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## Corrections to the Scaling Laws of Integrated Wavelets from Singularities of Mellin Transforms.

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**Abstract.** - By considering the Mellin transform of the Integrated Wavelet Transform (IWT), we show that correction to scaling laws for the IWT can be conveniently described in terms of the meromorphic spectrum of the Mellin transform. Relevant singularities are the same as the ones obtained with the energy integral formalism.

The wavelet transform has been widely used in recent times as a useful tool in analysing the local behaviour of systems exhibiting scaling laws [1,2]. Integrated wavelet transforms (IWT) were also recently introduced and it was shown that basically the same results as from correlation integrals, concerning global exponents, can be obtained [3,4].

The basic difference between the usual correlation integral and an integrated wavelet transform is the use of a smooth monotonic function rather than a stepwise function to analyse the behaviour of the invariant measure of a given dynamical system.

Since the scaling law and its corrections are conveniently described by considering the meromorphic spectrum of the Mellin transform of the correlation integrals (energy integrals) [5], it is natural to apply the same technique to the IWT. The result is that the singularities associated to the measure are the same as the one obtained for the energy integral (lying on the right half complex plane), while other singularities, related to the peculiar nature of the wavelet transform, are present but completely separated from the previous ones (left half complex plane).

The leading correction to the scaling law is determined by the former set of singularities while the second set gives completely negligible terms vanishing exponentially fast at short distance.

For self-similar systems like Cantor sets the correction term is periodic with the same period in both cases and it is shown that the integrated wavelet transform's Fourier components of order  $m$  are damped by a factor  $\exp[-cm]$ , where  $c$  is a constant. As a consequence the correction term of the IWT is much smoother with respect to the correlation integral as fully confirmed by numerical results.

We refer, for the class of wavelets considered here, to the definitions of ref. [2] and [4]. Given a compact subset  $J$  of a metric space (with the metric  $\|\cdot\|$ ) endowed with a probability Borel non-atomic measure  $\mu$  and a monotone decreasing function  $g: R \rightarrow R$  of class  $C^1$

( $g'(t) \leq 0$  for  $t \geq 0$ ), the integrated wavelet transform is defined by

$$T_p(l) = l^{-p} \int \int g\left(\frac{\|x-y\|}{l}\right) d\mu(x) d\mu(y), \quad (1)$$

where  $\lim_{l \rightarrow 0^+} l^{-p} g(r/l) = 0$  for  $p \geq 0$  and  $l > 0$ .

The Mellin transform of the IWT is given by

$$\Psi_p(\alpha) = \int_0^\infty l^{-\alpha} dT_p(l). \quad (2)$$

When  $p = 0$ , it can be checked that  $dT/dl > 0$  and therefore  $\Phi_0(\alpha)$  is the Mellin transform of a positive measure just as for the energy integral case.

Interchanging the integration orders, we can write

$$\Psi_p(\alpha) = \Phi(\alpha + p) \gamma_p(\alpha), \quad (3)$$

where  $\Phi(\alpha)$  is the energy integral

$$\Phi(\alpha) = \int_0^\infty l^{-\alpha} dC(l) \equiv \int_0^\infty \|x-y\|^{-\alpha} d\mu(x) d\mu(y) \quad (4)$$

and

$$\gamma_p(\alpha) = p \int t^{\alpha+p-1} g(t) dt + \int t^{\alpha+p} g'(t) dt. \quad (5)$$

In order to evaluate  $\Psi_p(\alpha)$ , we must observe that the integral in the r.h.s. of (4) is well defined for  $\Re\alpha + p < \bar{\alpha}$ , where  $\bar{\alpha} > 0$  is the so-called divergence abscissa and is a lower bound to the Hausdorff dimension of the set [6]. Conversely the integral in the r.h.s. of (5) is well defined for  $\Re\alpha + p > 0$ . As a consequence in the strip  $-p < \Re\alpha < -p + \bar{\alpha}$  both integrals are well defined and it is possible to continue analytically the result in the whole  $\alpha$ -plane.

It is evident that the dynamical singularities associated to the measure are in the right  $\alpha + p$ -plane, while the wavelet transform singularities are in the left plane. For simplicity we choose from now on the following scaling function:

$$g(t) = \exp[-t] \quad (6)$$

and explicitly verify, in the simple case of a system defined on  $[0, 1]$  with  $\mu(x)$  equal to the Lebesgue measure, that the limits interchange leading to (3) is correct. We have for  $p = 0$

$$C(l) = \begin{cases} 2l(1-l/2), & l < 1, \\ 1, & l > 1, \end{cases} \quad T_0(l) = 2l[1 + l(\exp[-1/l] - 1)], \quad (7)$$

and it is relevant to observe that  $dT_0(l)/dl$  behaves as a constant for  $l \rightarrow 0$  and as  $1/l^2$  for  $l \rightarrow \infty$ .

Computing  $\Psi_0(\alpha, R) = \int_0^R l^{-\alpha} dT_0(l)$  for finite  $R$ , one verifies that the previous behaviours allow the limit  $R \rightarrow \infty$  to be taken correctly. So we recover the result (3) which explicitly

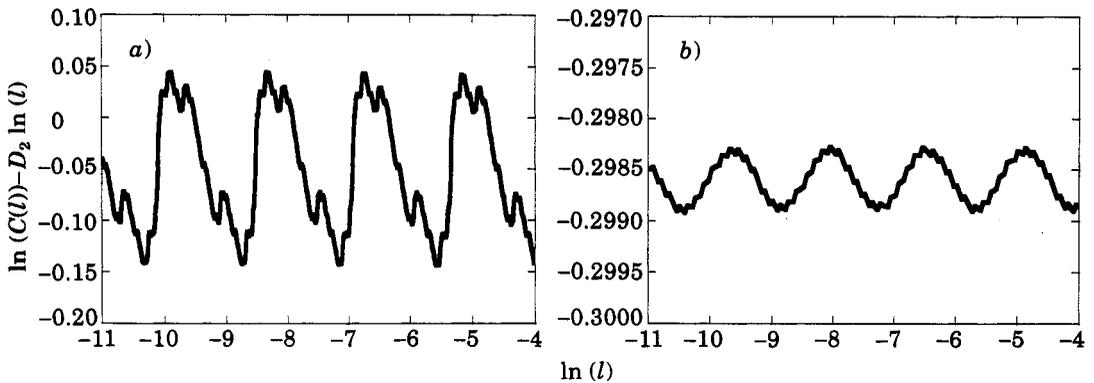


Fig. 1. - a)  $\ln C(l) - D_2 \ln(l)$  for the ternary Cantor set with  $p_1 = p_2 = 1/2$ . b)  $\ln |T_0(l)| - D_2 \ln(l)$  for the same Cantor set, using the scaling function  $g(t) = \exp[-t]$ .

reads in this case  $\Psi_0(\alpha) = \Phi(\alpha) \gamma_0(\alpha)$ , where  $\Phi(\alpha) = 2/(1 - \alpha) - 2/(2 - \alpha)$ ,  $\gamma_0(\alpha) = \Gamma(\alpha + 1)$  and  $\Gamma$  is the Euler function.

Another interesting case is given by a self-similar system like the Cantor with two equal scales  $\lambda_1 = \lambda_2 = \lambda$  and equal weights  $p_1 = p_2 = 1/2$ , where the singularities of  $\Phi(\alpha)$  are known to be equally spaced on a straight line parallel to the imaginary axis [5]:

$$\alpha_m = \frac{\ln 2}{\ln 1/\lambda} + \frac{i2\pi m}{\ln 1/\lambda}. \tag{8}$$

Even in this case  $\gamma_0(\alpha) = \Gamma(\alpha + 1)$  and it is easy to show that the residues  $\hat{r}_m$  of  $\Psi_0(\alpha)$  at the poles  $\alpha_m$  are related to the residues  $r_m$  of  $\Phi(\alpha)$  by

$$\hat{r}_m = r_m \Gamma(1 + \alpha_m) \approx r_m \exp \left[ -\frac{\pi^2}{\ln 1/\lambda} m \right]. \tag{9}$$

It was shown [7] that  $r_m$  decrease almost as  $m^{-1}$  while we can see here that the  $\hat{r}_m$  have a very fast exponential decay. As a consequence the behaviour of the IWT corrections to the scaling, which is still given by a periodic function in  $\ln \lambda$ , will be very smooth since only the first Fourier coefficient has a significant amplitude. Figure 1, where  $C(l)$  is compared with  $T_0(l)$  for the Cantor set mentioned above, fully confirms this result which is extensible to any linear Cantor and to Julia set [8].

We remark that the method illustrated in this letter can be easily extended to the study of local properties of the measure employing the usual wavelet transform instead of the integrated wavelet transform.

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