of the $200\mu m$ long wires are modulated quasi-periodically : the modulation consists in two wire sections (letters) A and B, of equal length but different width ; these letters are arranged in a finite sequence, S_j, obtained recursively using the following Fibonacci like algorithm : $S_{j>2} = [S_{j-1}S_{j-2}]$ and $S_1 = B$, $S_2 = A$ where $[S_{j-1}S_{j-2}]$ is the concatenation of the sequences S_{j-1} and S_{j-2}. The sequence S_∞ becomes rigorously quasi-periodic as j tends to infinity. We used a finite sized cavity, however, we observed the features of the fractal spectrum predicted by the theory : namely, gaps densely distributed and an integrated density of states $N(\epsilon)$ well described by a scaling form of the type $N(\epsilon) = \epsilon^{\ln a/\ln b}F(\frac{\ln\epsilon}{\ln b})$ and which follows the gap labelling theorem. The photon modes are described by a 2D scalar wave equation with vanishing boundary conditions on the boundary of the wire. This 2D problem is then reduced to a 1D Schroedinger equation with an effective quasi-periodic potential, translating the geometry of the cavity. The equation was then solved numerically (using the transfer matrix formalism) and some useful quantities were plotted : the IDOS, the participation ration, the RMS displacement and the time-averaged return probability. The latter is given fig. (5).

We found that the participation ratio and the return probability evidence a log-periodicity, which would be expected as a signature of the fractal spectrum.

V.

Let us now discuss the method used in a more general way, to highlight the connection between the presence of a scaling symmetry in a physical system and the typical form of certain quantities. We see from the calculation that the scaling property of the spectrum leads naturally to a specific behaviour of some physical quantities defined

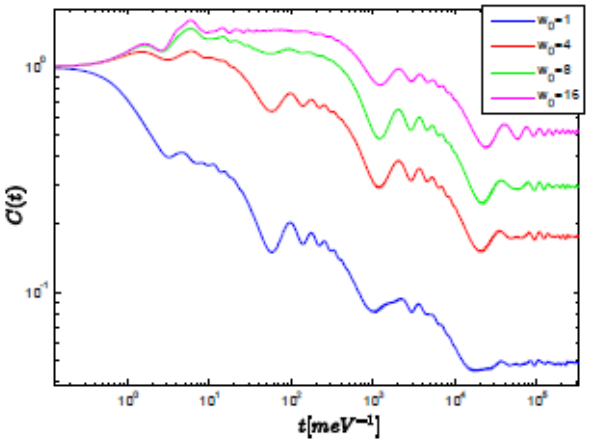

FIG. 5: Auto-correlation function of the wave paquet in the Fibonacci cavity ; numerical results for an initial gaussian wave paquet : $\psi(x,t=0) \sim e^{-2(x-x_0)^2/w_0^2}$. The graphs correspond to different values of w_0, with identical x_0.

as integrals over the spectrum - power law modulated by log-periodic oscillations.

Consider a system having a fractal spectrum, characterized by a scaling property, which is expressed through a condition of the kind (1) verified by the measure.

One can show (see annexe) that a function f verifying a functional equation of the type :

$$f(x) = \frac{1}{b}f(ax) + g(x)$$

is (under some conditions on g) of the form :

$$f(x) = x^{\ln(b)/\ln(a)}G\left(\frac{\ln x}{\ln a}\right) \tag{3}$$

with G a 1-periodic function.

Now, in the case of a fractal F described by a measure μ such that :

$$\int_F f(x)d\mu(x) = \sum_{n\in\mathbb{N}} \frac{1}{b_n} f(a_n x)d\mu(x)$$

for some $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}*$, then any function $\Psi(t)$ defined by an integral : $\Psi(t) = \int_F h(x,t)d\mu(x)$ will satisfy :

$$\Psi(t) = \sum_{n\in\mathbb{N}} \frac{1}{b_n} \int_F h(a_n x, t)d\mu(x)$$

If furthermore $h(x,t)$ can be expressed as a function of

$x^\alpha t^\beta$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$: $h(x,t) = \psi(x^\alpha t^\beta)$, then :

$$\begin{aligned}\Psi(t) &= \sum_{n\in\mathbb{N}} \frac{1}{b_n} \int_F \psi(a_n x^\alpha t^\beta) d\mu(x) \\ &= \sum_{n\in\mathbb{N}} \frac{1}{b_n} \int_F h(x, a_n^{1/\beta} t) d\mu(x) \\ &= \sum_{n\in\mathbb{N}} \frac{1}{b_n} \Psi(a_n^{1/\beta} t) \\ &= \frac{1}{b_j} \Psi(a_j^{1/\beta} t) + \Phi(t)\end{aligned}$$

with $\Phi(t) = \sum_{n\in\mathbb{N}-j} \frac{1}{b_n} \Psi(a_n^{1/\beta} t)$. Thus, if the Mellin transforms of $\Phi(t)$ and $\Psi(t)$ exist and have overlapping domains of definition, and if the Mellin transform of $\Phi(t)$ has no poles, then $\Psi(t)$ will be of the form :

$$\Psi(t) = t^{\ln(b_j)/\ln(a_j^{1/\beta})} G\left(\frac{\ln t}{\ln a_j^{1/\beta}}\right)$$

with G 1-periodic.

More generally, this also occurs whenever we can rearrange the integral in order to have : $\int_F h(\alpha x, t) d\mu(x) \propto \int_F h(x, \beta t) d\mu(x)$.

Thus, we see in the case of quantum dynamics that, since the energy ϵ and the time t are conjugate variables, it is likely that a time dependant quantity defined as an integral over the spectrum (through the spectral decomposition for instance) of a function defined using the evolution operator $e^{iHt/\hbar}$ or of any function which couples ϵ and t, will be of the form (5). This is well verified for $C(t)$.

Other dynamical quantities should also display this typical behavior ; we shall briefly discuss them.

We discuss first the time-averaged participation ratio, usually studied to get information on the dynamics of a quantum system. Moreover, this quantity was studied numerically for the Fibonacci cavity and the results support the idea of a log-periodic feature. By definition, the participation ratio is :

$$P_\psi = \frac{1}{\int_{espace} |\langle x|\psi\rangle|^4 dx}$$

This quantity reflects the localization in space of the wave function $\psi(x,t)$: if the state is localized, then $P_\psi \simeq 0$; if it is extended, then $P_\psi \simeq 1$ [9]. Taking its time-average, to have a time-dependent quantity, we define :

$$P_\psi(t) = \frac{1}{t} \int_0^t \frac{dt'}{\int_{espace} |\langle x|\psi(t')\rangle|^4 dx}$$

Now, using Green functions, we can show [10] that :

$$|\psi(x,t)| = \left| \int_{space} \int_{spectrum} e^{-i\epsilon t} \phi(x,\epsilon)^* \phi(x',\epsilon) \psi(x',0) dx' d\mu(\epsilon) \right|$$

$|\phi(\epsilon)\rangle$ being the eigenfunction of the Hamiltonian associated with the energy ϵ. The calculation has not yet completely been carried through, since sufficient information on the behaviour of the space correlation term $\phi(x,\epsilon)^*\phi(x',\epsilon)$ is still missing.

We shall now briefly discuss the RMS displacement, another quantity commonly studied both in quantum dynamics - spreading of the wave packet - and in diffusive processes. First note that, since the above method (which led to the expression of the auto-correlation function), uses the linearity of the integral, it is obvious that it is unlikely to work for the RMS

$$\Delta x = \sqrt{\langle (x(t) - x_0)^2 \rangle} = \sqrt{\int_{space} (x(t) - x_0)^2 |\psi(t,x)|^2 dx}$$

because of the square root. However, we should obtain interesting results for the moment of order 2 :

$$(\Delta x)^2 = \langle (x(t) - x_0)^2 \rangle = \int_{space} (x(t) - x_0)^2 |\psi(t,x)|^2 dx$$

Although a numerical study, in the case of the Fibonacci cavity, did not give encouraging results, note that this quantity is very close to

$$\begin{aligned}&\int_{space} (x(t) - x_0)^2 |\psi(t,x)|^2 dx \\ &= \left| \int_{space} \int_{spectrum} e^{-i\epsilon t} \phi(x,\epsilon)^* \phi(x',\epsilon) \psi(x',0) dx' d\mu(\epsilon) \right|^2\end{aligned}$$

and that using the assumptions discussed for the participation ratio about the behaviour of $\overline{\phi(\mathbf{r},\epsilon)^*\phi(\mathbf{r'},\epsilon)}$ should also lead to interesting results.

Furthermore, the case of a Hamiltonian with a one-scale Cantor set spectrum has been solved numerically in the tight-binding approximation [14], and it was found that the moments $\langle (x(t) - x_0)^\alpha \rangle$ have a log-periodic behaviour.

VI. II

In order to have a better understanding of the dynamics of our system, we consider the case of the tight-binding model. We shall use a different approach than previously, in the framework of dynamical systems.

We consider an infinite 1D lattice, with sites k labelled from 0 to ∞. We define a potential V on this system such that the discrete Schrodinger operator : $H = -\Delta + V$, which acts on $l_2(\mathbb{Z})$, has a triadic Cantor set spectrum. As in the previous paragraph, the eigenfunctions $\{\phi_\epsilon\}_{\epsilon\in K}$, evaluated on the sites k, can be seen as polynomials in ϵ :

$$\phi_\epsilon(k) = p_k(\epsilon) \tag{4}$$

We are interested of the evolution of a wave packet, described by the normalized wave function $\psi(t)$, evolving according to Schrodinger's equation :

$$i\frac{d\psi(t)}{dt} = H\psi(t)$$

The wave function is initially localized at the site 1 : $\langle k|\psi(t=0)\rangle = \delta(k-1)$. At time t, the system is in the state : $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$, and the value of the wave function at site k is given by :

$$|\psi(k,t)| = \int_K \phi_\epsilon(1)^*\phi_\epsilon(k)e^{-i\epsilon t}d\mu$$

Using (2), we obtain :

$$\begin{aligned}\psi(k,t) &= \int_K \phi_\epsilon(1)^*\phi_\epsilon(k)e^{-i\epsilon t}d\mu &(5)\\ &= \int_K p_1(\epsilon)p_k(\epsilon)e^{-i\epsilon t}d\mu &(6)\\ &= \sum_{j_k\le k} c_{j_k}\int_K \epsilon^{j_k}e^{-i\epsilon t}d\mu &(7)\end{aligned}$$

with $p_k(x) = \sum_{0\le j_k\le k} c_{j_k}x^{j_k}$ Thus, the dynamical behaviour of the system depends essentially on the functions :

$$h_k(t) = \int_K e^{-i\epsilon t}\epsilon^k d\mu(\epsilon)$$

aka the Fourier transforms on K of the moments x^k.

Study of $h_k(t)$

Using the Mellin transform (see annexe), we may express $h_k(t)$ as a series :

$$h_k(t) = \sum_{n\neq 0}\frac{(-it)^n}{n!}\mu_{n+k}$$

with $\mu_n = \int_K x^n d\mu$. (Note that we obtain the same expression by writing $e^{-i\epsilon t} = \sum_{n\le 0}\frac{(-i\epsilon t)^n}{n!}$ and reversing the signs sum and integral).

It is not clear, from this expression, whether or not $h_k(t)$ has a log-periodic feature, which is expected as a fingerprint of the scaling symmetry of the spectrum expressed in (1). We shall now show that there is, however, such a behaviour. We will do so by calculating the zeroes of $h_k(t)$ and the maxima of $|h_k(t)|$.

First note that $h_0(t)$ verifies the following functional equation, derived by direct application of (1) :

$$\begin{aligned}h_0(t) &= \int_K e^{-i\epsilon t}d\mu\\ &= \frac{1}{2}\int_K e^{-i\epsilon t/3}d\mu + \frac{e^{-i2t/3}}{2}\int_K e^{-i\epsilon t/3}d\mu\\ &= \frac{1+e^{-i2t/3}}{2}h_0(t/3)\end{aligned}$$

By repeating this process, and using $h_0(0) = \int_K 1d\mu = 1$, we find the exact expression of $h_0(t)$:

$$h_0(t) = \prod_{k\ge 1}\frac{1+e^{-i2t/3^k}}{2} \quad (8)$$

This is consistent with the expression derived in [18]. Furthermore, there is a perfect fit between the graph of the theoretical expression (2) and the one obtained from the direct calculation of $h_0(t)$ using :

$$h_0(t) = \lim_{n\to\infty}\left(\frac{3}{2}\right)^n \sum_{a_{j_n}\in K_n}\int_{a_{j_n}}^{a_{j_n}+3^{-n}} e^{-ixt}dx$$

with K_n the set of the left edges of the remaining segments after n iterations in the construction of the Cantor set from the segment $[0,1]$.

From (2), we easily deduce the zeroes of $h_0(t)$: it is the set :

$$Z_0 = \{t_m = \frac{3\pi}{2}(2m+1), m\in\mathbb{Z}\}$$

We also get the local maxima of $|h_0(t)|$:

$$M_0 = \{v_{n,m} = \pi m 3^n, m\in\mathbb{Z}, n\in\mathbb{N}^*\}$$

with the property :

$$h_0(v_{n,m}) = \prod_{k\ge n+1}\frac{1+e^{-i2t/3^{\overbrace{k-n}^{k'}}}}{2} = \prod_{k'\ge 1}\frac{1+e^{-i2t/3^{k'}}}{2} = h_0(v_{0,}$$

Furthermore, the product $\prod_{k'\ge 1}\frac{1+e^{-i2t/3^{k'}}}{2}$ is convergent and consists of nonzero terms, therefore it is nonzero. Taking $n\to\infty$, we can therefore deduce that $h_0(t)$ does not converge to zero when $t\to\infty$ (as also found in [18]).

Now, using (1) again, we derive the following recurrence formula for $h_k(t)$:

$$\begin{aligned}h_k(t) &= \int_K e^{-i\epsilon t}\epsilon^k d\mu\\ &= \frac{3^{-k}}{2}\int_K e^{-i\epsilon t/3}\epsilon^k d\mu + \frac{3^{-k}e^{-i2t/3}}{2}\int_K e^{-i\epsilon t/3}(\epsilon+2)^k d\mu\\ &= \frac{3^{-k}}{2}h_k(t/3) + \frac{3^{-k}}{2}e^{-i2t/3}\sum_{j=0}^{k}\binom{k}{j}h_j(t/3)2^{k-j}\end{aligned}$$

We will use this relation to show, by recurrence, that the set Z_k of zeroes of h_k, $k\neq 0$ contains the set :

$$\{t_{m,n} = \frac{\pi}{2}(2m+1)3^{k+1}, m\in\mathbb{Z}\}$$

First, we deduce that, if $t_{n,m} = \frac{\pi}{2}(2m+1)3^2,\ m \in \mathbb{Z}\}$, then :

$$h_1(t_{n,m}) = \frac{3^{-1}}{2}h_1(t_{n,m}/3) - \frac{3^{-1}}{2}(h_1(t_{n,m}/3) + \underbrace{h_0(t_{n,m}/3)}_{0})$$

since $t_{n,m}/3 \in Z_0$. Thus : $\{t_{m,n} = \frac{\pi}{2}(2m+1)3^2, m \in \mathbb{Z}\} \subset Z_1$. The generalization to higher k is trivially derived using the recurrence relation above.

It seems that there are no other zeroes for $h_k(t)$, e.g. that

$$Z_k = \{t_{m,n} = \frac{\pi}{2}(2m+1)3^{k+1}, m \in \mathbb{Z}\}$$

but it is not rigorously proven ; the numerics however seem to support this hypothesis (see fig. 1).

We therefore deduce that $h_k(t)$ has the same zeroes than $h_{k-1}(t/3)$.

Using a similar reasoning, we can study the points at which the module $|h_k(t)|$ has a local maximum. The maxima of $|h_0(t)|$ are reached at the points : $t_{m,n} = 3^n m\pi$. Using the recurrence relation between the h_k, one can see that

$$h_k(t_{m,n}) \approx 3^{-k}h_0(t_{m,n-1})$$

Thus, we may infer, with a good approximation, that :

$$h_k(t) \propto h_{k-1}(t/3)$$

This is confirmed by the numerics (see fig. 2).

To summarize, the functions $h_k(t)$ themselves are not log-periodic, and have periodically spaced zeroes of the form $t_{m,n} = \frac{\pi}{2}(2m+1)3^n$. The fingerprint of the fractal spectrum lies in the recurrence relation :

$$h_k(t) = \frac{3^{-k}}{2}h_k(t/3) + \frac{3^{-k}}{2}e^{-i2t/3}\sum_{j=0}^{k}\binom{k}{j}h_j(t/3)2^{k-j}$$

which, up to a good approximation, can be simplified into :

$$h_k(t) \propto h_{k-1}(t/3)$$

Note however that it is not a rigorous proportionality relation.

Application to the RMS and the participation ratio

In order to study how the fractal feature of the spectrum affects the dynamics of the system, we will now consider the RMS :

$$\Delta x(t)^2 = \sum_{k=1}^{\infty} k^2|\psi(k,t)|^2$$

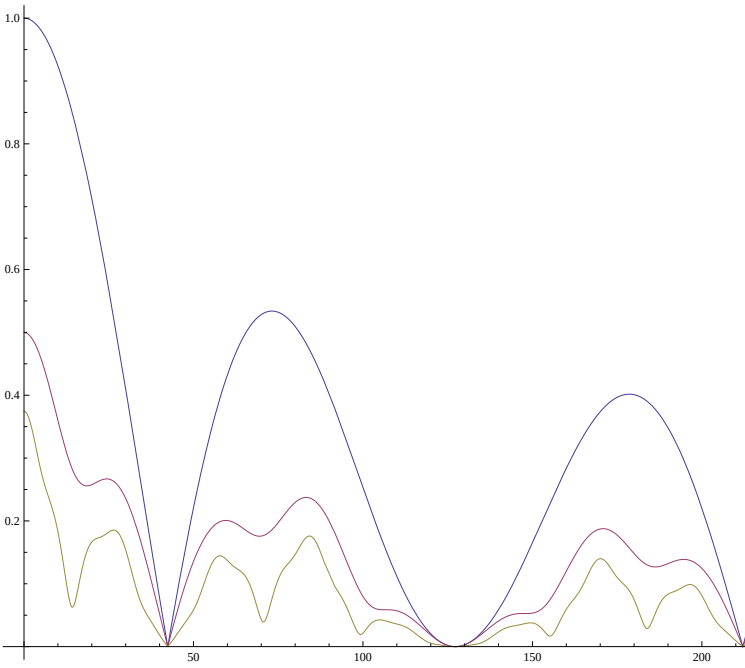

FIG. 6: Numerical curves of $|h_0(t/9)|$ (blue), $|h_1(t/3)|$ (red) and $|h_2(t)|$ (yellow).

and the participation ratio :

$$P_\psi = \frac{1}{\int_{espace}|\langle x|\psi\rangle|^4 dx}$$

Injecting (3) in the definition of the RMS, we obtain :

$$\begin{aligned}\Delta x(t) &= \sum_{k=1}^{\infty} k^2|\psi(k,t)|^2 \\ &= \sum_{k=1}^{\infty} k^2|\sum_{j_k\le k} c_{j_k}\int_K \epsilon^{j_k}e^{-i\epsilon t}d\mu|^2 \\ &= \sum_{k=1}^{\infty} k^2|\sum_{j_k\le k} c_{j_k}h_{j_k}(t)|^2\end{aligned}$$

Note that, even if the $h_k(t)$ have periodically spaced zeroes, this does not imply that $\Delta x(t)$ cancels for some t : in fact, since the zeroes of h_k are odd multiples of 3^{k+1}, the infinite sum in the expression of Δx insures that, for any time t, we are always before the first zero of infinitely many h_k.

Using $h_k(t) \propto h_{k-1}(t/3)$, we obtain in first approximation :

$$\Delta x(t) \propto \Delta x(t/3)$$

which leads to :

$$\Delta x(t) = t^\alpha G(\ln t)$$

with G periodic.

We can make a more precise calculation by using directly the recurrence relation $h_k(t) = \frac{3^{-k}}{2} h_k(t/3) + \frac{3^{-k}}{2} e^{-i2t/3} \sum_{j=0}^{k} \binom{k}{j} h_j(t/3) 2^{k-j}$:

$$\begin{aligned}
\Delta x(t) &= \sum_{k=1}^{\infty} k^2 |\sum_{j_k \leq k} c_{j_k} h_{j_k}(t)|^2 \\
&= \sum_{k=1}^{\infty} k^2 |\sum_{j_k \leq k} c_{j_k} \frac{3^{-j_k}}{2} h_{j_k}(t/3) + \frac{3^{-j_k}}{2} e^{-i2t/3} \sum_{l=0}^{j_k} \binom{j_k}{l} h_l(t/3) 2^{j_k - l}(t)|^2 \\
&\approx \frac{1}{b} \Delta x(t/3) + R(t)
\end{aligned}$$

where b can be approximated numerically (there does not seem to be a simple theoretical value for b) and $R(t)$ is a sum of terms of the form $k^2 h_j(t)^* h j'(t)$.

A similar reasoning seems to indicate that the participation ratio also has a log-periodic feature, however the calculation is more complicated and I have not yet finished it.

Study of the Green function

Knowing explicitely the eigenfunctions $\phi_\epsilon(k) = p_k(\epsilon)$ allows us to study the static and time dependent Green functions of the system. In fact, the Green functions can be expressed in terms of the functions $h_k(t)$ and the moments $\mu_k = \int_K x^k d\mu$.

The time dependent Green function, defined as $G(k,k',t) = \langle k|e^{-iHt}|k'\rangle$, is a linear combination of the $h_k(t)$:

$$\begin{aligned}
G(k,k',t) &= \langle k|e^{-iHt}|k'\rangle \\
&= \int_K e^{-i\epsilon t} \phi_\epsilon(k)^* \phi_\epsilon(k') d\mu(\epsilon) \\
&= \int_K e^{-i\epsilon t} p_k(\epsilon)^* p_{k'}(\epsilon) d\mu(\epsilon) \\
&= \int_K e^{-i\epsilon t} \sum_{j_k \leq k} \sum_{j'_k \leq k'} c_{j_k} c_{j'_k} \epsilon^{j_k + j_{k'}} d\mu(\epsilon) \\
&= \sum_{j_k \leq k} \sum_{j'_k \leq k'} h_{j_k + j_{k'}}(t)
\end{aligned}$$

The stationary Green function, aka the Fourier transform of the propagator, defined by $G(k,k',\epsilon) = \int_0^\infty dt G(k,k',t) e^{i\epsilon t}$, is given by :

$$\begin{aligned}
G(k,k',\epsilon) &= \int_0^\infty dt G(k,k',t) e^{i\epsilon t} \\
&= \int_K \frac{\phi_\epsilon(k)^* \phi_\epsilon(k')}{\epsilon - \epsilon'} d\mu(\epsilon') \\
&= \int_K \frac{p_k(\epsilon)^* p_{k'}(\epsilon)}{\epsilon - \epsilon'} d\mu(\epsilon') \\
&= \sum_{j_k \leq k} \sum_{j'_k \leq k'} c_{j_k} c_{j'_k} \int_K \frac{\epsilon^{j_k + j_{k'}}}{\epsilon - \epsilon'} d\mu(\epsilon')
\end{aligned}$$

For $\epsilon = 0$, we get :

$$\begin{aligned}
G(k,k',0) &= \sum_{j_k \leq k} \sum_{j'_k \leq k'} c_{j_k} c_{j'_k} \int_K \frac{\epsilon^{j_k + j_{k'}}}{\epsilon'} d\mu(\epsilon') \\
&= \sum_{j_k \leq k} \sum_{j'_k \leq k'} c_{j_k} c_{j'_k} \mu_{j_k + j_{k'}}
\end{aligned}$$

It can be shown (see [19]) that :

$$\mu_n \approx n^{-\ln 2/\ln 3} F(\ln n/\ln 3)(1 + O(\frac{1}{n}))$$

with F periodic of period one. Therefore :

$$G(k,k',0) \approx \sum_{j_k \leq k} \sum_{j'_k \leq k'} c_{j_k} c_{j'_k} (j_k + j_{k'})^{-\ln 2/\ln 3} F(\ln(j_k + j_{k'})/\ln 3)$$

VII. CONCLUSION

In summary, we derived the analytical expression of the auto-correlation function $C(t)$ in the case of a triadic Cantor set energy spectrum, and found that it is of the form : $C(t) = \left(\frac{t}{\tau}\right)^{-d_K} g(\ln(t/\tau))$ where $d_K = \frac{\ln(2)}{\ln(3)}$ is the fractal dimension of the triadic Cantor set, τ a time scale a g a periodic function. We generalize the method and discuss the connection between the presence of self-similar structures or features in a physical system and the typical log-periodic behaviour of certain quantities.

One can also use this method to derive the decay probability of a two level atom coupled to a fractal spectrum ([15]). We show in this case that Fermi's golden rule does not hold, and that the decay probability of the atom from an energy level E_i to an energy $E_f \pm \Delta E$ is of the form (5). In this case, the log-periodic oscillations coud be interpretated as a signature of the topological properties of the spectrum : a closed, nowhere dense set with no isolated point.

To finish, let us discuss briefly the question of turbulent diffusion, which was found to be related to fractals. A powerful tool for the study of turbulent diffusion is the method of breakdown coefficients, bdc, (see [16] and references therein). If a turbulent system happens to have a physically distinguished scale factor (for instance, unstable eddies breaking up in typically always the same number of smaller eddies), then the moments of the bdc's have a log-periodic behaviour (namelly, a power law modulated by log periodic oscillations) [16]. It is interesting to note that these log-periodic oscillations were derived with a very different approach - starting from a functionnal relation obtained in the frame of statistical physics. Thus, two very different physical problems (turbulence and quantum dynamics) led to a similar observation : there is a deep connection between the topological properties of self-similar sets and the occurence of anomalous log-periodic behaviour of statistical and dynamical quantities.

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Annexe

Scaling property of the measure $d\mu$

Let f be an integrable function defined on $[0,1]$, with real or complex values.

We derive here the following property :

$$\int_K f(x)d\mu(x) = \frac{1}{2}\int_K f\left(\frac{x}{3}\right)d\mu(x) + \frac{1}{2}\int_K f\left(\frac{x+2}{3}\right)d\mu(x)$$

By definition :

$$\int_K f(x)d\mu(x) = \lim_{n\to\infty} \underbrace{\left(\frac{2}{3}\right)^n \sum_{a_{j_n}\in P_n} \int_{a_{j_n}}^{a_{j_n}+3^{-n}} f(x)dx}_{I_n(f(x))}$$

with P_n the set of the left edges of the remaining segments at the n^{th} step of the construction of the triadic Cantor set. One can directly check that the sequence $(P_n)_n$ follows the recurrence property :

$$P_n = \frac{1}{3}P_{n-1} \cup \{\frac{2}{3} + \frac{1}{3}P_{n-1}\}$$

Therefore :

$$\begin{aligned}
I_n(f(x)) &= \left(\frac{2}{3}\right)^n \sum_{a_{j_n}\in P_n} \int_{a_{j_n}}^{a_{j_n}+3^{-n}} f(x)dx \\
&= \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}}\in P_{n-1}} \int_{a_{j_{n-1}}/3}^{a_{j_{n-1}}/3+3^{-n}} f(x)dx \\
&+ \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}}\in P_{n-1}} \int_{\frac{2}{3}+a_{j_{n-1}}/3}^{\frac{2}{3}+a_{j_{n-1}}/3+3^{-n}} f(x)dx \\
&= \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}}\in P_{n-1}} \int_{a_{j_{n-1}}}^{a_{j_{n-1}}+3^{-n+1}} f\left(\frac{u}{3}\right)\frac{du}{3} \\
&+ \left(\frac{2}{3}\right)^n \sum_{a_{j_{n-1}}\in P_{n-1}} \int_{a_{j_{n-1}}}^{a_{j_{n-1}}+3^{-n+1}} f\left(\frac{v+2}{3}\right)\frac{dv}{3} \\
&= \frac{3}{2}\frac{1}{3}\left[I_{n-1}\left(f\left(\frac{u}{3}\right)\right) + I_{n-1}\left(f\left(\frac{2+v}{3}\right)\right)\right]
\end{aligned}$$

But : $\lim_{n\to\infty} I_{n-1}\left(f\left(\frac{u}{3}\right)\right) = \int_K f\left(\frac{u}{3}\right)d\mu(u)$ and $\lim_{n\to\infty} I_{n-1}\left(f\left(\frac{2+v}{3}\right)\right) = \int_K f\left(\frac{2+v}{3}\right)d\mu(u)$, so finally :

$$\int_K f(x)d\mu(x) = \frac{1}{2}\int_K f\left(\frac{x}{3}\right)d\mu(x) + \frac{1}{2}\int_K f\left(\frac{x+2}{3}\right)d\mu(x)$$

This completes the proof.

Log-periodicity of functions verifying : $f(x) = \frac{1}{b}f(ax) + g(x)$

Let $f : \mathbb{R} \to \mathbb{C}$ verifying a functional equality of the kind :

$$f(x) = \frac{1}{b}f(ax) + g(x) \tag{9}$$

Let's show that f is of the form :

$$f(x) = x^{\ln(b)/\ln(a)} G\left(\frac{\ln x}{\ln a}\right)$$

with G 1-periodic.

We begin by taking the Mellin transform of (3) (assuming that the domain of definition of the Mellin transforms of f and g have a non empty intersection) :

$$\begin{aligned} M_f &= \int_0^\infty f(x)x^{s-1}dx \\ &= \frac{1}{b}\int_0^\infty f(ax)x^{s-1}dx + \int_0^\infty g(x)x^{s-1}dx \\ &= \frac{1}{b}\int_0^\infty f(u)\left(\frac{u}{a}\right)^{s-1}\frac{du}{a} + M_g(s) \end{aligned}$$

So :

$$M_f(s) = \frac{M_g(s)}{1-\frac{a^{-s}}{b}}$$

and by taking the inverse Mellin transform :

$$\begin{aligned} f(x) &= \tfrac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} M_f(s)x^{-s}ds \\ &= \tfrac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \tfrac{M_g(s)}{1-\frac{a^{-s}}{b}}x^{-s}ds \end{aligned}$$

Assuming now that M_g has no poles, the residue formula finally yields :

$$\begin{aligned} f(x) &= \sum_{n\in\mathbb{Z}}\left[\frac{(s-s_n)M_g(s)x^{-s}}{1-\frac{a^{-s}}{b}}\right]_{s=s_n} \\ &\quad (\ \ s_n = -\tfrac{\ln a}{\ln b} - \tfrac{2i\pi n}{\ln b}, \text{ poles of } \tfrac{M_g(s)}{1-\frac{a^{-s}}{b}}) \\ &= x^{\ln(b)/\ln(a)} \underbrace{\sum_{n\in\mathbb{Z}} c_n x^{2i\pi n/\ln b}}_{G(x)} \end{aligned}$$

which is what we wanted to prove.

Asymptotic behaviour of the coefficients

We prove here that the coefficients in the expression (??) decay at least like $\frac{1}{n^{3/2-d}}$:

$$|\gamma_n \sin\left(\frac{\pi}{2}(d+\frac{2i\pi n}{\ln(3)}-1)\right)\Gamma(d+\frac{2i\pi n}{\ln(3)}-1)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

To see this, let us start with $|\sin(\frac{\pi}{2}\left(d+\frac{2i\pi n}{\ln(3)}-1\right)\Gamma(d+\frac{2i\pi n}{\ln(3)}-1)|$; we have ([14]) :

$$\lim_{n\to\infty}|\Gamma(d+\frac{2i\pi n}{\ln(3)}-1)| = \sqrt{2\pi}e^{-(d-1)}e^{-\frac{2\pi^2 n}{2\ln(3)}}\left(\frac{2\pi n}{\ln(3)}\right)^{d-\frac{3}{2}}$$

Furthermore, since $|\sin(\frac{\pi}{2}(d+\frac{2i\pi n}{\ln(3)}-1))| \sim_{n\to\infty} e^{\frac{2\pi^2 n}{2\ln(3)}}e^{1-d}/2$, one has :

$$|\sin(\frac{\pi}{2}(d+\frac{2i\pi n}{\ln(3)}-1))\Gamma(d+\frac{2i\pi n}{\ln(3)}-1)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

Let us now study γ_n. By definition :

$$\begin{aligned} \gamma_n &= \int_{K\times K}\frac{d\mu(x)d\mu(y)}{|2-x+y|^{d_n}} \\ &= \lim_{k\to\infty}\left(\frac{3}{2}\right)^{2k}\sum_{a_{j_k},a_{j'k}\in P_k}\int_{a_{j_k}}^{a_{j_k}+3^{-k}}\int_{a_{j'_k}}^{a_{j'_k}+3^{-k}}\frac{dxdy}{|2-x+y|^{d_n}} \end{aligned}$$

with $d_n = \frac{\ln 2}{\ln 3} + \frac{2i\pi n}{\ln 3}$ and $P_k = \{a_{j_k}\}$ the set of the left edges of the remaining segments after k iterations in the construction of the triadic Cantor set K. Since K is embedded in $[0,1]$, and since for all $x, y \in [0,1]$ the inequality $2 > x - y$ holds, one can suppress the absolute value in the integral. Now, for all $x, y \in [0,1]$, one has : $1 \le 2 - x + y \le 3$, so $|\frac{1}{|2-x+y|^{d_n}}| = \frac{1}{|2-x+y|^d} < 1$.

Thus :

$$|\gamma_n| < \int_{K\times K}\frac{d\mu(x)d\mu(y)}{|(2-x+y)^{d_n}|} < \int_{K\times K} d\mu(x)d\mu(y) = 1$$

and so : $|\gamma_n| = O(1)$. Numerically, it seems that we even have : $|\gamma_n| = O\left(\frac{1}{n}\right)$, but this has not been proven rigorously.

We therefore obtain finally that, at least :

$$|\gamma_n \sin\left(\frac{\pi}{2}(d+\frac{2i\pi n}{\ln(3)}-1)\right)\Gamma(d+\frac{2i\pi n}{\ln(3)}-1)| = O\left(\frac{1}{n^{3/2-d}}\right)$$

[1] obtained by iteration of the following process : start with a full square, divide it in 9 equal squares and keep only the four at the corners of the initial square ; see fig.(2)