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# Integrated wavelets on fractal sets: I. The correlation dimension

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**Abstract.** We define the integrated wavelet transform of a measure on a set  $J$  and, using the thermodynamic formalism, we rigorously show that, for a large class of dynamical systems, it gives the correlation dimension of  $J$ . We recover qualitatively the same result analysing the Mellin transform of the wavelet. We apply this method to the numerical analysis of some hyperbolic and non-hyperbolic invariant sets.

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## 1. Introduction

The wavelet transform, although recently introduced (see [1] and references therein), has already a large field of applications. Notably, interest in using the wavelet transform to study many physical phenomena involving a scale invariance has considerably increased. In particular, it allows the computation of the local exponents associated with the singularities of multifractal measures [2, 3]. The aim of this paper is to prove that, defining an integrated wavelet transform (iwr), one can obtain in fact a new method to determine a global fractal exponent, which is nothing but the correlation dimension. In analogy to the Hausdorff dimension, this exponent arises as a transition point between two asymptotic behaviours (0 and  $\infty$ ) of the iwr.

(a) We first recall the definition and some rigorous properties of the wavelet transform used in the study of fractal sets. Given a compact subset  $J$  of a metric space (with the metric  $\| \cdot \|$ ) endowed with a probability Borel non-atomic measure  $\mu$  and a function  $g: \mathbb{R} \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$  and also satisfying the properties

(i)  $\lim_{a \rightarrow 0^+} a^{-p} g(r/a) = 0$  pointwise for  $r > 0$  and for any  $p \geq 0$ ,

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(ii)  $g'(r) < 0$  for  $r \in (0, \alpha)$ ,  $g'(r) > 0$  for  $r \in (\alpha, +\infty)$  and  $r^\gamma g'(r)$  summable on  $[0, \infty)$  for any  $\gamma > -1$ ,  
 the wavelet transform of the measure  $\mu$  is defined as

$$T_p(a, b) = \frac{1}{a^p} \int_J g\left(\frac{\|x - b\|}{a}\right) d\mu(x). \tag{1.1}$$

In [2] we proved that

$$\inf\left\{p; \limsup_{a \rightarrow 0^+} |T_p(a, b)| > 0\right\} \geq \beta \tag{1.2}$$

where

$$\beta_- = \liminf_{t \rightarrow 0^+} \frac{\log[\mu(B(b, t))]}{\log t}$$

is the local scaling exponent at position  $b$ ,  $B(b, t)$  being a ball of radius  $t$  and centre  $b \in J$ .

(b) We now introduce the IWT averaging over all the positions  $b$  as

$$T_p(a) = \frac{1}{a^p} \int_J \int_J g\left(\frac{\|x - b\|}{a}\right) d\mu(x) d\mu(b) \tag{1.3}$$

and we get the following results:

(R1) For a large class of dynamic invariant sets, described in section 2 (conformal mixing repellers), we prove that if  $\liminf_{a \rightarrow 0^+} |T_p(a)|$  and  $\limsup_{a \rightarrow 0^+} |T_p(a)|$  are finite and non-zero for a certain  $p$ , then this  $p$  is unique and coincides with the correlation dimension of  $J$ , defined as a root of the topological pressure.

(R2) We show that the situation described in (R1) occurs in full generality for the dynamical systems whose correlation integral scales with a power law. A rigorous computation can be performed on the multifractal ternary Cantor set. The link between the IWT and the correlation integral is given by the Mellin transform of the derivative of the wavelet.

The proofs of (R1) and (R2) require assumption (i) but we have to substitute (ii) by a stronger one:

(ii)'  $g(r)$  is monotone for  $r \geq 0$ .

(R3) We report explicit numerical computations of the IWT in the case of linear Cantor sets and for non-hyperbolic invariant sets.

(c) We now want to compare the use of the wavelet transform applied to fractal sets with the more general wavelet transforms of signals and this discussion will also motivate and justify assumptions (i) and (ii)'. Given a real signal  $s(x)$ ,  $x \in \mathbb{R}$ , with a certain degree of smoothness (e.g. Hölder regularity or local differentiability), the wavelet transform of  $s(x)$  is defined as in (1.1) but setting  $p = 1$ ,  $d\mu(x) = s(x) dx$  and the argument of  $g$  being the real number  $(x - b)/a$ . Then, under some assumptions on  $g$ , among which is the condition of zero mean,  $\int_{-\infty}^{+\infty} g(r) dr = 0$ , it is possible to establish some necessary and sufficient conditions for the asymptotic behaviour of the wavelet transform which reflect the local or global regularity of the signal (see the article by Holschneider and Tchamitchian in [3] for a detailed exposition of these results). In the case of fractal measures  $\mu$ , the situation changes considerably. First of all the signal  $s(x)$  is replaced by the highly irregular Radon–Nykodim density,  $d\mu(x)/dx$ , where  $x$  belongs to a metric space  $J$  which is not necessarily a subset of  $\mathbb{R}$ ; the signal is now a positive quantity and the argument of the wavelet is

a distance  $\|x - b\|/a$ , so that we only need to consider it on the positive semiaxis. Thus, there are now no more reasons for the assumptions required for signals, in particular the zero mean condition, to be really necessary in analysing fractal measures.

We in fact proceed in a different manner. First we are able, for the conformal repellers, to characterize the asymptotic transition point of the iwt as a fractal dimension;† moreover, if the wavelet is adapted (see below for the definition), the limit (3.1) holds, linking the asymptotic behaviour of the iwt with the preceding fractal dimension. To achieve these results, the crucial assumption is (i), the fast decay of the wavelet at infinity and its localization around zero. The hypothesis of monotonicity is of a technical nature and some numerical investigations with the so-called ‘Mexican hat’,  $g(r) = (1 - r^2) \exp(-r^2/2)$ , show that it could be relaxed.

We then formally show that the iwt can be written as the Mellin transform of the derivative of the wavelet, at least if the scaling (4.4) for the correlation integral holds with some sort of uniformity. In this derivation the monotonicity of the wavelet is not necessary but, to put the method on a rigorous basis, we would need a general mathematical theory of the correlation integral, which is still missing. Note, however, that the condition of a wavelet of zero mean never enters in either the rigorous or formal arguments quoted earlier.

## 2. The dynamical system

The dynamical systems we consider are the disconnected conformal mixing repellers [4–8]; they comprise a large class of dynamic invariant sets, including the hyperbolic disconnected Julia sets [9] and the expanding maps  $T$  of the interval  $I$  for which  $T^{-1}(I)$  is a disjoint union of subsets of  $I$  (‘cookie-cutter’ Cantor sets—see [6–8] for a detailed analysis).

We summarize a few properties of these systems used in the following proofs (see also [4, 6, 7]). If  $J$  denotes the invariant set under the mapping  $T$  (usually  $J$  is a subset of a Riemannian manifold, with the metric  $\|\cdot\|$  eventually adapted), then there exists a closed connected set  $V \supset J$ , which is the closure of its interior, where  $T$  is of the class  $C^{1+\varepsilon}$ ,  $\varepsilon > 0$ , and the derivative  $D_x T$  (or  $DT(x)$ ) at the point  $x$  is conformal (a scalar times an isometry) and uniformly expanding, that is,  $\|D_x T^n\| > \lambda^n$ ,  $\lambda > 1$ , for any  $x \in V$  and  $n \geq 0$ . We denote by the same symbol  $\|\cdot\|$  the distance taken on  $V$ . Then the pre-image of each subset of  $V$  is the disjoint union of  $s$  (the degree of the map) subsets of  $V$ : we denote by  $T_i^{-1}$ ,  $i = 1, \dots, s$  the inverse branches of  $T$  globally defined on  $V$ . We set  $A_{k_1, \dots, k_n} = T_{k_n}^{-1} \dots T_{k_1}^{-1} V$  and  $\tilde{A}_{k_1, \dots, k_n} = A_{k_1, \dots, k_n} \cap J$ , where  $k_i = 1, \dots, s$ , and in the following we often interchange the two since the measures considered are concentrated on  $J$ ; moreover, we define the partition of  $J$  of order  $n$  as

$$\mathcal{A}^{(n)} = \bigcup_{\substack{k_1, \dots, k_n \\ k_i = 1, \dots, s}} \tilde{A}_{k_1, \dots, k_n}$$

We finally obtain  $J = \bigcap_{n=0}^{\infty} \mathcal{A}^{(n)}$ , where  $\mathcal{A}^{(0)} = V$ .

On each set  $A_{k_1, \dots, k_n}$  the derivative of  $T^n$  is uniformly bounded (by a standard

† This requires the introduction of the non-negative parameter  $p$  in the definition of the wavelet transform.

distortion argument [4]), that is,

$$C^{-1} \leq \frac{\|D_x T^n\|}{\|D_y T^n\|} < C \quad \text{for any } (x, y) \in A_{k_1, \dots, k_n} \text{ and } n > 0. \quad (2.1)$$

For these sets, the action of  $T$  is topologically mixing [4]; this allows us to find an uncountable set of ergodic measures  $\mu_\beta$ ,  $\beta \in \mathbb{R}$ , with support  $J$  and mutually singular (Gibbs measures), each of which being equivalent (with a continuous Radon–Nykodim derivative) to a probability Borel measure  $\nu_\beta$  satisfying the following remarkable condition [10, 7]:

$$\nu_\beta(TA) = \int_A \frac{\|D_x T\|^\beta}{\exp(-P(\beta))} d\nu_\beta(x) \quad (2.2)$$

where  $A$  is any measurable subset of  $J$  where  $T$  is injective and  $P(\beta)$  is the topological pressure of the function,  $-\beta \log \|D_x T\|$  [4, 11]. In the following we interchange  $\mu_\beta$  and  $\nu_\beta$ , as they are (uniformly) equivalent. The generalized dimensions  $D_\beta(q)$ ,  $q \in \mathbb{Z}$ , of these measures, defined according to the standard partition function approach [12], have been proved to be the solution of the equation [5–8, 13]

$$P(\beta q - D_\beta(q)(q - 1)) = qP(\beta). \quad (2.3)$$

We are now ready to formulate the main result of this paper.

### 3. Rigorous results

In this section we exclusively refer to the systems considered in section 2.

*Definition.* We say that the wavelet  $g$  is  $p$ -adapted to the measure  $\mu$ , or simply  $p$ -adapted, for  $p \geq 0$ , if both the limits  $\liminf_{a \rightarrow 0^+} |T_p(a)|$ ,  $\limsup_{a \rightarrow 0^+} |T_p(a)|$  are different from 0 and  $+\infty$ .

*Theorem.* Let us suppose that the wavelet  $g$  is  $p$  adapted to  $\mu_\beta$ ; then the value of  $p$  is unique and is equal to  $D_\beta(2)$  or, equivalently, it is the unique solution of  $P(2\beta - p) = 2P(\beta)$ .

*Remark 1.* Sometimes we say that a wavelet is adapted if it is  $p$  adapted for a (unique)  $p$ . The dimension  $D_\beta(2)$  is usually called the ‘correlation dimension’: it is a lower bound to the Hausdorff dimension of the set  $d_H(J)$  and, for several systems (see also section 4), is equal to the scaling exponent given by the correlation integral method developed extensively in [14, 15]. In particular, for  $\beta = 0$ , the measure  $\mu_0$  (‘balanced measure’) has a useful time series representation [4, 9]. In section 4, the connection with the correlation integral will be made more explicit.

*Remark 2.* The Gibbs measures  $\mu_\beta$  are dense in the set  $I$  of invariant measures on the repeller in the norm topology [4]; since  $D_\beta(2) \leq d_H(J)$ , for any  $\beta$ , we conjecture that, whenever  $\limsup_{a \rightarrow 0^+} |T_p(a)| < +\infty$  for a measure  $\mu \in I$ ,  $p \leq d_H(J)$ .

Conditions like that stated in remark 2 are investigated in [19].

As an easy consequence of the theorem, we have the following necessary condition for the adaptedness of  $g$ , which is particularly interesting in view of the numerical applications.

Corollary. If the wavelet is  $D_\beta(2)$  adapted, then

$$\lim_{a \rightarrow 0^+} \frac{\log |T_p(a)|}{\log a} = D_\beta(2) - p \quad \text{for any } p \geq 0. \tag{3.1}$$

*Proof of the theorem.* For any  $n \geq 0$ , the integral in (1.3) is done on the Cartesian product  $\mathcal{A}^{(n)} \times \mathcal{A}^{(n)}$ ; we split the integral in two parts, the first computed on the diagonal elements of  $\mathcal{A}^{(n)} \times \mathcal{A}^{(n)}$ , which are the Cartesian products of two equal sets and the other computed on the products of two different elements of  $\mathcal{A}^{(n)}$ , which are disjoint.

The second part of the integral, which we call  $B_n(a)$ , is zero in the limit  $a \rightarrow 0^+$ , since the argument of  $g$  is bounded away from 0 and  $\infty$  (being  $J$  compact) and by property (i) of the wavelet. By (i) and (ii)' we can assume, without restriction,  $g > 0$  on  $\mathbb{R}^+$ . So we have (since we replace  $\mu_\beta$  with  $\nu_\beta$ , we use the symbol  $\cong$ , meaning 'uniformly bounded from above and below by a constant independent of  $n$ ')

$$T_p(a) \cong B_n(a) + \sum_{\substack{k_1, \dots, k_n \\ k_i=1, \dots, s}} \int_{A_{k_1 \dots k_n}} \int_{A_{k_1 \dots k_n}} \frac{1}{a^p} g\left(\frac{\|x - y\|}{a}\right) d\nu_\beta(x) d\nu_\beta(y).$$

Applying property (2.2)  $n$  times, we get

$$T_p(a) \cong B_n(a) + \sum_{k_1, \dots, k_n} \iint_J d\nu_\beta(x) d\nu_\beta(y) \frac{1}{a^p} g\left(\frac{\|T_{k_n}^{-1} \dots T_{k_1}^{-1}x - T_{k_n}^{-1} \dots T_{k_1}^{-1}y\|}{a}\right) \\ \times \frac{\exp 2n(-P(\beta))}{\prod_{i=0}^{n-1} \|DT(T_{k_{n-i}}^{-1} \dots T_{k_1}^{-1}x)\|^\beta \prod_{i=0}^{n-1} \|DT(T_{k_{n-i}}^{-1} \dots T_{k_1}^{-1}y)\|^\beta}.$$

The term  $\|T_{k_n}^{-1} \dots T_{k_1}^{-1}x - T_{k_n}^{-1} \dots T_{k_1}^{-1}y\|$  is bounded from above and below by an expression of the type  $\prod_{i=0}^{n-1} \|DT_{k_{n-i}}^{-1}(T_{k_{n-i-1}}^{-1} \dots T_{k_1}^{-1}\xi)\| \|x - y\|$ , where we set  $T_{k_0}^{-1} T_{k_1}^{-1} \xi = \xi$  and  $\xi \in V$  takes the values for which the product is respectively maximum and minimum. We note that the norm of the product of derivatives is equal to the product of the norms since the derivatives are conformal; still from the conformal property of the derivatives, the products in  $\xi$  can be arranged as

$$\prod_{i=0}^{n-1} \|DT(T_{k_{n-i}}^{-1} \dots T_{k_1}^{-1}\xi)\|^{-1} = \|DT^n(\tilde{\eta}_{k_1 \dots k_n})\|^{-1} \tag{3.2}$$

where

$$\tilde{\eta}_{k_1 \dots k_n} = T_{k_n}^{-1} \dots T_{k_1}^{-1} \xi \in A_{k_1 \dots k_n}.$$

By the distortion argument, we can replace, up to a uniform bound in  $n$ ,  $\tilde{\eta}_{k_1 \dots k_n}$  with any other point in  $A_{k_1 \dots k_n} \cap J$ ; we call  $\eta_{k_1 \dots k_n}$  this point. By property (ii)', we can bound  $g$  from above and below by an expression of the type

$$g\left(C^{\pm 1} \|DT^n(\eta_{k_1 \dots k_n})\|^{-1} \frac{\|x - y\|}{a}\right)$$

where  $C^{\pm 1}$  are the factors given by (2.1); we will not write them explicitly since they will be absorbed by the rescaled  $a$  which we are going to define. Then

$$T_p(a) \cong B_n(a) + \sum_{k_1, \dots, k_n} \frac{1}{a^p} \iint_J g\left(\|DT^n(\eta_{k_1 \dots k_n})\|^{-1} \frac{\|x - y\|}{a}\right) \\ \times \exp(-2nP(\beta)) \frac{1}{\|DT^n(\eta_{k_1 \dots k_n})\|^{2\beta}} d\nu_\beta(x) d\nu_\beta(y).$$

Observing that, for a rescaled  $a$ , the integral on the right-hand side gives itself an iwr, we get†

$$T_p \left( \prod_{J_1 \dots J_n} \|DT^n(\eta_{J_1 \dots J_n})\|^{-1} a \right) \cong B_n \left( \prod_{J_1 \dots J_n} \|DT^n(\eta_{J_1 \dots J_n})\|^{-1} a \right) + \sum_{k_1 \dots k_n} \frac{\exp(-2nP(\beta))}{\|DT^n(\eta_{k_1 \dots k_n})\|^{-(p-2\beta)}} T_p \left( \prod_{\substack{J_1 \dots J_n \\ J_1 \neq k_1 \\ \vdots \\ J_n \neq k_n}} \|DT^n(\eta_{J_1 \dots J_n})\|^{-1} a \right).$$

We then factorize in the sums  $\sum_{k_1 \dots k_n}$  respectively the maximum and the minimum of  $T_p$ . For sufficiently small values of  $a$ , these terms can be replaced, up to a uniform constant, by  $T_p(\alpha a)$ , where  $\alpha$  is now a particular factor  $\prod_{J_1 \dots J_n} \|DT^n(\eta_{J_1 \dots J_n})\|^{-1}$  fixed once and for all; we use here the adaptedness of the wavelet. We then get

$$1 \cong \exp n \left( -2P(\beta) + \frac{1}{n} \log \sum_{k_1 \dots k_n} \|DT^n(\eta_{k_1 \dots k_n})\|^{p-2\beta} \right).$$

When  $n \rightarrow +\infty$ , the term  $(1/n) \log \sum_{k_1 \dots k_n} \|DT^n(\eta_{k_1 \dots k_n})\|^{p-2\beta}$  converges to  $P(2\beta - p)$  [4, 6–8, 10]; since  $p$  is clearly independent of  $n$  and the preceding relation is true for any  $n$ , then  $p$  must satisfy the equation

$$P(2\beta - p) = 2P(\beta).$$

By the convexity properties of the functional  $P$  [4, 11] it follows that  $p$  is unique and, comparing with (2.3), we finally have  $p = D_\beta(2)$ .  $\square$

#### 4. Examples

For smooth domains  $J$  in  $\mathbb{R}^n$ , our method recovers the integer dimension of  $J$ . Thus, it is sufficient to compute the transition point for an  $n$ -unit hypercube in  $\mathbb{R}^n$  with respect to the Lebesgue measure. In fact we have the following.

*Proposition.* Let  $J = [0, 1]^n$  be the unit hypercube in  $\mathbb{R}^n$  and  $\mu(d\mathbf{x}) = dx_1 \dots dx_n$  the  $n$ -dimensional Lebesgue measure. Then, if the wavelet  $g$  is  $p$ -adapted to  $\mu$ ,  $p = n$ .

*Proof.* We decompose  $J$  in  $2^n$  hypercubes  $\tilde{J}_i$  of sides  $\frac{1}{2}$ , just dividing each side of  $J$  in two equal parts. Put  $\tilde{J}_1$  the cube with a vertex at  $\mathbf{0} \equiv (0, \dots, 0)$ . Since

$$\int_{\tilde{J}_i} \int_{\tilde{J}_i} g \left( \frac{\|\mathbf{x} - \mathbf{y}\|}{a} \right) \mu(d\mathbf{x}) \mu(d\mathbf{y}) = \int_{\tilde{J}_i} \int_{\tilde{J}_i} g \left( \frac{\|\mathbf{x} - \mathbf{y}\|}{a} \right) \mu(d\mathbf{x}) \mu(d\mathbf{y})$$

where  $i = 2, \dots, 2^n$  and  $\|\cdot\|$  is any norm in  $\mathbb{R}^n$ , dilatating  $\tilde{J}_1$  up to  $J$ , we easily get the Jacobian of the transformation being  $2^{-2n}$ ,

$$T_p(a) = \frac{2^n}{a^p} \int_{\tilde{J}_i} \int_{\tilde{J}_i} g \left( \frac{\|\mathbf{x} - \mathbf{y}\|}{a} \right) \mu(d\mathbf{x}) \mu(d\mathbf{y}) = \frac{2^{-n}}{a^p} \int_J \int_J g \left( \frac{\|\mathbf{x} - \mathbf{y}\|}{2a} \right) \mu(d\mathbf{x}) \mu(d\mathbf{y})$$

from which  $T_p(a/2) = 2^{p-n} T_p(a)$ , which gives the result.  $\square$

† We are not able to achieve this step if the repeller is connected; a suitable use of the Markov partition of  $J$  should allow us to overcome this difficulty.

In this case the adaptedness of some wavelet can be easily proved. Let, for example,  $g(r)$  be equal to  $e^{-r}$ ; using for the computation of the iWT the norm  $\|\mathbf{x} - \mathbf{y}\| = \sum_{i=1}^n |x_i - y_i|$ , we easily get

$$T_p(a) = \frac{2^n}{a^{p-n}} + o(1) \quad a \rightarrow 0^+ \tag{4.1}$$

which shows a transition point at  $p = n$ , where the iWT has a finite positive limit.

The next step is to compute the iWT for fractal sets. We proceed as follows: we show, with an heuristic argument, that, for a large class of fractals, precisely those satisfying the scaling (4.4) below, the wavelets considered in section 1 are adapted. We can only compute the iWT in a rigorous way for a class of multifractal ternary Cantor sets, for a particular subsequence  $a_n \rightarrow 0^+$ , and show that it gives the correlation dimension, in accordance with the theorem. Finally we give some examples on the numerical utilization of the iWT.

### 4.1. A qualitative analysis

By a useful trick already used in [15] in a different context, we can write (1.3) as

$$T_p(a) = \frac{1}{a^p} \int_0^\Delta g\left(\frac{r}{a}\right) dC(r) = \frac{1}{a^p} \left[ g\left(\frac{r}{a}\right) C(r) \right]_0^\Delta - \frac{1}{a^{p+1}} \int_0^\Delta C(r) g'\left(\frac{r}{a}\right) dr \tag{4.2}$$

where  $\Delta$  is the diameter of  $J$  and  $dC(r)$  is the Stieltjes measure defined by the monotone increasing positive function

$$C(r) = \int_J \int_J \theta(r - \|x - y\|) d\mu(x) d\mu(y) \tag{4.3}$$

where  $\theta$  is the Heaviside function. From now on we neglect the term in the brackets on the right-hand side of (4.2), since it vanishes in the limit  $a \rightarrow 0^+$  by property (i). For an invariant ergodic measure  $\mu$ , we can express  $C(r)$  as a double time series and change it in the correlation integral mentioned in the remark of section 3, whose scaling for several fractal sets has been found to obey [14]

$$C(r) \sim r^v \quad r \rightarrow 0^+ \tag{4.4}$$

where  $v$  is called the correlation dimension of  $J$ . The scaling (4.4) was rigorously proved for a few smooth sets and a formal expression was computed for a class of multifractal Cantor sets, using the theory of the energy integral [15]. In all these cases the scaling exponent  $v$  coincides with  $D_B(2)$  given by (2.3).

Extending (4.4) up to  $\Delta$ , we can express the iWT approximately as the Mellin transform of  $g'$ , namely

$$|T_p(a)| \sim \frac{1}{a^{p-v}} \left| \int_0^{\Delta/a} g'(y) y^v dy \right|. \tag{4.5}$$

By properties (i) and (ii)', the integral on the right-hand side is finite when  $a \rightarrow 0^+$ . For example, it is equal to  $v\Gamma(v)$  for  $g(r) = e^{-r}$  ([17], p 312). We can also write (4.5) as

$$\log |T_p(a)| \underset{a \rightarrow 0^+}{\sim} (v - p) \log a + \log |\psi(a)| \tag{4.6}$$



which is consistent with the corollary of the theorem and where  $\psi(a)$  is a function of order 1 for  $a \rightarrow 0^+$ . †

*Remark.* Until now we have used the scaling of the correlation integral to compute the iwr by means of representation (4.2). But the procedure could be inverted: if one knows the asymptotic structure of the iwr for  $a \rightarrow 0^+$ , one could try to compute the correlation integral just taking a suitable inverse wavelet transform of  $T_p(a)$ . We think that this would be an interesting subject to investigate.

4.2. *Multifractal ternary Cantor sets*

We recall that the ternary Cantor set on the unit interval is generated by two inverse branches with the same slope  $\lambda = \frac{1}{3}$ , but the following considerations also hold for  $0 < \lambda < \frac{1}{3}$ . To make these sets multifractals and then distinguish among the generalized dimensions, we endow them with the balanced measure  $\eta$  of positive weights  $p_1, p_2, p_1 \neq p_2$ , which is an invariant and ergodic probability measure satisfying a condition like (2.2) (see [8] and references therein), namely

$$\eta(T_i^{-1}A) = p_i \eta(A) \quad i = 1, 2 \tag{4.7}$$

where  $p_1 + p_2 = 1$  and  $A$  is any measurable subset of the unit interval.

The generalized dimensions  $D_\eta(q)$  corresponding to this measure satisfy [8]

$$D_\eta(q)(q - 1) = \frac{\log(p_1^q + p_2^q)}{\log \lambda}. \tag{4.8}$$

Clearly, our theorem works as well for this class of measures; moreover, we can prove the following:

*Proposition.* For the ternary Cantor set on the unit interval with the measure  $\eta$  of weights  $p_1, p_2$ , the iwr can be computed with the subsequence  $a = \lambda^n$ , and the transition point is  $D_\eta(2)$ .

*Proof.* Applying (4.7) to the integral (4.3) for  $0 < \lambda \leq \frac{1}{3}$ , we get

$$C(\lambda r) = (p_1^2 + p_2^2)C(r) \quad \text{or, iterating,} \quad C(\lambda^n r) = (p_1^2 + p_2^2)^n C(r). \tag{4.9}$$

Computing the iwr in the form (4.2) with the subsequence  $a = \lambda^n$ , neglecting again the term into the brackets and using the scaling (4.9), we have

$$|T_p(\lambda^n)| = (p_1^2 + p_2^2)^n \lambda^{-np} \left| \int_0^{1/\lambda^n} C(r)g'(r) dr \right|. \tag{4.10}$$

When  $n \rightarrow +\infty$ , the integral in (4.10) is finite by (i) and (ii)'; the same is true for  $|T_p(\lambda^n)|$  if  $p$  is equal to  $\log(p_1^2 + p_2^2)/\log \lambda$ , that is,  $D_\eta(2)$  given by (4.8). □

*Remark.* In the particular case of the linear Cantor sets with the balanced measure, a different average of the wavelet transform was proposed in [18] to detect the fractal dimensions.

† We must observe that the scaling (4.4) is usually modulated by periodic factors in  $\log a$  ('lacunarity', see [16]), which could induce damped oscillations in the asymptotic behaviour of  $T_p(a)$ ; for example,  $\psi(a)$  could be itself a periodic function of  $\log a$ . We will return to this point in part 4.2 of this section.

4.3. Numerical computations

We now want to illustrate the numerical applications of the iwr.

We checked the scaling (4.6) on the linear Cantor set on  $[0, 1]$  generated by two inverse branches of slope  $\lambda_1 = \frac{1}{2}$  and  $\lambda_2 = \frac{1}{4}$ , with the balanced measure of weights  $p_1 = p_2 = \frac{1}{2}$ . We have a general implicit formula for the correlation dimension  $D_2$ , similar to (4.8), namely

$$\lambda_1^{-D_2} p_1^2 + \lambda_2^{-D_2} p_2^2 = 1. \tag{4.11}$$

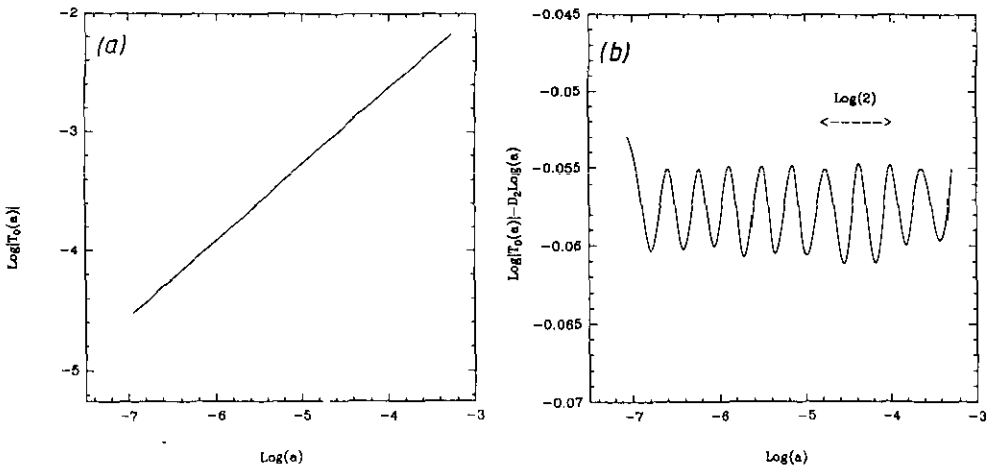
In our case, we get

$$D_2 = \frac{\log[(-1 + \sqrt{17})/2]}{\log 2} = 0.642\,981\,36\dots$$

We performed the numerical computation of the iwr with the functions  $g_1(r) = e^{-r}$  and  $g_2(r) = e^{-r^2}$ , averaging over the predecessors of any point in the unit interval with the same probability, up to 14 iterations, for which the computation becomes stable.

Figures 1(a) and 2(a) show the results as log-log plots for both the wavelets. In particular, the value detected for  $\nu$  is 0.6553... with  $g_1$  and 0.6487... with  $g_2$  (compare with the true solution of (4.11) computed above).

As predicted in the last footnote, the asymptotic behaviour of  $\log |T_p(a)|$  is affected by oscillations. Their amplitude is very small and does not appear in figures 1(a) and 2(a), but it is particularly evident in figures 1(b) and 2(b), where we plot  $\log |T_0(a)| - D_2 \log a$ ,  $D_2$  being the true value for the dimension. The period of the oscillations has been found to be  $(\log 2)/2$ . We performed the same computations for the ternary Cantor set with balanced measure and the functions  $g_1$  and  $g_2$ . We report the results in figures 3 and 4; the period of the oscillations is  $\log 3$ , which is the same as found for the oscillations of the correlation integral in [16] and for the oscillations of the usual wavelet in [3].



**Figure 1.** (a)  $\log |T_0(a)|$  versus  $\log a$  for the Cantor set with scales  $\lambda_1 = \frac{1}{2}$  and  $\lambda_2 = \frac{1}{4}$ , with  $g(r) = e^{-r}$ ; The value for the correlation dimension is  $\nu = 0.6553\dots$  (b)  $\log |T_0(a)| - D_2 \log a$  versus  $\log a$  for the Cantor set of part (a) with  $g(r) = e^{-r}$ .

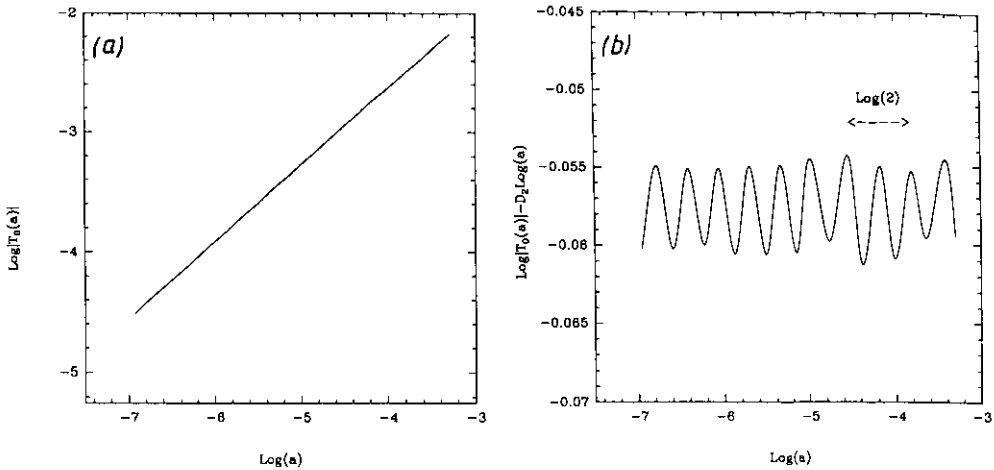


Figure 2. (a) As figure 1(a) for  $g(r) = e^{-r^2}$ ; the value for the correlation dimension is  $\nu = 0.6487\dots$  (b), As figure 1(b) for  $g(r) = e^{-r^2}$ .

Figures 3(b) and 4(b) show that the wavelet  $g_2$  works better than  $g_1$  in the range of small values of  $a$ , comparable with the scales of the set, determined by the largest order of iteration. This is certainly related to the rate of decay of the iwr, which is measured by the quantity (see (4.5))

$$\left| \int_{\Delta/a}^{+\infty} g'(r)r^\nu dr \right| \quad a \rightarrow 0^+. \tag{4.12}$$

These oscillations can be easily explained by an argument similar to that proposed in [16] for the correlation integral. As said in the proof of the theorem, we can apply the balance property (4.7) only to the Cartesian product of equal sets in order to

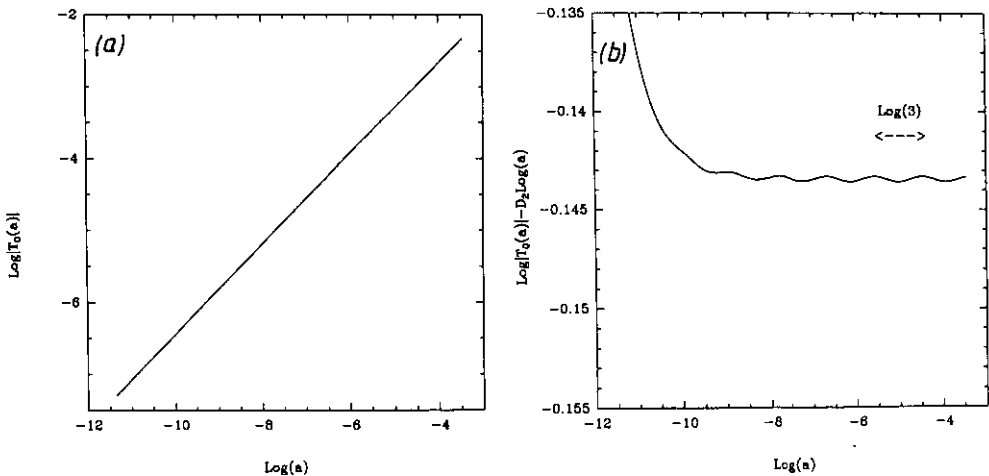


Figure 3. (a) As figure 1(a) for the ternary Cantor set, with  $g(r) = e^{-r}$ ; the value for the correlation dimension is  $\nu = 0.6378\dots$  (b) As figure 1(b) for the ternary Cantor set, with  $g(r) = e^{-r}$ .

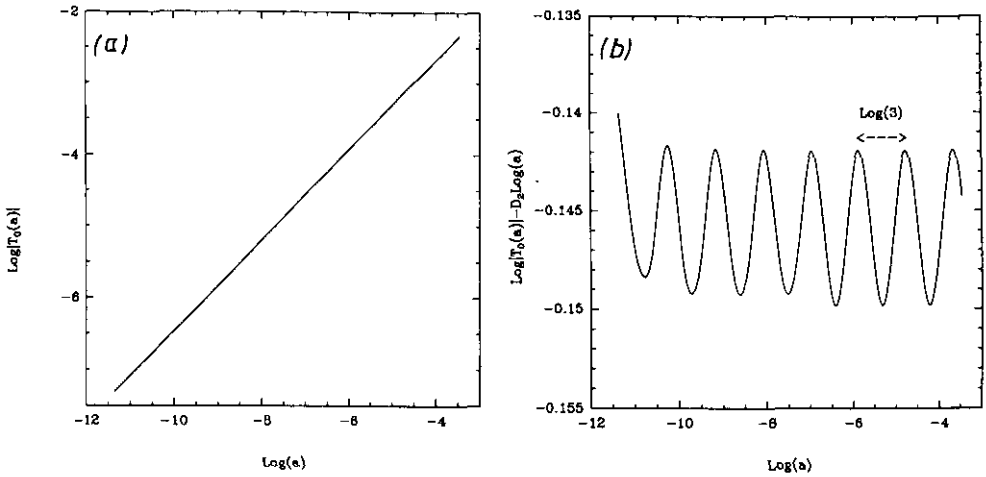


Figure 4. (a) As figure 3(a) for  $g(r) = e^{-r^2}$ ; the value for the correlation dimension is  $\nu = 0.6392\dots$  (b) As figure 3(b) for  $g(r) = e^{-r^2}$ .

compute the IWT: the term we neglect converges to zero for  $a \rightarrow 0^+$  faster than any power of  $a$  by property (i). For the linear Cantor set with scales  $\lambda_1 = \frac{1}{2}$  and  $\lambda_2 = \frac{1}{4}$  and the measure of weights  $p_1 = p_2 = \frac{1}{2}$ , we thus get

$$T_0(a) = \frac{1}{4}T_0(2a) + \frac{1}{4}T_0(4a) + o(1) \quad a \rightarrow 0^+ \tag{4.13}$$

It is easy to check that the general real solution of (4.13) without the  $o(1)$  term is of the form  $a^\alpha \psi(\log a)$ , where  $\psi$  is a continuous periodic function of  $\log a$  of period  $(\log 2)/k$ ,  $k \in \mathbb{N}$  and  $\alpha$  is just given by (4.11). For the ternary Cantor set with equal scales  $\frac{1}{3}$ , we get

$$T_0(a) = \frac{1}{2}T_0(3a) + o(1) \quad a \rightarrow 0^+ \tag{4.14}$$

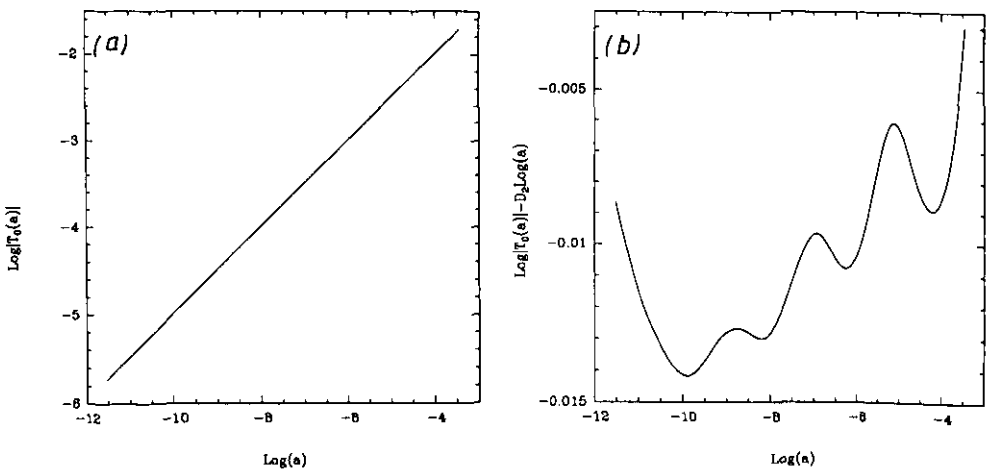
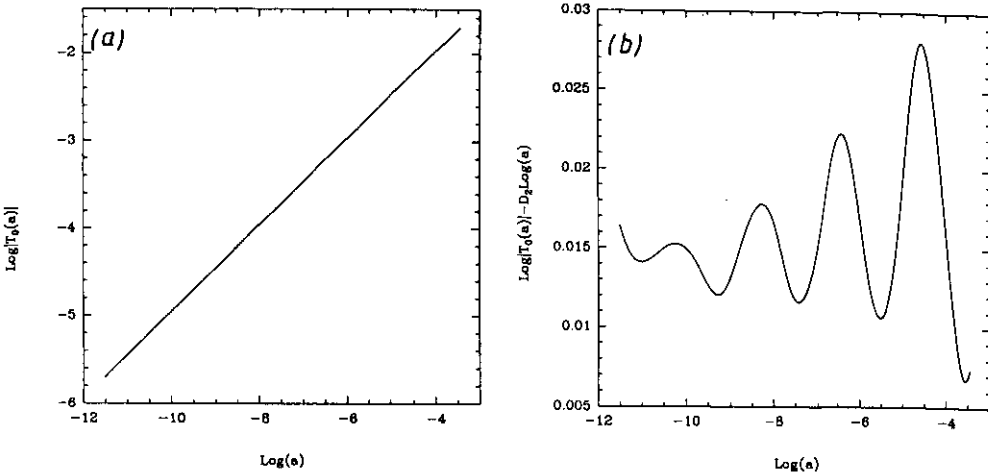


Figure 5. (a) As figure 1(a) for the period-doubling Cantor set with  $g(r) = e^{-r}$ ; the value for the correlation dimension is  $\nu = 0.4976\dots$  (b) As figure 1(b) for the period-doubling Cantor set, with  $g(r) = e^{-r}$ ; here and for figures 6(b), 7(b) and 8(b), we use for  $D_2$  the value of  $\nu$  found numerically.



**Figure 6.** (a) As figure 5(a) for  $g(r) = e^{-r^2}$ ; the value for the correlation dimension is  $\nu = 0.4951\dots$  (b) As figure 5(b) for  $g(r) = e^{-r^2}$ .

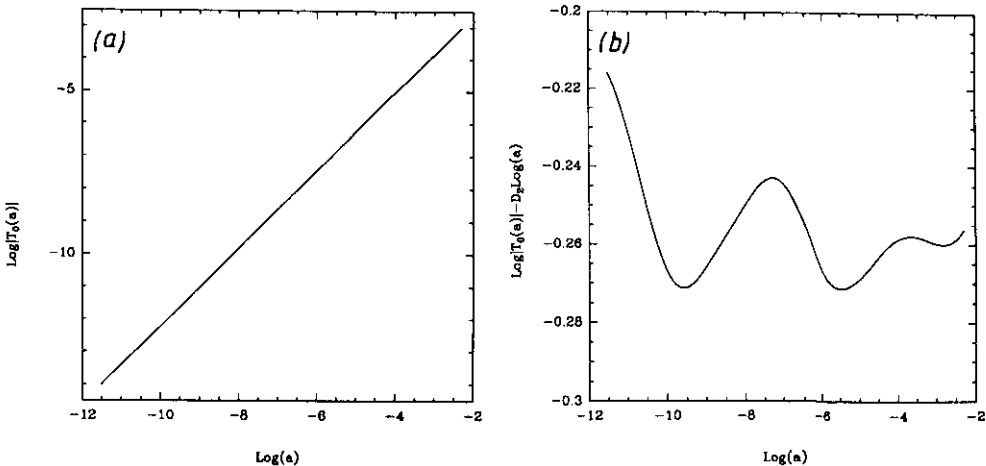
which admits a general solution of the form (without the  $o(1)$  term)  $a^\alpha \psi(\log a)$ , where  $\psi$  is a continuous periodic function of  $\log a$  of period  $(\log 3)/k$ ,  $k \in \mathbb{N}$  and  $\alpha = \log 2/\log 3$ .

The next applications show that the IWT works as well for non-hyperbolic sets.

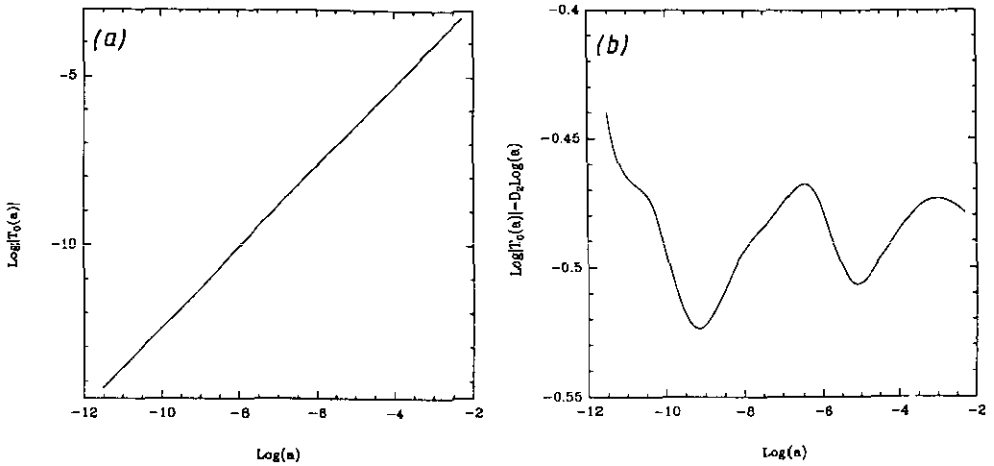
We first consider the period-doubling Cantor set generated by the logistic map

$$x' = ax(1 - x) \tag{4.15}$$

where  $a = a_\infty = 3.5699456$  is the point of onset of chaos via period-doubling bifurcations (see, for instance, [12, 14]). The value of the correlation dimension found by the correlation integral technique [14] is  $\nu = 0.500 \pm 0.005$ , which is in good agreement with the value computed by means of the IWT, which gives



**Figure 7.** (a) As figure 1(a) for the Hénon attractor, with  $g(r) = e^{-r}$ ; the value for the correlation dimension is  $\nu = 1.1977\dots$  (b) As figure 1 (b) for the Hénon attractor, with  $g(r) = e^{-r}$ .



**Figure 8.** (a) As figure 7(a) for  $g(r) = e^{-r^2}$ ; the value for the correlation dimension is  $\nu = 1.1942\dots$  (b) As figure 7(b) for  $g(r) = e^{-r^2}$ .

$\nu = 0.4976\dots$ , for  $g(r) = e^{-r}$  and  $\nu = 0.4951\dots$ , for  $g(r) = e^{-r^2}$ . The iwt was computed giving equal weights to the elements of the partition constructed, iterating  $2^{15}$  times the critical point of the mapping, as explained in [3] and [12]; the iterates of the critical point  $x = \frac{1}{2}$  accumulate in fact on the attracting set of (4.15). In figures 5(b) and 6(b) we show the oscillations of the iwt for decreasing values of  $a$ .

In figures 7 and 8 we report the computation of the iwt for the Hénon attractor generated by the mapping  $x' = 1 - ax^2 + y$ ;  $y' = bx$ , with  $a = 1.4$  and  $b = 0.3$ . The preceding values found for the correlation dimension were, for example,  $\nu = 1.21 \pm 0.01$  in [14] and  $\nu = 1.199 \pm 0.003$  in [20]. The iwt gave us  $\nu = 1.1977\dots$ , for  $g(r) = e^{-r}$  and  $\nu = 1.1942\dots$ , for  $g(r) = e^{-r^2}$ . The iwt was computed assuming, as usual, the existence of an ergodic physical measure on the attractor and then approximating the integral by an ergodic mean, with a number of iterations equal to 10 000. The oscillations are shown very clearly by our method, although they appear more irregular than for the one-dimensional Cantor sets described above. The same difficulty arises using the correlation integral technique, and this is assumed to be related to the non-hyperbolicity of the mapping, more precisely to the presence of homoclinic tangencies (see [20] and references therein).

### 5. Conclusion

In conclusion, we think that the iwt is a powerful tool with which to analyse fractal sets and that its range of applicability could be extended beyond the mixing repellers or the deterministic invariant sets.

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